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Wolfgang Schmid

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Postal address:

Mathematical Statistics
Dept. of Mathematics
Stockholm University
SE-106 91 Stockholm
Sweden

Internet:

http://www.math.su.se

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Tests for the weights of the global minimum variance portfolio in a high-dimensional setting

Taras Bodnar[a, *], Solomiia Dmytriv[b], Nestor Parolya[c, d] and Wolfgang Schmid[b]

[a]Department of Mathematics, Stockholm University, Stockholm, Sweden

[b]Department of Statistics, European University Viadrina, Frankfurt(Oder), Germany

[c]Institute of Statistics, Leibniz University Hannover, Hannover, Germany

[d]Department of Economics, Heidelberg University, Heidelberg, Germany

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Abstract: In this paper we construct two tests for the weights of the global minimum variance portfolio (GMVP) in a high-dimensional setting, namely when the number of assets p depends on the sample size n such that $\frac{p}{n} \to c \in (0,1)$ as n tends to infinity. The considered tests are based on the sample estimator and on the shrinkage estimator of the GMVP weights. The asymptotic distributions of both test statistics under the null and alternative hypotheses are derived. Moreover, we provide a simulation study where the power functions of the proposed tests are compared with other existing approaches. A good performance of the test based on the shrinkage estimator is observed even for values of c close to 1.

Keywords: Portfolio analysis; Global minimum variance portfolio; Statistical test; Shrinkage estimator; Random matrix theory.

*Corresponding Author: Taras Bodnar. E-Mail: taras.bodnar@math.su.se. Tel: +46 8 164562. Fax: +46 8 612 6717.

1 Introduction

There is a rapid development of financial markets over the last period of time and the amount of money invested in risky assets extremely increases. Due to this fact an investor requires the knowledge of optimal portfolio proportions in order to receive a large expected return and, at the same time, to reduce the level of the risk associated with the investment decision.

Since Markowitz (1952) presented his mean-variance analysis, a lot of works about optimal portfolio selection were published. However, one is faced with some difficulties in the practical implementation of these investing theories, since the sampling error is present when the unknown theoretical quantities are estimated.

In the classical asymptotic analysis it is almost always assumed that the sample size increases while the size of the portfolio, namely the number of included assets p, remains constant (e.g., Jobson and Korkie (1981), Okhrin and Schmid (2006)). Nowadays, this case is often called standard asymptotics (see Cam and Yang (2000)). Here, the traditional plug-in estimator of optimal portfolio, the so-called sample estimator, is consistent and asymptotically normally distributed. However, in many applications the number of assets in a portfolio is large in comparison to the sample size, i.e., the portfolio dimension p and the sample size n tend to infinity simultaneously, such that $\frac{p}{n}$ tends to the concentration ratio $c > 0$. In that case, we are faced with the so-called high-dimensional asymptotics or "Kolmogorov" asymptotics (see Bühlmann and Van De Geer (2011), Bai and Shi (2011), Cai and Shen (2011)). Whenever the dimension of the data is large, the classical limit theorems are no longer suitable, because the traditional estimators will result in a serious departure from the optimal ones under the high-dimensional asymptotics (Bai and Silverstein (2010)). They fail to provide consistent estimators of the unknown parameters of the asset returns, i.e., the mean vector and the covariance matrix. Generally, the greater the concentration ratio c, the worse are the sample estimators. In these cases new test statistics must be developed and completely new asymptotic techniques must be applied for their derivations. There are several papers dealing with the high-dimensional asymptotics in portfolio theory using results from random matrix theory (c.f., Frahm and Jaekel (2008), Laloux et al. (2000)). Recently, Bodnar, Parolya and Schmid (2017) present a shrinkage-type estimator for the global minimum variance portfolio (GMVP) weights, while Bodnar, Okhrin and Parolya (2017) derived the optimal shrinkage estimator of the mean-variance portfolio.

The problem of testing the efficiency of a portfolio is a classical problem in finance. The former literature focuses on the case of standard asymptotics or consider exact tests where both p and n are fixed. For example, Gibbons, Ross and Shanken (1989) provided an exact F-tests for testing the efficiency of a given portfolio, while Britten-Jones (1999) derived inference procedures on the efficient portfolio weights based on the application of linear regression. More recently, Bodnar and Schmid (2008) presented a test for the general linear hypothesis of the portfolio weights in the case of elliptically contoured distributions. The contribution of this paper is the derivation of statistical techniques for testing the efficiency of a portfolio under the high-dimensional asymptotics. Two statistical tests are considered. While the first approach is based on the asymptotic distribution of the test statistic suggested by Bodnar and Schmid (2008) in the high-dimensional setting, the second test makes use of the shrinkage estimator of the GMVP weights and provides a powerful alternative to the existing methods. To the best of our knowledge, it is the first time that the shrinkage approach is applied to statistical test theory.

The paper is structured as follows. In Section 2, we discuss main results on distributional properties for optimal portfolio weights, presented by Okhrin and Schmid (2006). In Section 3, a new test based on the shrinkage estimator for the GMVP weights is derived and the high-dimensional version of the test based on the test statistics given in Bodnar and Schmid (2008) is proposed. The asymptotic distributions of the test statistics under both the null hypothesis and the alternative hypothesis are obtained and the corresponding power functions of both tests are presented. In Section 4, the power functions of the proposed tests are compared with each other for different values of $c \in (0,1)$. In our comparison study a test of Glombeck (2014) is considered as well. We conclude in Section 5. All proofs are given in the Appendix.

2 Estimation of Optimal Portfolio Weights

We consider a financial market consisting of p risky assets. Let $\mathbf{X}_t$ denote the p-dimensional vector of the returns on risky assets at time t. Suppose that $E(\mathbf{X}_t) = \boldsymbol{\mu}$ and $Cov(\mathbf{X}_t) = \mathbf{\Sigma}$. The covariance matrix $\mathbf{\Sigma}$ is assumed to be positive definite.

Let us consider a single period investor, who invests into the GMVP, one of the mostly used portfolios (see, e.g., Memmel and Kempf (2006), Frahm and Memmel (2010), Okhrin and Schmid (2006), Bodnar and Schmid (2008), Glombeck (2014) and others). It exhibits the smallest attainable portfolio variance $\mathbf{w}'\mathbf{\Sigma}\mathbf{w}$ under the constraint $\mathbf{w}'\mathbf{1} = 1$, where $\mathbf{1} = (1, \ldots, 1)'$ denotes the p-dimensional vector of ones and $\mathbf{w}$ stands for the vector of portfolio weights. The weights of GMVP are given by

$$\mathbf{w}_{GMVP} = \frac{\mathbf{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}}. \tag{1}$$

The practical implementation of the mean-variance framework in the spirit of Markowitz (1952) relies on estimating the first two moments of the asset returns. Because we do not know the true covariance matrix, it is usually replaced by its sample estimator which is based on a sample of $n > p$ historical asset returns $\mathbf{X}_1, \ldots, \mathbf{X}_n$ given by

$$\hat{\mathbf{\Sigma}}_n = \frac{1}{n-1} \sum_{j=1}^{n} \left(\mathbf{X}_j - \bar{\mathbf{X}}_n\right) \left(\mathbf{X}_j - \bar{\mathbf{X}}_n\right)' \text{ with } \bar{\mathbf{X}}_n = \frac{1}{n} \sum_{v=1}^{n} \mathbf{X}_v. \tag{2}$$

Replacing $\mathbf{\Sigma}$ in (1) by the sample estimator $\hat{\mathbf{\Sigma}}_n$, we get an estimator of the GMVP weights expressed as

$$\hat{\mathbf{w}}_n = \frac{\hat{\mathbf{\Sigma}}_n^{-1}\mathbf{1}}{\mathbf{1}'\hat{\mathbf{\Sigma}}_n^{-1}\mathbf{1}}. \tag{3}$$

Note that the estimator of the GMVP weights is exclusively a function of the estimator $\hat{\mathbf{\Sigma}}_n$ of the covariance matrix .

Assuming that the asset returns $\{\mathbf{X}_t\}$ follow a stationary Gaussian process with mean $\boldsymbol{\mu}$ and covariance matrix $\mathbf{\Sigma}$, Okhrin and Schmid (2006) proved that the vector of estimated optimal portfolio weights is asymptotically normal. Under the additional assumption of independence they derived the exact distribution of $\hat{\mathbf{w}}_n$. Okhrin and Schmid (2006) showed that the distribution of arbitrary $p-1$ components of $\hat{\mathbf{w}}_n$ is a $(p-1)$- dimensional t-distribution with $n-p+1$ degrees of freedom and

$$E(\hat{\mathbf{w}}_n) = \mathbf{w}_{GMVP}, Cov(\hat{\mathbf{w}}_n) = \mathbf{\Omega} = \frac{1}{n-p-1}\frac{\mathbf{Q}}{\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}}, \mathbf{Q} = \mathbf{\Sigma}^{-1} - \frac{\mathbf{\Sigma}^{-1}\mathbf{1}\mathbf{1}'\mathbf{\Sigma}^{-1}}{\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}}. \tag{4}$$

Consequently, if $\hat{\mathbf{w}}_n^*$ and $\mathbf{w}_{GMVP}^*$ are obtained by deleting the last element of $\hat{\mathbf{w}}_n$ and $\mathbf{w}_{GMVP}$ and if $\mathbf{\Omega}^*$ and $\mathbf{Q}^*$ are consisting of the first $(p-1)\times(p-1)$ elements of $\mathbf{\Omega}$ and $\mathbf{Q}$, then $\hat{\mathbf{w}}_n^*$ has a $(p-1)$- variate t-distribution with $n-p+1$ degrees of freedom and parameters $\mathbf{w}_{GMVP}^*$ and $\frac{1}{n-p+1}\frac{\mathbf{Q}^*}{\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}}$. It is denoted by $\hat{\mathbf{w}}_n^* \sim t_{p-1}(n-p+1, \mathbf{w}_{GMVP}^*, \frac{n-p-1}{n-p+1}\mathbf{\Omega}^*)$, since $\frac{n-p-1}{n-p+1}\mathbf{\Omega}^* = \frac{1}{n-p+1}\frac{\mathbf{Q}^*}{\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}}$.

3 Test Theory for the GMVP in High Dimensions

At each time point, an investor is interested to know whether the holding portfolio coincides with the true GMVP or it has to be reconstructed. For that reason we consider the following testing problem

$$H_0 : \mathbf{w}_{GMVP} = \mathbf{r} \qquad \text{against} \qquad H_1 : \mathbf{w}_{GMVP} \neq \mathbf{r}, \tag{5}$$

where $\mathbf{r}$ with $\mathbf{r}'\mathbf{1} = 1$ is a known vector, for example, the weights of the holding portfolio. This means that it is analyzed whether the true GMVP weights are equal to some given values.

Bodnar and Schmid (2008) analyzed a general linear hypothesis for the GMVP portfolio weights and introduced an exact test assuming that the asset returns are independent and elliptically contoured distributed. Moreover, they derived the exact distribution of the test statistic under the null hypothesis and under the alternative hypothesis.

The main focus of this paper lies on high-dimensional portfolios. We want to consider the testing problem (5) in a high-dimensional environment, i.e. assuming that $\frac{p}{n} \to c \in (0,1)$ as $n \to \infty$. Note that in this case H_0 and H_1 depend on n as well. Thus it would be more precisely to write $H_{0,n} : \mathbf{w}^*_{GMVP,n} = \mathbf{r}^*_n$ and $H_{1,n} : \mathbf{w}^*_{GMVP,n} \neq \mathbf{r}^*_n$. In the following, we will ignore this fact in order to simplify our notation. Moreover, it turns out that the sample covariance matrix is no longer a good estimator of the covariance matrix (see, Bai and Silverstein (2010), Bai and Shi (2011)). For that reason it is unclear how good the test of Bodnar and Schmid (2008) behaves in that context. First, we study its behavior under the high-dimensional asymptotics and after that propose an alternative test which makes use of the shrinkage estimator for the portfolio weights, cf., Bodnar, Parolya and Schmid (2017).

In the last years, several papers deal with estimators of unknown portfolio parameters under high-dimensional asymptotics with applications to portfolio theory. Glombeck (2014) formulated tests for the portfolio weights, variances of the excess returns and the Sharpe ratios of the GMVP for $c \in (0,1)$. Bodnar, Parolya and Schmid (2017) and Bodnar, Okhrin and Parolya (2017) derived the shrinkage estimators for the GMVP and for the mean-variance portfolio, respectively, under the Kolmogorov asymptotics for $c \in (0,\infty)$.

3.1 A Test Based on the Mahalanobis Distance

Bodnar and Schmid (2008) proposed a test for a general linear hypothesis of weights of the global minimum variance portfolio. Here we are interested in the special case (5). For this case the test statistic is given by

$$T_n = \frac{n-p}{p-1}(\mathbf{1}'\hat{\mathbf{\Sigma}}^{-1}\mathbf{1})(\hat{\mathbf{w}}^*_n - \mathbf{r}^*)'(\hat{\mathbf{Q}}^*_n)^{-1}(\hat{\mathbf{w}}^*_n - \mathbf{r}^*), \tag{6}$$

where $\hat{\mathbf{Q}}^*_n$ consists of the first $(p-1)\times(p-1)$ elements of $\hat{\mathbf{Q}}_n = \hat{\mathbf{\Sigma}}_n^{-1} - \frac{\hat{\mathbf{\Sigma}}_n^{-1}\mathbf{1}\mathbf{1}'\hat{\mathbf{\Sigma}}_n^{-1}}{\mathbf{1}'\hat{\mathbf{\Sigma}}_n^{-1}\mathbf{1}}$ and the number of assets p in the portfolio is fixed. It was shown that under the null hypothesis $T_n \sim F_{p-1,n-p}$. Moreover, the density of T_n under the alternative hypothesis H_1 is equal to

$$\begin{aligned} f_{T_n}(x) &= f_{p-1,n-p}(x)\,(1+\lambda)^{-(n-1)/2} \\ &\times \;{}_2F_1\left(\frac{n-1}{2}, \frac{n-1}{2}, \frac{p-1}{2}; \frac{(p-1)x}{n-p+(p-1)x}\frac{\lambda}{1+\lambda}\right) \end{aligned} \tag{7}$$

where

$$\lambda = \mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}(\mathbf{w}^*_{GMVP} - \mathbf{r}^*)'(\mathbf{Q}^*)^{-1}(\mathbf{w}^*_{GMVP} - \mathbf{r}^*) \tag{8}$$

and ${}_2F_1$ stands for the hypergeometric function (cf., Abramowitz and Stegun (1964), chap. 15), i.e.

$${}_2F_1(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{i=0}^{\infty} \frac{\Gamma(a+i)\Gamma(b+i)}{\Gamma(c+i)} \frac{z^i}{i!}.$$

Thus, the exact power function of the test is known. Note that this result is also valid for matrix-variate elliptically contoured distributions (see Bodnar and Schmid (2008)). On the other hand, several numerical difficulties appear when the power function of the test is calculated for large values of p and n, since it involves the hypergeometric function whose computation is very challenging for large values of p and n. In order to deal with this problem, we derive the asymptotic distribution of T_n for a high-dimensional setting. This result is given in Theorem 1. The proof is shifted to the Appendix. Since λ depends on p (i.e., on n) through $\mathbf{\Sigma}$, we write λ_n in the rest of the paper.

Theorem 1 *Let $p \equiv p(n)$ and $c_n = \frac{p}{n} \to c \in (0,1)$. Assume that $\{\mathbf{X}_t\}$ is a sequence of independent and normally distributed p-dimensional random vectors with mean $\boldsymbol{\mu}$ and covariance matrix $\mathbf{\Sigma}$ which is assumed to be positive definite. Let*

$$C_n^2 = 2 + 2\frac{\lambda_n^2}{c} + 4\frac{\lambda_n}{c} + 2\frac{c}{1-c}\left(1 + \frac{\lambda_n}{c}\right)^2.$$

Then it holds that

$$\sqrt{p-1}\left(\frac{T_n - 1 - \lambda_n \frac{n-1}{p-1}}{C_n}\right) \xrightarrow{d} \mathcal{N}(0,1)$$

for $p/n \to c \in (0,1)$ as $n \to \infty$. Under the null hypothesis, $\sqrt{p-1}\ (T_n - 1) \xrightarrow{d} \mathcal{N}(0, 2/(1-c))$ for $p/n \to c \in (0,1)$ as $n \to \infty$.

The results of Theorem 1 lead to an asymptotic expression of the power function given by

$$\begin{aligned} & P\left(\frac{\sqrt{p-1}\,(T_n-1)}{\sqrt{2/(1-c)}} > z_{1-\alpha}\right) \\ = \;& 1 - P\left(\frac{\sqrt{p-1}\left(T_n - 1 - \lambda_n\frac{n-1}{p-1}\right)}{C_n} \le \frac{\sqrt{2/(1-c)}z_{1-\alpha} - \sqrt{p-1}\lambda_n\frac{n-1}{p-1}}{C_n}\right) \\ \approx \;& 1 - \Phi\left(\frac{\sqrt{2/(1-c)}z_{1-\alpha} - \sqrt{p-1}\frac{\lambda_n}{c}}{C_n}\right), \end{aligned} \tag{9}$$

where $z_{1-\alpha}$ is the $(1-\alpha)$-quantile of the standard normal distribution.

In Figure 1 we plot the power function (9) as a function of λ_n for several values of c and n (solid line). In addition, the empirical power of the test is shown for the same values of c and n (dashed line) which is equal to the relative number of rejections of the null hypothesis obtained via a simulation study. It is remarkable that following the proof of Theorem 1, the considered simulation study can be considerably simplified. Instead of generating a $p \times n$ random matrix of asset returns in each simulation run, we simulate four independent random variables from standard univariate distributions and then compute the statistic T_n for the given value of λ_n following the stochastic representation (25) in the Appendix. Namely, the simulation study is performed in the following way:

(i) Generate four independent random variables $\omega_1^{(b)} \sim \mathcal{N}(0,1)$, $\xi_2^{(b)} \sim \chi^2_{n-p}$, $\xi_3^{(b)} \sim \chi^2_{n-1}$, and $\xi_4^{(b)} \sim \chi^2_{p-2}$;

(ii) For fixed λ_n, compute

$$T_n^{(b)} \stackrel{d}{=} \frac{n-p}{p-1} \frac{(\sqrt{\lambda_n \xi_3^{(b)}} + \omega_1^{(b)})^2 + \xi_4^{(b)}}{\xi_2^{(b)}}$$

(iii) Repeat steps (a) and (b) for $b = 1, ..., B$ and calculate the empirical power by

$$\hat{P} = \frac{1}{B} \sum_{b=1}^{B} \mathbb{1}_{(z_{1-\alpha}, +\infty)} \left(\frac{\sqrt{p-1} \left(T_n^{(b)} - 1 \right)}{\sqrt{2/(1-c)}} \right), \tag{10}$$

where $\mathbb{1}_{\mathcal{A}}(.)$ is the indicator function of the set $\mathcal{A}$.

In Figure 1 we observe a good performance of the asymptotic approximation of the power function. It works perfect for both small and large values of c.

3.2 Test Based on a Shrinkage Estimator

In most cases the unknown parameters of the asset return distribution are replaced by their sample counterparts when an optimal portfolio is constructed. In recent years, however, other type of estimators like, e.g., shrinkage estimators have been discussed as well (cf., Okhrin and Schmid (2007), Bodnar, Parolya and Schmid (2017)). The shrinkage methodology was introduced by Stein (1956). His results were extended by Efron and Morris (1976) to the case in which the covariance matrix is unknown. The shrinkage methodology can be applied to the expected asset returns (e.g., Jorion (1986)) and the covariance matrix (Bodnar, Gupta and Parolya (2014, 2016)). Both of them appears to be very successful in reducing damaging influences on the portfolio selection. A shrinkage estimator was applied directly to the portfolio weights by Golosnoy and Okhrin (2007) and Okhrin and Schmid (2008). They showed that the shrinkage estimators of the portfolio weights lead to a decrease of the variance of the portfolio weights and to an increase of the utility.

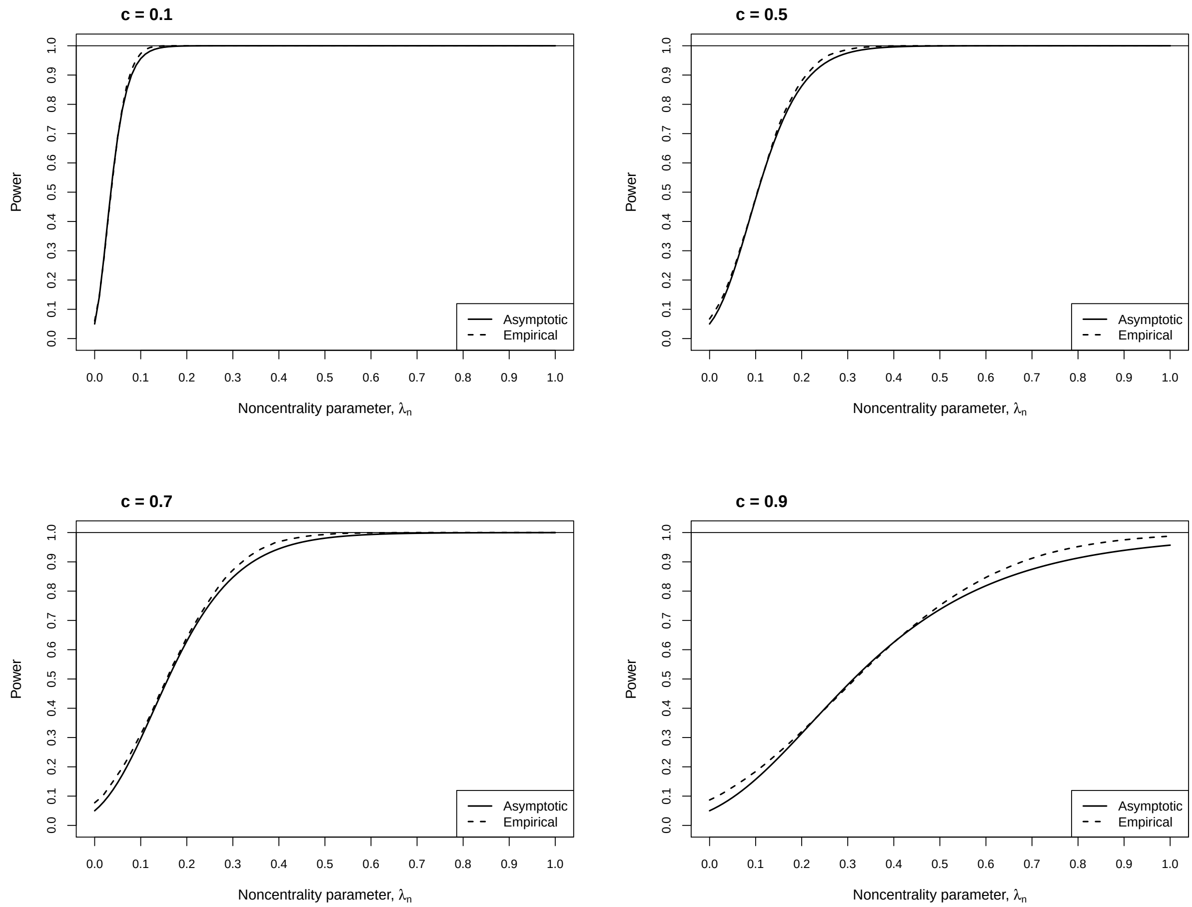

Figure 1: Asymptotic power function vs empirical power function for different values as a function of λ_n in (7) for various values of $c \in \{0.1, 0.5, 0.7, 0.9\}$ and a level of significance of 5%.

Bodnar, Parolya and Schmid (2017) proposed a new shrinkage estimator for the weights of the GMVP which turns out to provide better results in the high-dimensional case than the existing estimators. It is based on the convex combination between the sample estimator of the GMVP weights and an arbitrary constant vector expressed as

$$\hat{\mathbf{w}}_{n;GSE} = \alpha_n \frac{\hat{\mathbf{\Sigma}}_n^{-1}\mathbf{1}}{\mathbf{1}'\hat{\mathbf{\Sigma}}_n^{-1}\mathbf{1}} + (1-\alpha_n)\mathbf{b}_n \qquad \text{with} \qquad \mathbf{b}_n'\mathbf{1} = 1. \tag{11}$$

Here the index GSE stands for general shrinkage estimator. It is assumed that $\mathbf{b}_n \in \mathbb{R}^p$ is a vector of constants satisfying that $\mathbf{b}_n'\mathbf{\Sigma}\mathbf{b}_n$ is uniformly bounded. Bodnar, Parolya and Schmid (2017) proposed to determine the optimal shrinkage intensity α_n for a given target portfolio $\mathbf{b}_n$ such that the out-of-sample risk is minimal, i.e.

$$L = (\hat{\mathbf{w}}_{n;GSE} - \mathbf{w}_{GMVP})'\mathbf{\Sigma}(\hat{\mathbf{w}}_{n;GSE} - \mathbf{w}_{GMVP}) \tag{12}$$

is minimized with respect to α_n. This leads to

$$\hat{\alpha}_n = \frac{(\mathbf{b}_n - \hat{\mathbf{w}}_n)' \, \boldsymbol{\Sigma} \, \mathbf{b}_n}{(\mathbf{b}_n - \hat{\mathbf{w}}_n)' \, \boldsymbol{\Sigma} \, (\mathbf{b}_n - \hat{\mathbf{w}}_n)}. \tag{13}$$

They showed that the optimal shrinkage intensity $\hat{\alpha}_n$ is almost surely asymptotically equivalent to a nonrandom quantity $\tilde{\alpha}_n \in [0,1]$ when $\frac{p}{n} \to c \in (0,1)$ as $n \to \infty$ which is given by

$$\tilde{\alpha}_n = \frac{(1-c)R_{\mathbf{b}_n}}{c + (1-c)R_{\mathbf{b}_n}}, \tag{14}$$

where

$$R_{\mathbf{b}_n} = \frac{\sigma^2_{\mathbf{b}_n} - \sigma^2_n}{\sigma^2_n} = \mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}\mathbf{b}_n'\boldsymbol{\Sigma}\mathbf{b}_n - 1$$

is the relative loss of the target portfolio $\mathbf{b}_n$, $\sigma^2_{\mathbf{b}_n} = \mathbf{b}_n'\boldsymbol{\Sigma}\mathbf{b}_n$ is the variance of the target portfolio and $\sigma^2_n = 1/\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}$ is the variance of the GMVP. This result provides an estimator of the optimal shrinkage intensity given by

$$\hat{\tilde{\alpha}}_n = \frac{(1-\frac{p}{n})\hat{R}_{\mathbf{b}_n}}{\frac{p}{n} + (1-\frac{p}{n})\hat{R}_{\mathbf{b}_n}}, \;\; \hat{R}_{\mathbf{b}_n} = (1-\frac{p}{n})\mathbf{b}_n'\hat{\boldsymbol{\Sigma}}_n\mathbf{b}_n\mathbf{1}'\hat{\boldsymbol{\Sigma}}_n^{-1}\mathbf{1} - 1. \tag{15}$$

Using the estimated shrinkage intensity $\hat{\tilde{\alpha}}_n$ the corresponding portfolio weights are given by

$$\hat{\mathbf{w}}_{n;ESI} = \hat{\tilde{\alpha}}_n\hat{\mathbf{w}}_n + (1-\hat{\tilde{\alpha}}_n)\mathbf{b}_n. \tag{16}$$

Bodnar, Parolya and Schmid (2017) proved that the ratio $\dfrac{\hat{\tilde{\alpha}}_n}{\tilde{\alpha}_n} \to 1$ if $\frac{p}{n} \to c \in (0,1)$ as $n \to \infty$. In Theorem 2 we show that the estimated intensity is asymptotically normally distributed. The proof of Theorem 2 is given in the Appendix.

Theorem 2 *Let $p \equiv p(n)$ and $c_n = \frac{p}{n} \to c \in (0,1)$. Assume that $\{\mathbf{X}_t\}$ is a sequence of independent and normally distributed p-dimensional random vectors with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$, which is assumed to be positive definite. Then*

$$\sqrt{n}\frac{\hat{\tilde{\alpha}}_n - A_n}{B_n} \xrightarrow{d} \mathcal{N}(0,1) \;\; \text{for } p/n \to c \in (0,1) \;\; \text{as } n \to \infty, \tag{17}$$

where

$$\begin{aligned} A_n &= \frac{(1-c_n)R_{\mathbf{b}_n}}{c_n + (1-c_n)R_{\mathbf{b}_n}}, \\ B_n^2 &= 2\frac{c_n^3(1-c_n)(1+R_{\mathbf{b}_n})^2}{(c_n + (1-c_n)R_{\mathbf{b}_n})^4}. \end{aligned}$$

Next we introduce a test based on the estimated shrinkage intensity. The motivation is based on the following equivalences

$$\tilde{\alpha}_n = 0 \iff R_{\mathbf{b}_n} = 0 \iff \sigma^2_{\mathbf{b}_n} = \sigma^2_n .$$

This means that $\tilde{\alpha}_n = 0$ if and only if the variance of the portfolio based on $\mathbf{b}_n$ is equal to the variance of the GMVP. This means that $\mathbf{b}_n'\mathbf{\Sigma}\mathbf{b}_n = 1/\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1} = \min_{\mathbf{w}:\, \mathbf{w}'\mathbf{1}=1} = \mathbf{w}'\mathbf{\Sigma}\mathbf{w} = \mathbf{w}'_{GMVP}\mathbf{\Sigma}\, \mathbf{w}_{GMVP}$. Since the GMVP weights are uniquely determined this is only valid if and only if $\mathbf{b}_n = \mathbf{w}_{GMVP}$. Choosing $\mathbf{b}_n = \mathbf{r}$ it holds that

$$\mathbf{w}_{GMVP} = \mathbf{r} \iff \tilde{\alpha}_n = 0.$$

Thus it is possible to get a test for the structure of the GMVP using the shrinkage intensity with the hypothesis given by

$$H_0 : \tilde{\alpha}_n = 0 \qquad \text{against} \qquad H_1 : \tilde{\alpha}_n > 0. \tag{18}$$

Note that $\hat{\hat{\alpha}} = \hat{\hat{\alpha}}(\mathbf{b}_n)$. Let $S_n = \sqrt{n}\, \hat{\hat{\alpha}}(\mathbf{b}_n = \mathbf{r})$. For testing (18) we use the test statistic S_n.

Theorem 3 *Suppose that the conditions of Theorem 2 are satisfied. Then*

$$\frac{S_n - \sqrt{n}A_n}{B_n} \xrightarrow{d} \mathcal{N}(0,1) \quad \text{for } p/n \to c \in (0,1) \quad \text{as } n \to \infty ,$$

where A_n and B_n are given in the statement of Theorem 2. Under the null hypothesis, $S_n \xrightarrow{d} \mathcal{N}\left(0, 2(1-c)/c\right)$ for $p/n \to c \in (0,1)$ as $n \to \infty$.

The proof of Theorem 3 follows directly from Theorem 2. This result gives us a promising new approach how to detect deviations of the true portfolio weights from given quantities. Using Theorem 3 we are able to make a statement about the power function of this test. Since A_n and B_n depend on $\mathbf{b}_n$, we only have to replace this quantity by $\mathbf{r}$. It holds that

$$\begin{aligned} P\left(\frac{S_n}{\sqrt{2\frac{1-c}{c}}} > z_{1-\alpha}\right) &= 1 - P\left(\frac{S_n - A_n(\mathbf{b}_n = \mathbf{r})}{B_n(\mathbf{b}_n = \mathbf{r})} \leq \frac{\sqrt{2\frac{1-c}{c}}z_{1-\alpha} - A_n(\mathbf{b}_n = \mathbf{r})}{B_n(\mathbf{b}_n = \mathbf{r})}\right) \\ &\approx 1 - \Phi\left(\frac{\sqrt{2\frac{1-c}{c}}z_{1-\alpha} - A_n(\mathbf{b}_n = \mathbf{r})}{B_n(\mathbf{b}_n = \mathbf{r})}\right). \end{aligned} \tag{19}$$

Note that the approximation given in (19) is purely a function of $R_{\mathbf{b}_n = \mathbf{r}}$. This is a main difference to the test discussed in Section 3.1 where the power function is a function of λ_n. These properties are very useful to analyze the performance of both tests and simplify the power analysis.

In *Figure 3* the power of the test is shown as the function of $R_{\mathbf{b}_n}$ and n. It can be seen that the test performs better for smaller values of c. With increasing values of c,

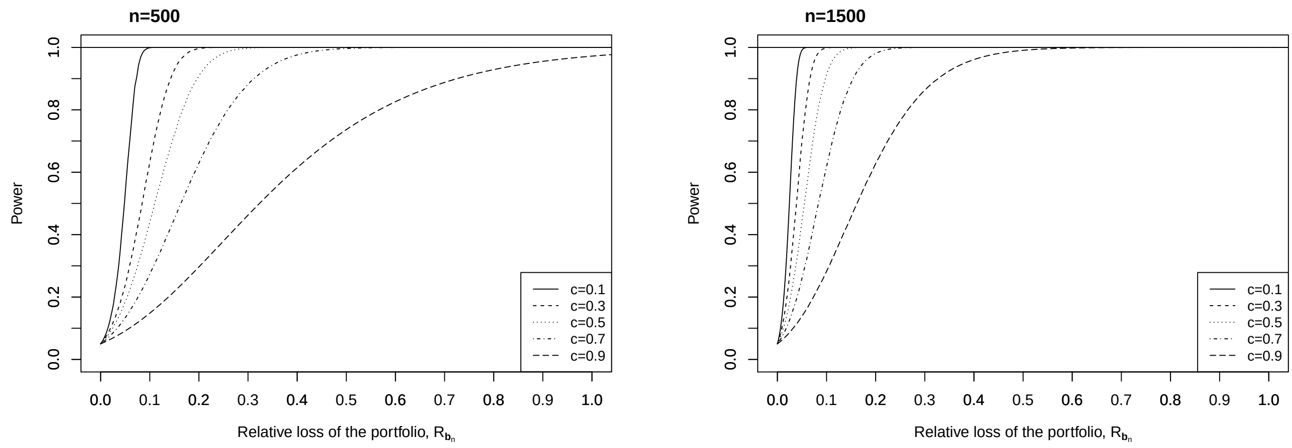

Figure 2: The asymptotic power function of the test in (18) as the function of $R_{\mathbf{b}_n}$ for various values of $c \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$ and 5% level of significance. The number of observation is $n = 500$.

the power of the test decreases. We determine the power function for two different sample numbers $n = 500$ and $n = 1500$. As expected, the test shows a better performance for larger values of n , since $A_n(\mathbf{b}_n = \mathbf{r})$ increases and the numerator of the expression in the cumulative distribution function in (19) becomes negative and the whole expression tends to 1.

4 Comparison Study

The aim of this section is to compare several tests for the weights of the GMVP with each other.

In the preceding two subsections we considered two tests for the weights of the GMVP. For the test based on the empirical portfolio weights the exact distribution of the test statistic is known. In Section 3.1 the asymptotic power function of the test proposed by Bodnar and Schmid (2008) is derived in a high-dimensional setting. In Section 3.2 a new test is proposed and its asymptotic power function which purely depends on $R_{\mathbf{b}_n=\mathbf{r}}$ is determined. The fact that both tests depend on different quantities complicates the comparison of both tests. Note that

$$R_{\mathbf{b}_n=\mathbf{r}} = \mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}\ \mathbf{r}'\mathbf{\Sigma}\mathbf{r} - 1 = \lambda_n \frac{\mathbf{r}'\mathbf{\Sigma}\mathbf{r}}{(\mathbf{w}^*_{GMVP} - \mathbf{r}^*)'(\mathbf{Q}^*)^{-1}(\mathbf{w}^*_{GMVP} - \mathbf{r}^*)} - 1.$$

In Section 4.1 it is explained how both tests are compared with each other. Additionally, we include the test presented by Glombeck (2014, Theorem 10) in our comparison study.

4.1 Design of the Comparison Study

Let $\mathbf{\Sigma}$ be a $p \times p$ positive definite covariance matrix of asset returns, n the number of samples and $p \equiv p(n)$. The structure of the covariance matrix is chosen in the following way: one-ninth of the eigenvalues are set equal to 2, four-ninths equal to 5, and the rest equal to 10. Doing this, we can ensure that the eigenvalues are not very dispersed and if p increases, then the spectrum of the covariance matrix does not change its behavior. Then the covariance matrix is determined as follows

$$\mathbf{\Sigma} = \Theta \mathbf{\Lambda} \Theta^{'},$$

where $\mathbf{\Lambda}$ is the diagonal matrix whose diagonal elements are the predefined eigenvalues and Θ is the $p \times p$ matrix of eigenvectors obtained from the spectral decomposition of a standard Wishart-distributed random matrix.

We consider the following scenario for modeling the changes. Under alternative hypothesis the covariance matrix is defined by

$$\mathbf{\Sigma}_1 = \Delta \mathbf{\Sigma} \Delta, \tag{20}$$

where Δ denotes the changes in the standard deviations and it is given by

$$\Delta = \left(\begin{array}{c|c} D_m & \mathbf{0} \\ \hline \mathbf{0} & I_{p-m} \end{array} \right), \tag{21}$$

with $D_m = diag(a)$ and $a = 1 + 0.1k$, $k \in \{1, 2, \ldots, 15\}$, $m \in \{0.1p, 0.2p, 0.5p, 0.8p\}$.

In order to demonstrate the influence of Δ, we built the ℓ_1 norm of the difference between $\mathbf{w}_{GMVP}(\mathbf{\Sigma}_1)$ and $\mathbf{w}_{GMVP}(\mathbf{\Sigma})$ as a function of a. This difference relates to the proportional transaction costs for moving to the new optimal portfolio weights $\mathbf{w}_{GMVP}(\mathbf{\Sigma}_1)$. In *Figure 3* we can see that the biggest influence on the portfolio composition is observed if $m = 0.5p$. For m larger than $0.5p$, the influence decreases. The results, obtained for this scenario, present almost a linear relationship between the ℓ_1 norm of the difference vector and the size of the change. It is worth mentioning that the differences are all zero when no changes occur, i.e., under H_0.

4.2 Comparison of the Tests

In this section we present the results of a simulation study to compare the power of the three tests. It is based on 10^5 independent realizations of Δ. The significance level α is chosen to be 5% and the concentration ratio c takes a value within the set $\{0.1, 0.5, 0.7, 0.9\}$.

In order to illustrate the performance of the tests based on the shrinkage approach,

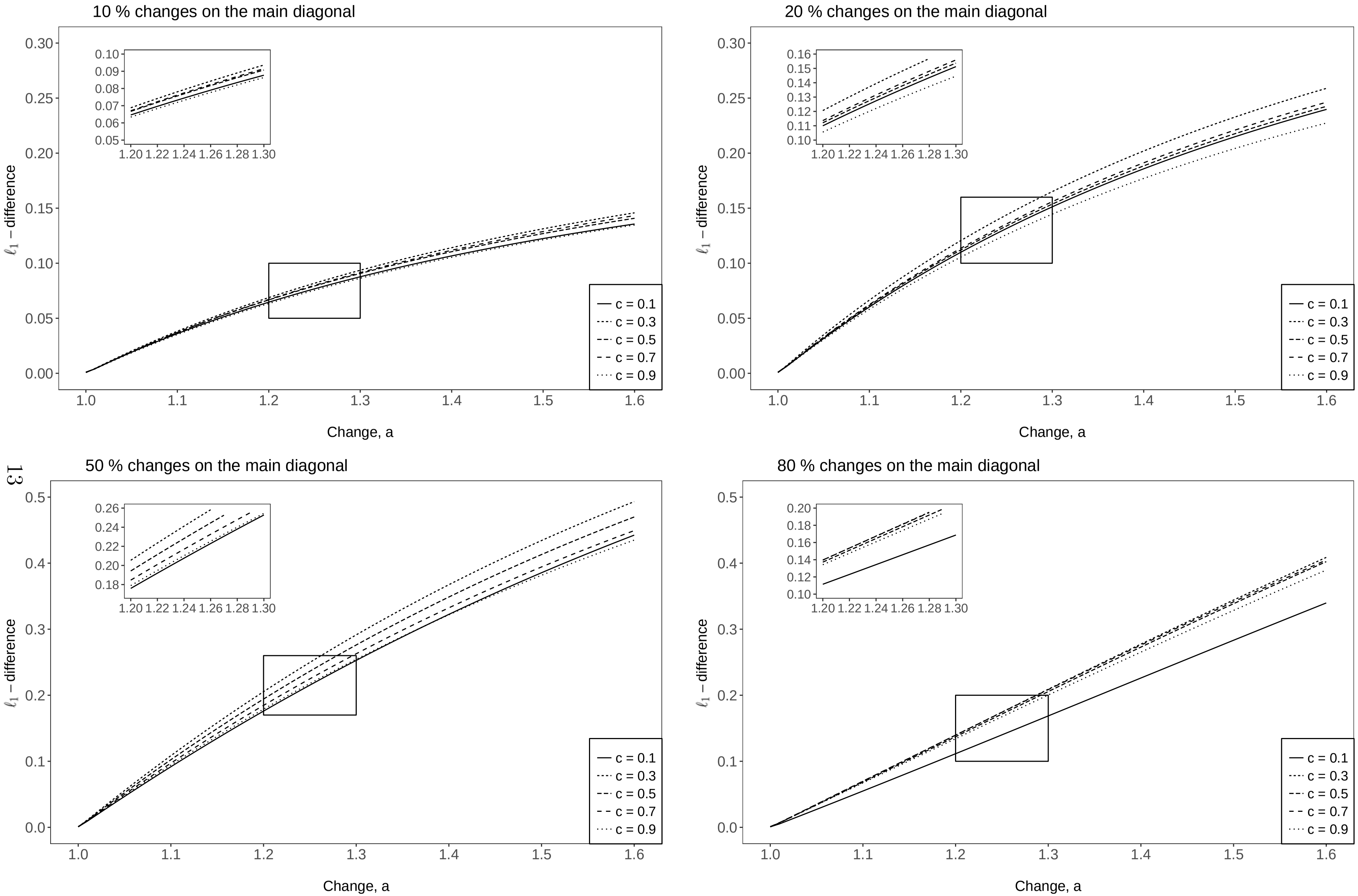

Figure 3: ℓ_1 - differences $\sum_{i=1}^{p}\left|w_{i,GMVP}(\mathbf{\Sigma}_1) - w_{i,GMVP}(\mathbf{\Sigma})\right|$ as a function of change, a. Here $n = 500$.

the test based on the statistic of Bodnar and Schmid (2008) and the test proposed by Glombeck (2014), the empirical power functions for the general hypothesis were evaluated for $m = 0.2p$ (*Figure 4a*) and $m = 0.5p$ (*Figure 4b*).

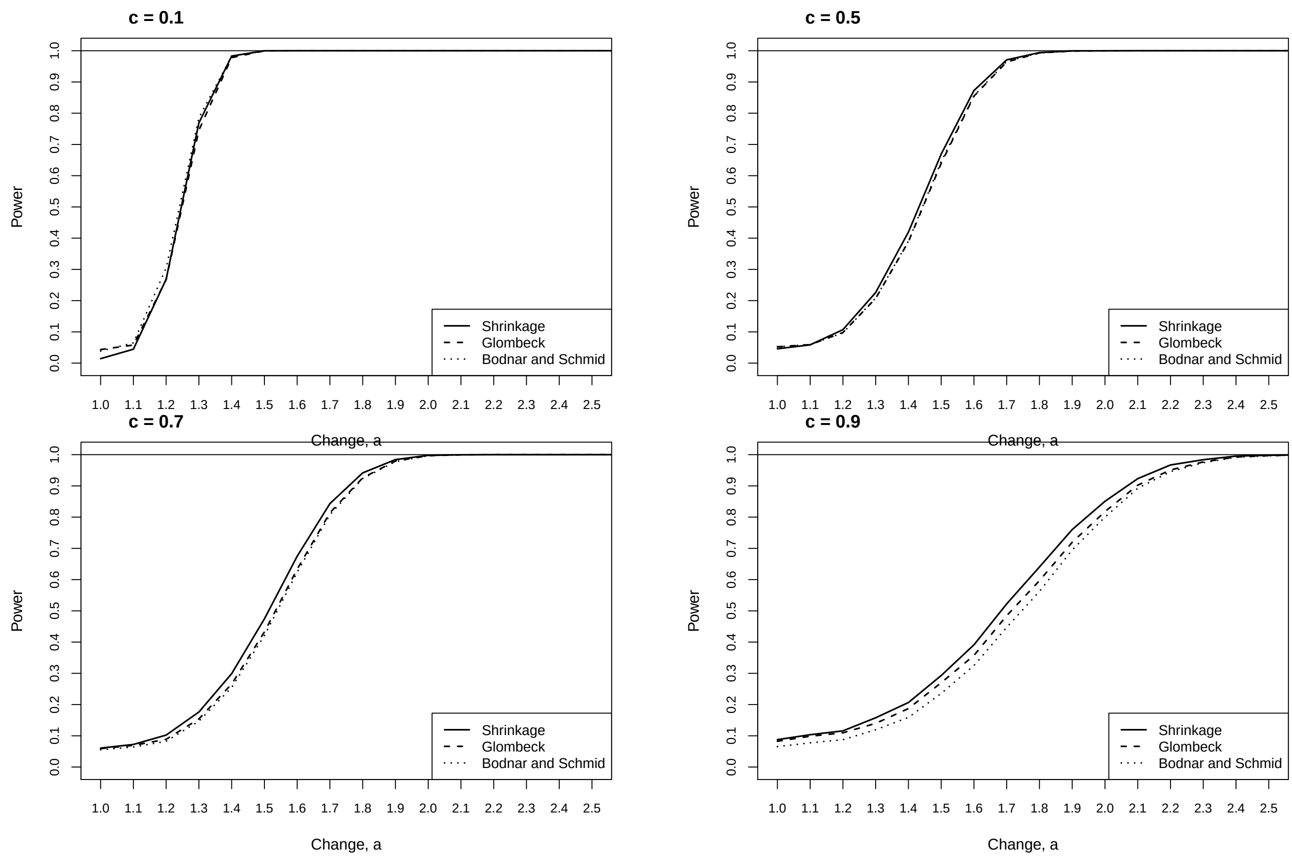

Figure 4a: Empirical power functions of the three tests for different values of c and 20% changes on the main diagonal according to scenario given in (20) and $n = 500$.

In *Figure 4a*, where 20% of elements in the main diagonal of the covariance matrix are contaminated, we observe a slow increase of the power functions for $c = 0.9$ and a better behavior for smaller values of c. In the case $c = 0.1$, there is no significant difference in the performance of the tests. For $c = 0.5$ and $c = 0.7$ the power curves of Glombeck's test and the test of Bodnar and Schmid (2008) are very close to each other, while the test presented by Glombeck (2014) shows a little bit better performance than the one presented by Bodnar and Schmid (2008) when $c = 0.9$. For $c = \{0.5, 0.7, 0.9\}$ the test based on the shrinkage approach outperforms its competitors.

Figure 4b illustrates the behavior of the tests for $m = 0.5p$. We detect an improvement in the performances of the tests for all values of c. In the case $c = 0.1$, the test of Bodnar and Schmid (2008) outperforms both the shrinkage approach and Glombeck's test, whereas for $c = 0.9$ this test is appeared to be the worst one. For $c = 0.5$ and

$c = 0.7$ the same situation as for $m = 0.2p$ is present, where the power of Glombeck's test and the test of Bodnar and Schmid (2008) almost coincide with each other. The test based on the estimated shrinkage intensity lies above the rest of competitors for $c = 0.5$, $c = 0.7$, and $c = 0.9$.

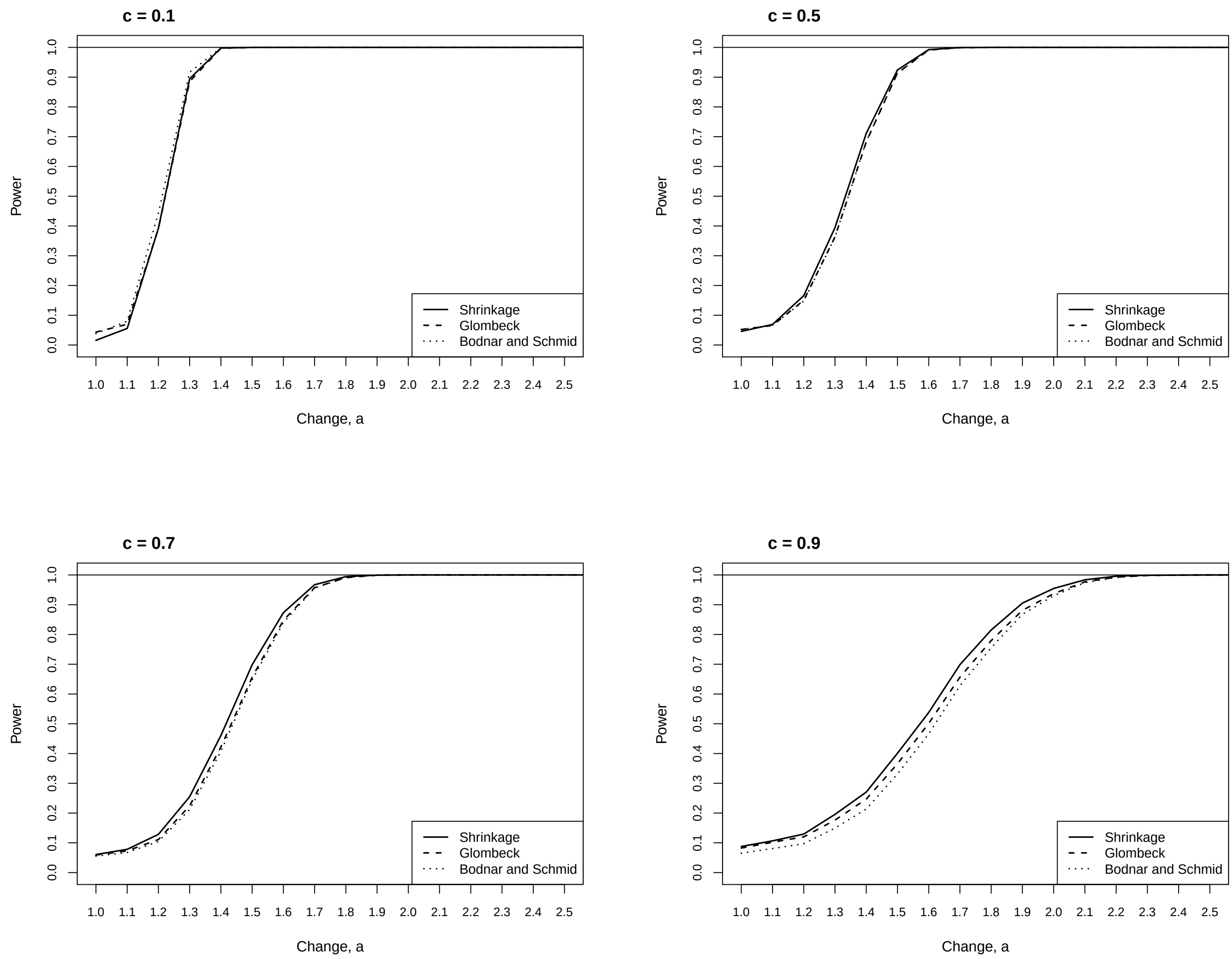

Figure 4b: Empirical power functions of the three tests for different values of c and 50% changes on the main diagonal according to scenario given in (20) and $n = 500$.

5 Summary

The main focus of this paper lies on the inference of the GMVP weights. After constructions of an optimal portfolio, an investor is interested to know whether at a fixed time point the weights of holding portfolio are still optimal or not. For that reason we investigate several asymptotic and exact statistical procedures for detecting deviations in the weights of the GMVP. One test is based on the sample estimator of the GMVP weights, while another uses its shrinkage estimator. To the best of our knowledge, the shrinkage approach, which is very popular in point estimation, is applied in test theory for the first time. The asymptotic distribution of both test statistics are obtained under the null and alternative hypothesis in a high-dimensional setting. This is a great advantage with respect to other approaches appeared in literature where no statements about the distribution under the alternative hypothesis are made (e.g. Glombeck (2014)).

In order to compare the performance of the proposed procedures the empirical power functions of the derived tests are determined. It is shown that the test based on the shrinkage approach performs uniformly better than the other tests considered in the paper for moderate and large values of the concentration ratio c. It seems to be a very promising approach for testing portfolio weights in a high-dimensional situation.

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Appendix

In this section, the proof of Theorems 1 and 2 are given.

Proof of Theorem 1:

We first derive a stochastic representation for T_n. Let the symbol $\stackrel{d}{=}$ denote the equality in distribution. Then, it holds that (see, the proof of Theorem 2 in Bodnar and Schmid (2008))

$$T_n \stackrel{d}{=} \frac{n-p}{p-1}\frac{\xi_1}{\xi_2}, \tag{22}$$

where

$$\begin{aligned} \xi_2 &= (n-1)\frac{\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\hat{\mathbf{\Sigma}}^{-1}\mathbf{1}} \sim \chi^2_{n-p}, \\ \xi_1|\hat{\mathbf{Q}}_n^* &= (n-1)\left(\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}\right)(\hat{\mathbf{w}}_n^*-\mathbf{r}^*)'\,(\hat{\mathbf{Q}}_n^*)^{-1}\,(\hat{\mathbf{w}}_n^*-\mathbf{r}^*) \sim \chi^2_{p-1,\lambda_n(\hat{\mathbf{Q}}_n^*)}, \end{aligned} \tag{23}$$

with

$$\lambda_n(\hat{\mathbf{Q}}_n^*) = (n-1)\left(\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}\right)(\mathbf{w}_{GMVP}^*-\mathbf{r}^*)'\,(\hat{\mathbf{Q}}_n^*)^{-1}\,(\mathbf{w}_{GMVP}^*-\mathbf{r}^*)\,,$$

and ξ_2 is independent of both ~~and~~ $\hat{\mathbf{Q}}_n^*$ and $(\hat{\mathbf{Q}}_n^*)^{-1}(\hat{\mathbf{w}}_n^* - \mathbf{r}^*)$. Moreover, in using $(n-1)(\hat{\mathbf{Q}}_n^*)^{-1} \sim \mathcal{W}_p(n-1, (\mathbf{Q}^*)^{-1})$ (cf., Muirhead (1982, Theorems 3.2.10 and 3.2.11)) we get

$$\lambda_n(\hat{\mathbf{Q}}_n^*) = \lambda_n \frac{(n-1)\,(\mathbf{w}_{GMVP}^* - \mathbf{r}^*)'\,(\hat{\mathbf{Q}}_n^*)^{-1}\,(\mathbf{w}_{GMVP}^* - \mathbf{r}^*)}{(\mathbf{w}_{GMVP}^* - \mathbf{r}^*)'\,(\mathbf{Q}^*)^{-1}\,(\mathbf{w}_{GMVP}^* - \mathbf{r}^*)} \sim \lambda_n \xi_3\,, \tag{24}$$

where $\xi_3 \sim \chi^2_{n-1}$.

The last equality shows that the conditional distribution of ξ_1 given $\hat{\mathbf{Q}}_n^*$ depends only on $\hat{\mathbf{Q}}_n^*$ over ξ_3 and, consequently, the conditional distribution $\xi_1|\hat{\mathbf{Q}}_n^*$ coincides with $\xi_1|\xi_3$. Using the distributional properties of the non-central F-distribution, we get the following stochastic representation for ξ_1 given by

$$\xi_1 \stackrel{d}{=} (\sqrt{\lambda_n \xi_3} + \omega_1)^2 + \xi_4$$

and, hence, the stochastic representation of T_n is expressed as

$$T_n \stackrel{d}{=} \frac{n-p}{p-1} \frac{(\sqrt{\lambda_n \xi_3} + \omega_1)^2 + \xi_4}{\xi_2}, \tag{25}$$

where $\omega_1 \sim \mathcal{N}(0,1)$, $\xi_2 \sim \chi^2_{n-p}$, $\xi_3 \sim \chi^2_{n-1}$, and $\xi_4 \sim \chi^2_{p-2}$; ω_1, ξ_2, ξ_3, ξ_4 are independent.

Applying (25) we get

$$\begin{aligned}
&\frac{n-p}{\xi_2}\sqrt{p-1}\left(\frac{\lambda_n\xi_3 + 2\sqrt{\lambda_n\xi_3}\omega_1 + \omega_1^2 + \xi_4}{p-1} - \left(1 + \lambda_n\frac{n-1}{p-1}\right)\frac{\xi_2}{n-p}\right) \\
= \ &\frac{n-p}{\xi_2}\left(\lambda_n\frac{n-1}{p-1}\sqrt{p-1}\left(\frac{\xi_3}{n-1} - 1\right) + \sqrt{p-1}\left(\frac{\xi_4}{p-1} - 1\right)\right. \\
- \ &\left.\left(1 + \lambda_n\frac{n-1}{p-1}\right)\sqrt{p-1}\left(\frac{\xi_2}{n-p} - 1\right) + 2\sqrt{\lambda_n}\sqrt{\frac{\xi_3}{p-1}}\omega_1 + \frac{\omega_1^2}{\sqrt{p-1}}\right).
\end{aligned}$$

Using the asymptotic properties of a χ^2-distribution with infinite degrees of freedom and the independence of ω_1, ξ_2, ξ_3, ξ_4, the application of Slutsky's lemma (see, e.g., Theorem 1.5 in DasGupta (2008)) leads to

$$\sqrt{p-1}\left(\frac{T_n - 1 - \lambda_n\frac{n-1}{p-1}}{C_n}\right) \xrightarrow{d} \mathcal{N}(0,1)\,,$$

where

$$C_n^2 = 2 + 2\frac{\lambda_n^2}{c} + 4\frac{\lambda_n}{c} + 2\frac{c}{1-c}\left(1 + \frac{\lambda_n}{c}\right)^2.$$

$\square$

Proof of Theorem 2:

In order to stress the dependence on n we use the notation $\mathbf{\Sigma}_n$ in the following.

a) Using Proposition 3 of Glombeck (2014) we have

$$\sqrt{n}\begin{pmatrix} \dfrac{\mathbf{b}_n'\hat{\mathbf{\Sigma}}_n\mathbf{b}_n}{\mathbf{b}_n'\mathbf{\Sigma}_n\mathbf{b}_n}-1 \\ \dfrac{\mathbf{1}'\hat{\mathbf{\Sigma}}_n^{-1}\mathbf{1}}{\mathbf{1}'\mathbf{\Sigma}_n^{-1}\mathbf{1}}-\dfrac{1}{1-c_n} \end{pmatrix} \xrightarrow{d} \mathcal{N}\left[\begin{pmatrix}0\\0\end{pmatrix}, 2\begin{pmatrix}1 & -\frac{1}{1-c}\\ -\frac{1}{1-c} & \frac{1}{(1-c)^3}\end{pmatrix}\right], \tag{26}$$

if $n\to\infty$. We can rewrite $\hat{R}_{\mathbf{b}_n}$ as

$$\begin{aligned}\hat{R}_{\mathbf{b}_n} &= (1-c_n)\frac{\mathbf{b}_n'\hat{\mathbf{\Sigma}}_n\mathbf{b}_n}{\mathbf{b}_n'\mathbf{\Sigma}_n\mathbf{b}_n}\frac{\mathbf{1}'\hat{\mathbf{\Sigma}}_n^{-1}\mathbf{1}}{\mathbf{1}'\mathbf{\Sigma}_n^{-1}\mathbf{1}}\mathbf{b}_n'\mathbf{\Sigma}_n\mathbf{b}_n\mathbf{1}'\mathbf{\Sigma}_n^{-1}\mathbf{1}-1\\ &= \Delta_n(1-c_n)(D_nE_n+\frac{D_n}{1-c_n}+E_n)+\Delta_n-1\end{aligned}$$

with

$$\Delta_n=\mathbf{b}_n'\mathbf{\Sigma}_n\mathbf{b}_n\,\mathbf{1}'\mathbf{\Sigma}_n^{-1}\mathbf{1},\quad D_n=\frac{\mathbf{b}_n'\hat{\mathbf{\Sigma}}_n\mathbf{b}_n}{\mathbf{b}_n'\mathbf{\Sigma}_n\mathbf{b}_n}-1,\quad E_n=\frac{\mathbf{1}'\hat{\mathbf{\Sigma}}_n^{-1}\mathbf{1}}{\mathbf{1}'\mathbf{\Sigma}_n^{-1}\mathbf{1}}-\frac{1}{1-c_n}.$$

Using (26) it follows that

$$\begin{aligned}\sqrt{n}\frac{\hat{R}_{\mathbf{b}_n}-\Delta_n+1}{\Delta_n} &= \sqrt{n}(1-c_n)(D_nE_n+D_n/(1-c_n)+E_n)\\ &= (1-c_n)\sqrt{n}(D_n/(1-c_n)+E_n)+o_p(1)\\ &= (1\quad 1-c_n)\,\sqrt{n}\,\begin{pmatrix}D_n\\E_n\end{pmatrix}+o_p(1)\xrightarrow{d}\mathcal{N}\left(0,2\frac{c}{1-c}\right)\end{aligned}$$

since $\sqrt{n}D_n\xrightarrow{d}\mathcal{N}(0,2)$ and $\sqrt{n}E_n\xrightarrow{d}\mathcal{N}(0,\frac{2}{(1-c)^3})$.

b) Since

$$\hat{\hat{\alpha}}_n=\frac{(1-c_n)\left(\frac{\hat{R}_{\mathbf{b}_n}-\Delta_n+1}{\Delta_n}+\frac{\Delta_n-1}{\Delta_n}\right)}{\frac{c_n}{\Delta_n}+(1-c_n)\left(\frac{\hat{R}_{\mathbf{b}_n}-\Delta_n+1}{\Delta_n}+\frac{\Delta_n-1}{\Delta_n}\right)}$$

it follows that

$$\sqrt{n}\left(\hat{\hat{\alpha}}_n-\frac{(1-c_n)(\Delta_n-1)}{c_n+(1-c_n)(\Delta_n-1)}\right)=I_n+II_n$$

with

$$
\begin{aligned}
I_n &= \sqrt{n}\,\frac{(1-c_n)\left(\frac{\hat{R}_{\mathbf{b}_n}-\Delta_n+1}{\Delta_n}\right)}{\frac{c_n}{\Delta_n}+(1-c_n)\left(\frac{\hat{R}_{\mathbf{b}_n}-\Delta_n+1}{\Delta_n}+\frac{\Delta_n-1}{\Delta_n}\right)} \\
&= \sqrt{n}\,\frac{(1-c_n)\left(\frac{\hat{R}_{\mathbf{b}_n}-\Delta_n+1}{\Delta_n}\right)}{\frac{c_n}{\Delta_n}+(1-c_n)\frac{\Delta_n-1}{\Delta_n}}\quad\frac{\frac{c_n}{\Delta_n}+(1-c_n)\frac{\Delta_n-1}{\Delta_n}}{\frac{c_n}{\Delta_n}+(1-c_n)\left(\frac{\hat{R}_{\mathbf{b}_n}-\Delta_n+1}{\Delta_n}+\frac{\Delta_n-1}{\Delta_n}\right)} \\
&= \sqrt{n}\,\frac{(1-c_n)\left(\frac{\hat{R}_{\mathbf{b}_n}-\Delta_n+1}{\Delta_n}\right)}{1-c_n-\frac{1-2c_n}{\Delta_n}}\quad\frac{1}{1+\frac{1-c_n}{\sqrt{n}(1-c_n-\frac{1-2c_n}{\Delta_n})}\sqrt{n}\frac{\hat{R}_{\mathbf{b}_n}-\Delta_n+1}{\Delta_n}}.
\end{aligned}
$$

Furthermore

$$
\begin{aligned}
II_n &= \sqrt{n}(1-c_n)(1-\frac{1}{\Delta_n})\left(\frac{1}{\frac{c_n}{\Delta_n}+(1-c_n)\left(\frac{\hat{R}_{\mathbf{b}_n}-\Delta_n+1}{\Delta_n}+\frac{\Delta_n-1}{\Delta_n}\right)}-\frac{1}{\frac{c_n}{\Delta_n}+(1-c_n)\frac{\Delta_n-1}{\Delta_n}}\right) \\
&= \sqrt{n}\frac{1-c_n}{1-c_n-\frac{1-2c_n}{\Delta_n}}\;(1-\frac{1}{\Delta_n})\;\left(\frac{1}{1+\frac{1-c_n}{1-c_n-\frac{1-2c_n}{\Delta_n}}\frac{\hat{R}_{\mathbf{b}_n}-\Delta_n+1}{\Delta_n}}-1\right) \\
&= -\frac{(1-c_n)^2}{(1-c_n-\frac{1-2c_n}{\Delta_n})^2}\;(1-\frac{1}{\Delta_n})\;\frac{\sqrt{n}\frac{\hat{R}_{\mathbf{b}_n}-\Delta_n+1}{\Delta_n}}{1+\frac{1-c_n}{\sqrt{n}(1-c_n-\frac{1-2c_n}{\Delta_n})}\sqrt{n}\frac{\hat{R}_{\mathbf{b}_n}-\Delta_n+1}{\Delta_n}}.
\end{aligned}
$$

Consequently

$$
\begin{aligned}
I_n+II_n &= \sqrt{n}\frac{\hat{R}_{\mathbf{b}_n}-\Delta_n+1}{\Delta_n}\frac{c_n(1-c_n)}{(1-c_n-\frac{1-2c_n}{\Delta_n})^2}\;\frac{1}{\Delta_n}\;\frac{1}{1+\frac{1-c_n}{\sqrt{n}(1-c_n-\frac{1-2c_n}{\Delta_n})}\sqrt{n}\frac{\hat{R}_{\mathbf{b}_n}-\Delta_n+1}{\Delta_n}} \\
&= \sqrt{n}\frac{\hat{R}_{\mathbf{b}_n}-\Delta_n+1}{\Delta_n}\frac{c_n(1-c_n)\Delta_n}{(c_n+(\Delta_n-1)(1-c_n))^2}\;(1+o_p(1)) \\
&\overset{d}{\approx}\; \mathcal{N}\left(0,2\frac{c_n^3(1-c_n)\Delta_n^2}{(c_n+(\Delta_n-1)(1-c_n))^4}\right)
\end{aligned}
$$

if $\sqrt{n}(1-c_n-\frac{1-2c_n}{\Delta_n})\to\infty$ as $n\to\infty$. Since $\mathbf{b}_n'\mathbf{\Sigma}_n\mathbf{b}_n \geq \min_{\mathbf{w}}\mathbf{w}'\mathbf{\Sigma}_n\mathbf{w} = \frac{1}{\mathbf{1}_n'\mathbf{\Sigma}_n^{-1}\mathbf{1}_n}$ it holds that $\Delta_n \geq 1$ and thus this condition is fulfilled.

$\square$