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Stanislas Muhinyuza
Taras Bodnar
Mathias Lindholm

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Postal address:
Mathematical Statistics
Dept. of Mathematics
Stockholm University
SE-106 91 Stockholm
Sweden

Internet:
http://www.math.su.se

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A test on the location of the tangency portfolio on the set of feasible portfolios

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Stanislas Muhinyuza[a,b,1], Taras Bodnar[a], Mathias Lindholm[a]

[a] *Department of Mathematics, Stockholm University, Roslagsvägen 101, SE-10691 Stockholm, Sweden*

[b] *Department of Mathematics, College of Science and Technology, University of Rwanda, P.O. Box 3900, Kigali-Rwanda*

Abstract

Due to the problem of parameter uncertainty, specifying the location of the tangency portfolio(TP) on the set of feasible portfolios becomes a challenging task. The set of feasible portfolios is a parabola in the mean-variance space with optimal portfolios lying on its upper part. Using statistical test theory, we want to decide if the tangency portfolio is mean-variance efficient, i.e. if it belongs to the upper limb of the efficient frontier. In the opposite case, the investor would prefer to invest into the risk-free asset or into the global minimum variance portfolio which lies in the vertex of the set of feasible portfolios. Assuming that the portfolio asset returns are independent and multivariate normally distributed, we suggest a test on the location of the tangency portfolio on the set of feasible portfolios. The distribution of the test statistic is derived under both hypotheses, which we use to assess the power of the test and construct a confidence interval. Moreover, out-of-sample performance of the test is evaluated based on real data. The robustness to the assumption of normality is investigated via an extensive simulation study where we show that the new test is robust to the violation of the normality assumption and can also be used for heavy-tailed stochastic models. Moreover, in an empirical study we apply the developed theory to real data. We find that when the sample size is relatively large and a stable period is present on the market, then the mean-variance efficiency of the tangency portfolio can be statistically justified.

Keywords: tangency portfolio, feasible portfolios, test theory, power function, out-of-sample performance

[1]Corresponding author. E-mail address: stanislas.muhinyuza@math.su.se. This research was partly supported by the Swedish International Development Cooperation Agency (SIDA) through the UR-Sweden Programme for Research, Higher Education and Institutional Advancement

1 Introduction

The question of wealth allocation is relevant for both individuals, e.g. retirement savings, as well as for banks and other institutional investors. How this should be done in practice, however depend on a multitude of factors, not the least the investors view on risk in relation to return. The most influential approach to deal with this problem is the mean-variance analysis proposed by Markowitz (1952). Following Markowitz (1952), the optimal portfolio weights are found by minimizing the risk, i.e. the variance, of the portfolio for a given level of the expected return.

In the case without a risk-free asset, Merton (1972) showed that all optimal solutions of Markowitz's optimization problem lie on the upper limb of the parabola in the mean-variance space. This parabola is known as the efficient frontier and given by

$$V = \frac{a - 2bR + cR^2}{ac - b^2}, \tag{1}$$

where $a = \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$, $b = \mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$, $c = \mathbf{1}\boldsymbol{\Sigma}^{-1}\mathbf{1}$; $R = \mathbf{w}'\boldsymbol{\mu}$ is the expected return of the portfolio with the weights $\mathbf{w}$; $V = \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}$ is its variance; $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are the expected return vector and the covariance matrix of the asset returns, respectively. The symbol $\mathbf{1}$ denotes the vector of ones of an appropriate order. Unfortunately, the set of parameters $\{a, b, c\}$, known as the efficient set of constants, does not possess an appropriate financial meaning.

Rewriting (1) we obtain an alternative expression of the efficient frontier

$$(R - R_{GMV})^2 = s(V - V_{GMV}) \tag{2}$$

where

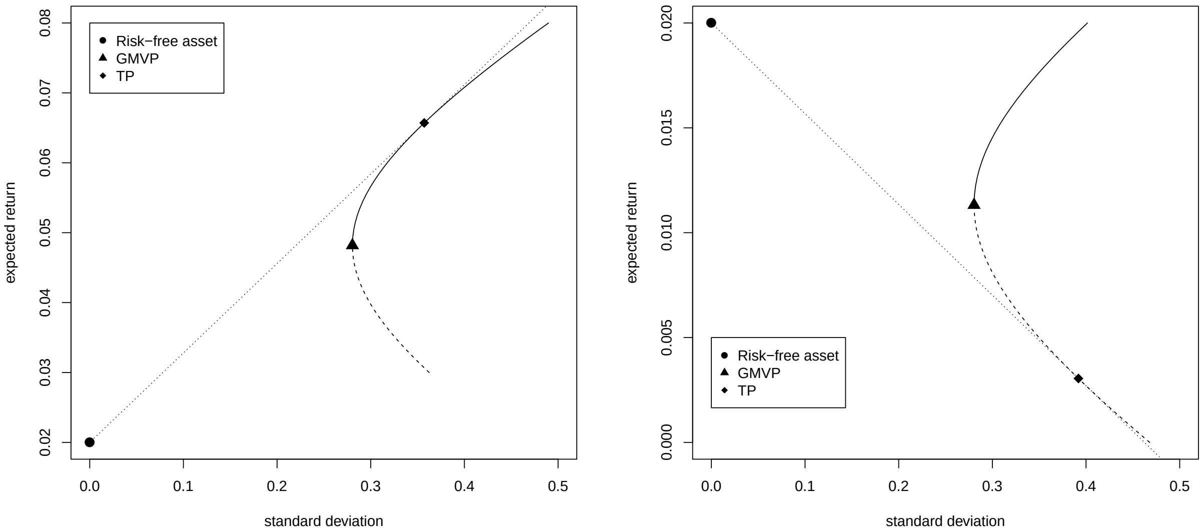
$$R_{GMV} = \frac{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}} \quad \text{and} \quad V_{GMV} = \frac{1}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}} \tag{3}$$

are the expected return and the variance of the global minimum variance portfolio (GMVP), that is, the portfolio with the smallest variance among the efficient portfolios (see, e.g., Frahm (2010); Glombek (2014); Bodnar et al. (2017a,b)). The parameter

$$s = \boldsymbol{\mu}'\mathbf{R}\boldsymbol{\mu} \quad \text{with} \quad \mathbf{R} = \boldsymbol{\Sigma}^{-1} - \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}\mathbf{1}'\boldsymbol{\Sigma}^{-1}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}} \tag{4}$$

stands for the slope coefficient of the parabola in the mean-variance space. The properties of the efficient frontier together with the statement about the distribution of the sample efficient frontier were discussed in detail by Bodnar and Schmid (2008); Kan and Zhou (2008); Bodnar and Schmid (2009).

If there is a possibility to invest into a risk-free asset, then the efficient frontier becomes a tangent line in the mean-variance space which is drawn from the return of the risk-free asset to the parabola (2). The tangent point is known as the tangency portfolio



(a) $R_{GMV} \geq r_f$ (b) $R_{GMV} < r_f$

Figure 1: Location of the tangency portfolio on the set of feasible portfolios in the two cases: Figure 1(a) $R_{GMV} \geq r_f$ and Figure1(b) $R_{GMV} < r_f$.

(TP), see e.g., Ingersoll (1987). This portfolio maximizes the Sharpe ratio (SR), $SR = (\mathbf{w}'\boldsymbol{\mu} - r_f)/(\sqrt{\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}})$, and it has recently received a lot of attention in the literature. For instance, its statistical properties under different assumptions imposed on the distribution of the asset returns were discussed in Lo (2002), whereas Britten-Jones (1999) derived an exact test on the TP weights and showed that it is not possible to reject the null hypothesis that the weight of the US market is equal to one in an international portfolio. Further, while Okhrin and Schmid (2006) showed that the estimated weights of this portfolio do not possess the first moment, Schmid and Zabolotskyy (2008) proved that the unbiased estimator of the TP weights does not exist at all. Recently, Bodnar and Zabolotskyy (2017) investigated the risk properties of the TP and showed that this portfolio is a very risky investment opportunity which should be carefully considered in practice.

The location of the TP portfolio on the set of feasible portfolio depends crucially on the relation between the expected return of the GMVP and the return of the risk-free asset (see Figure 1). The TP is mean-variance efficient, i.e. it belongs to the upper part of the efficient frontier as in Figure 1(a) only if the expected return of the GMVP is greater than the return on the risk-free asset return (see, e.g., Ingersoll (1987, chapter 4)). On the other hand, this consideration may not be appropriate in many practical situations where the expected return of the GMVP is inferior to the return of the risk-free asset. In this case the tangent line drawn to the set of feasible portfolios from the return of the risk-free rate has no joint point with the efficient frontier and, consequently, the TP belongs to the set of the feasible portfolios which are located on the lower part of the parabola as shown in Figure 1(b). The investor would then prefer to invest into the risk-free asset or in the GMVP which lies in the vertex of the efficient frontier. We contribute to the existing literature on the TP by deriving an exact test on its location on the set of feasible portfolios. The distribution of the suggested test statistic is obtained under both hypotheses. Moreover, out-of-sample performance of the portfolio determined by implementing the derived test is assessed.

The remainder of the paper is organised as follows. Section 2 contains a detailed description of statistical test theory for the location of the tangency portfolio on the set of feasible portfolios. We concentrate on the derivation of the test statistic, its distribution under both hypotheses, the analysis of the power function, and the construction of a confidence interval. In Section 3, out-of-sample performance is presented. In Section 4, the numerical procedure for investigating the robustness of normality assumption are provided, while empirical results are discussed in Section 5. Final remarks are presented in Section 6. All proofs are found in the appendix.

2 Finite-sample test on the location of the tangency portfolio

The location of the tangency portfolio on the set of feasible portfolios depends on the relation between the risk-free rate r_f and the expected return on the GMVP(R_{GMV}) as shown in Figure 1. If the investor wants to be sure in the investment into the TP, (s)he has to check if $R_{GMV} > r_f$. This problem can be formulated as a statistical test with the hypotheses given by

$$H_0: \ R_{GMV} \leq r_f \quad \text{against} \quad H_1: \ R_{GMV} > r_f. \tag{5}$$

The rejection of the null hypothesis means that the TP lies on the upper part of the efficient frontier as shown in Figure 1(a). In contrast, if the null hypothesis in (5) cannot be rejected, then the investor cannot be certain of the optimality of the TP and allocation into the risk-free asset could be considered as a suitable alternative.

Let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ denote an independent k-dimensional sample of the asset returns, where $\mathbf{E}[\mathbf{X}_t] = \boldsymbol{\mu}$ and $\mathbf{cov}[\mathbf{X}_t] = \boldsymbol{\Sigma}$, for $t = 1, \ldots, n$. The test statistic for testing (5) is obtained following the derivation in Bodnar and Schmid (2009) and is given by

$$T = \frac{\sqrt{n-k}}{\sqrt{n-1}} \frac{\hat{R}_{GMV} - r_f}{\sqrt{1 + \frac{n}{n-1}\hat{s}}\sqrt{\frac{\hat{V}_{GMV}}{n}}}, \tag{6}$$

where $\hat{R}_{GMV}$, $\hat{V}_{GMV}$, and $\hat{s}$ are the sample estimators for R_{GMV}, V_{GMV}, and s given by

$$\hat{R}_{GMV} = \frac{\mathbf{1}'\hat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\mu}}}{\mathbf{1}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}} \quad \text{and} \quad \hat{V}_{GMV} = \frac{1}{\mathbf{1}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}} \tag{7}$$

and

$$\hat{s} = \hat{\boldsymbol{\mu}}'\hat{\mathbf{R}}\hat{\boldsymbol{\mu}}, \quad \hat{\mathbf{R}} = \hat{\boldsymbol{\Sigma}}^{-1} - \frac{\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}\mathbf{1}'\hat{\boldsymbol{\Sigma}}^{-1}}{\mathbf{1}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}} \tag{8}$$

where

$$\hat{\boldsymbol{\mu}} = \frac{1}{n}\sum_{t=1}^{n}\mathbf{X}_t \quad \text{and} \quad \hat{\boldsymbol{\Sigma}} = \frac{1}{n-1}\sum_{t=1}^{n}(\mathbf{X}_t - \hat{\boldsymbol{\mu}})(\mathbf{X}_t - \hat{\boldsymbol{\mu}})'$$

are the sample mean vector and the sample covariance matrix, respectively. Further, the distribution of T is given by

Proposition 1. *Let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ be independent random vectors of asset returns with $\mathbf{X}_t \sim N_k(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ for $t = 1, \ldots, n$. Assume that $\boldsymbol{\Sigma}$ is positive definite and $n > k$. Then the density of T is given by*

$$f_T(x) = \frac{n(n-k+1)}{(k-1)(n-1)} \int_0^\infty f_{t_{n-k,\delta(y)}}(x) f_{F_{k-1,n-k+1,ns}}\left(\frac{n(n-k+1)}{(k-1)(n-1)}y\right) dy \tag{9}$$

where

$$\delta(y) = \sqrt{\frac{n}{1 + y(n/(n-1))}} S_{GMV} \quad \text{and where} \quad S_{GMV} = \frac{R_{GMV} - r_f}{\sqrt{V_{GMV}}}$$

is the Sharpe ratio of the GMVP. The slope parameter s is defined in (4).

The proof of Proposition 1 follows from Proposition 1 in Bodnar and Schmid (2009). Hence, from Proposition 1 it is seen that the test statistic T may be represented as a mixture of a non-central t distribution with $n-k$ degrees of freedom and a non-centrality parameter $\delta(y)$. Further, by using Proposition 1 it is possible to derive the critical value for the test (5) at significance level α. The result of this is stated in Proposition 2, whose proof is given in the appendix.

Proposition 2. *Under the conditions of Proposition 1, it holds that*

$$\sup_{V_{GMV}>0, s\geq 0, R_{GMV}\leq r_f} G_{T,\alpha,t_{n-k,1-\alpha}}(S_{GMV}, s) \leq \mathbb{P}_{H_0: R_{GMV}=r_f}(T > t_{n-k,1-\alpha}) = \alpha,$$

where

$$G_{T,\alpha,c}(S_{GMV}, s) = \mathbb{P}(T > c) = \int_c^\infty f_T(x)dx\,.$$

Thus, from Proposition 2 it is seen that the test of (5) rejects H_0 in favour of H_1 as soon as $T \geq t_{n-k,1-\alpha}$. Another important characteristic of a statistical test is its power function. It turns out that the power function of the test (5) only depends on $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ in terms of S_{GMV} and s and is given by

$$\begin{aligned} &G_{T,\alpha,t_{n-k,1-\alpha}}(S_{GMV}, s) = \mathbb{P}(T > t_{n-k,1-\alpha}) \\ &= \frac{n(n-k+1)}{(k-1)(n-1)} \int_0^\infty \left(1 - F_{t_{n-k,\delta(y)}}(t_{n-k,1-\alpha})\right) f_{F_{k-1,n-k+1,ns}}\left(\frac{n(n-k+1)}{(k-1)(n-1)}y\right) dy. \end{aligned}$$

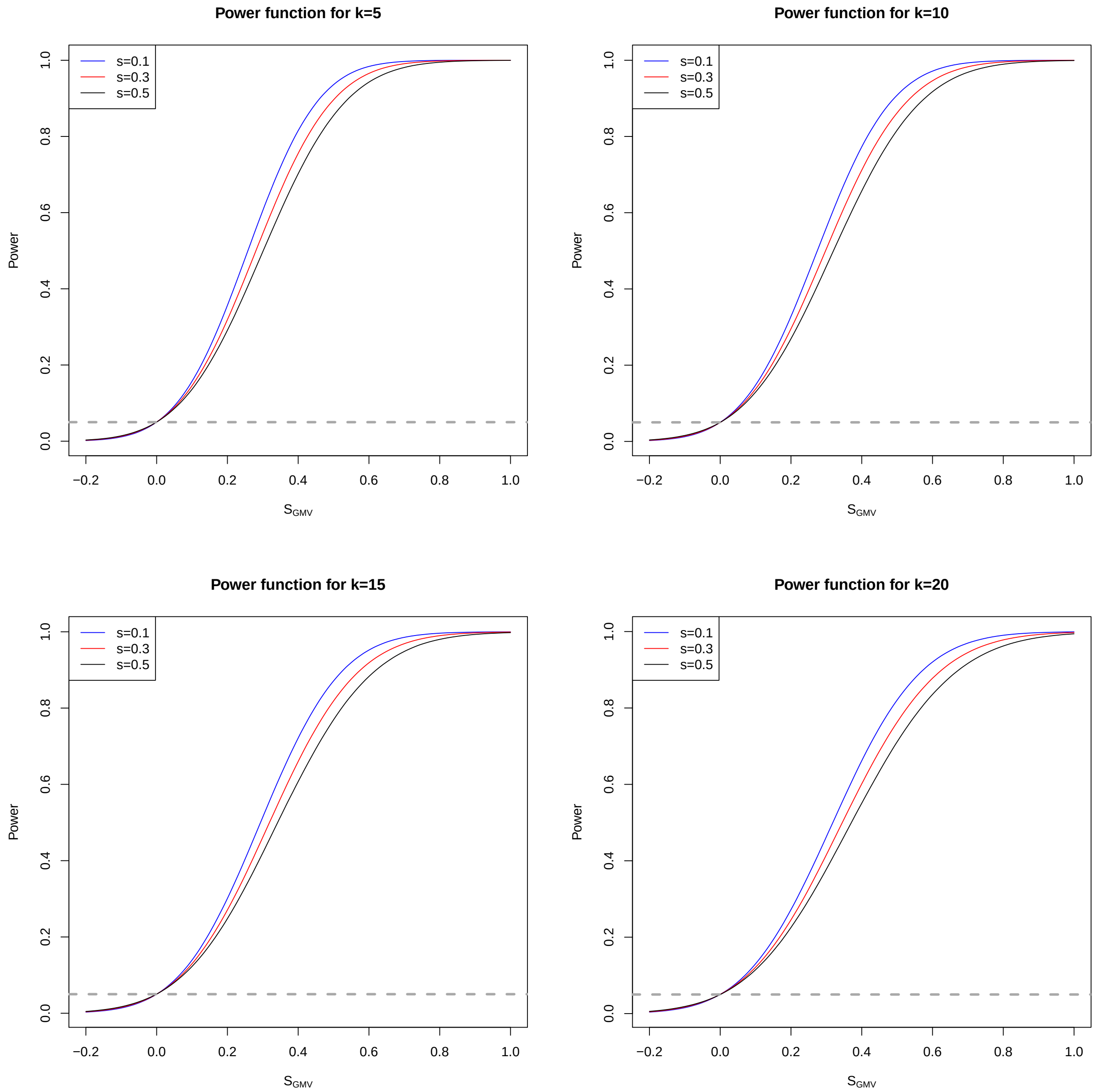


Figure 2: Power function of test (5) for portfolio dimension $k \in \{5, 10, 15, 20\}$ and sample size $n = 50$

This is a nice property of the suggested test which allows us to visualize its power function for fixed values of k and n as a function of s and S_{GMV} only. In Figures 2 and 3, we present the power of the test (5) for $k \in \{5, 10, 15, 20\}$, $n \in \{50, 250\}$, and $s = \{0.1, 0.3, 0.5\}$. The values of S_{GMV} smaller than or equal to 0 corresponds to the null hypothesis. We observe that the power increases rapidly as S_{GMV} becomes larger than zero. It reaches one already for moderate values of S_{GMV}. For example, it is close one for S_{GMV} around 0.2 when $n = 250$ corresponding to approximately one year of daily market observations or five years of weekly data. Furthermore, we note that the power increases if s decreases. This result is in line with the behaviour of the non-central F-distribution whose distribution function is decreasing in the non-centrality parameter. This result also

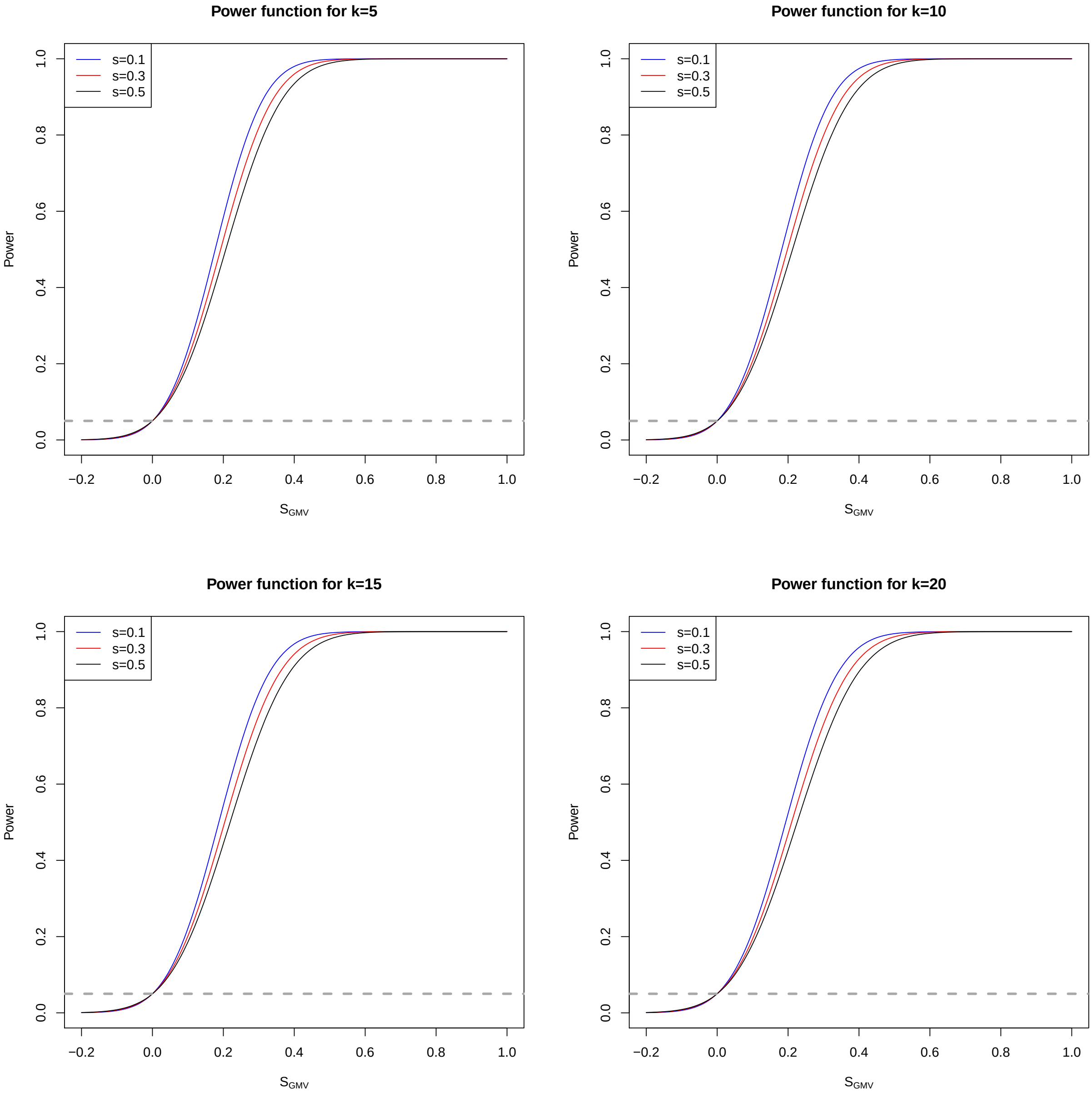


Figure 3: Power function of test (5) for portfolio dimension $k \in \{5, 10, 15, 20\}$ and sample size $n = 100$

has an interesting financial interpretation. If the slope parameter s is smaller, then the optimal portfolio with the same Sharpe ratio and the excess expected return as one in the case of larger s has a higher variance. Consequently, it deviates from the GMVP stronger than in the case of larger s and thus can be easier detected by the test (5).

We conclude this section with the two important remarks:

Remark 1. *Performing a statistical test on the hypotheses* (5), *one can only draw conclusions about investing into the TP. However, if the null hypothesis cannot be rejected, then we still have no statistical justification about avoiding the wealth allocation into the TP. In order to be sure that the TP belongs to the lower part of the parabola as in Figure 1(b), one has to perform the lower one-sided test with the hypotheses given by*

$$\tilde{H}_0: \ R_{GMV} \geq r_f \quad \text{against} \quad \tilde{H}_1: \ R_{GMV} < r_f. \tag{10}$$

This test reject the null hypothesis, i.e. it confirms that the TP is not efficient, as soon as $T < t_{n-k,\alpha}$ where the statistic T is given in (6).

The power function of the test (10) *is obtained similarly to the power function of the test* (5) *and is given by*

$$\begin{aligned}
&\tilde{G}_{T,\alpha,t_{n-k,\alpha}}(S_{GMV}, s) = \mathbb{P}(T < t_{n-k,\alpha}) \\
&= \frac{n(n-k+1)}{(k-1)(n-1)} \int_0^\infty \left(F_{t_{n-k,\delta(y)}}(t_{n-k,\alpha})\right) f_{F_{k-1,n-k+1,ns}}\left(\frac{n(n-k+1)}{(k-1)(n-1)} y\right) dy.
\end{aligned}$$

which also only depends on $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ through S_{GMV} and s.

Remark 2. *Using a one-to-one correspondence between a statistical test and an interval estimation, we can draw a further important conclusion by using the suggested two tests. Namely, it is possible to specify a $(1-\alpha)$ one-sided confidence interval for the risk-free rate such that if r_f belongs to this interval, a conclusion about the investment into the TP can be drawn.*

In the case of the upper one-sided test this interval is given by

$$I_{1-\alpha} = \left[\hat{R}_{GMV} - t_{n-k,1-\alpha} \frac{\sqrt{n-1}}{\sqrt{n-k}} \sqrt{1 + \frac{n}{n-1}\hat{s}} \sqrt{\frac{\hat{V}_{GMV}}{n}}, +\infty\right),$$

while for the lower one-sided we get

$$\tilde{I}_{1-\alpha} = \left(-\infty, \hat{R}_{GMV} - t_{n-k,\alpha} \frac{\sqrt{n-1}}{\sqrt{n-k}} \sqrt{1 + \frac{n}{n-1}\hat{s}} \sqrt{\frac{\hat{V}_{GMV}}{n}}\right],$$

Hence, for all $r_f \notin I_{1-\alpha}$ we conclude that the TP belongs to the efficient frontier and for all $r_f \notin \tilde{I}_{1-\alpha}$ the TP lies on the lower part of the set of feasible portfolios.

3 Out-of-sample performance

In this section we investigate the behaviour of the realized expected return of the GMVP in the period $n+1$ given by $\hat{R}_{GMV,n+1} = \hat{\mathbf{w}}'_{GMV}\mathbf{X}_{n+1}$ where $\mathbf{X}_{n+1}$ is the vector of asset returns at time point $n+1$ and $\hat{\mathbf{w}}_{GMV} = \hat{\mathbf{\Sigma}}^{-1}\mathbf{1}/(\mathbf{1}'\hat{\mathbf{\Sigma}}^{-1}\mathbf{1})$ are the estimated weights of the GMVP by using asset returns $\mathbf{X}_1, ..., \mathbf{X}_n$. The aim is to provide statements about the two conditional probabilities:

$$P_1 = \mathbb{P}\left(\hat{R}_{GMV,n+1} > r_f | \hat{R}_{GMV} > r_f\right) \tag{11}$$

and

$$P_2 = \mathbb{P}\left(\hat{R}_{GMV,n+1} > r_f | T > t_{n-k,1-\alpha}\right) \tag{12}$$

While the probability in (11) can be considered as a naive approach about forecasting the efficiency of the TP at time point $t+1$ given that the estimated expected return of the GMVP is larger than the return of the risk-free asset, the second probability provides a similar statement which is based on the result of the statistical test developed in Section 2.

In order to determine the conditional probabilities in (11) and (12), we first derive the joint distributions $(\hat{R}_{GMV,n+1}, \hat{R}_{GMV})$ and $(\hat{R}_{GMV,n+1}, T)$ in Theorem 1 presented in terms of their stochastic representations which is a very popular tool in computational statistics (Givens and Hoeting (2012)), frequentist statistics (Gupta et al. (2013)) and Bayesian statistics (Bodnar et al. (2017a)). Let the symbol $\stackrel{d}{=}$ denote equality in distribution. Then we get the following results.

Theorem 1. *Let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ be independent random vectors of asset returns with $\mathbf{X}_t \sim N_k(\boldsymbol{\mu}, \mathbf{\Sigma})$ for $t = 1, \ldots, n$. Assume that $\mathbf{\Sigma}$ is positive definite and $n > k$. Then:*

(a) the stochastic representation for $(\hat{R}_{GMV}, \hat{R}_{GMV,n+1})$ is given by

$$\hat{R}_{GMV} \stackrel{d}{=} R_{GMV} + \frac{\sqrt{V_{GMV}}}{\sqrt{n}} z_4 + \sqrt{\frac{1}{n}\xi_3 + \frac{1}{n}\left(\sqrt{ns} + z_5\right)^2}\sqrt{V_{GMV}}\frac{z_1}{\sqrt{\xi_1}} \tag{13}$$

and

$$\begin{aligned}\hat{R}_{GMV,n+1} &\stackrel{d}{=} R_{GMV} + \sqrt{V_{GMV}} z_6 + \sqrt{V_{GMV}}\left(\frac{\sqrt{s}(\sqrt{ns} + z_5)}{\sqrt{\xi_3 + (\sqrt{ns} + z_5)^2}} + z_7\right)\frac{z_1}{\sqrt{\xi_1}} \\ &+ \sqrt{V_{GMV}}\sqrt{\xi_4}\left(\frac{z_3}{\sqrt{\xi_2}}\frac{z_1}{\sqrt{\xi_1}} + \frac{z_2}{\sqrt{\xi_2}}\right)\end{aligned} \tag{14}$$

where $z_1, z_2, z_3, z_4, z_5, z_6, z_7 \sim \mathcal{N}(0,1)$, $\xi_1 \sim \chi^2_{n-k+1}$, $\xi_2 \sim \chi^2_{n-k+2}$, $\xi_3 \sim \chi^2_{k-2}$, $\xi_4 | z_5, \xi_3 \sim \chi^2_{k-2;\delta^2(s,\xi_3,z_5)}$ with $\delta^2(s,\xi_3,z_5) = \frac{s\xi_3}{\xi_3 + \left(\sqrt{ns}+z_5\right)^2}$; $z_1, z_2, z_3, z_4, z_6, z_7, \xi_1, \xi_2, (z_5, \xi_3, \xi_4)$ are mutually independent.

(b) the stochastic representation for $(T, \hat{R}_{GMV,n+1})$ *is given by* (14) *and*

$$T \stackrel{d}{=} \frac{\sqrt{n-k}}{\sqrt{\xi_5}} \frac{1}{\sqrt{1+\frac{\xi_3+\left(\sqrt{ns}+z_5\right)^2}{\xi_1}}} \left(\sqrt{n}\frac{R_{GMV}-r_f}{\sqrt{V_{GMV}}} + z_4 + \sqrt{\frac{\xi_3+\left(\sqrt{ns}+z_5\right)^2}{\xi_1}} z_1 \right) \tag{15}$$

where $\xi_5 \sim \chi^2_{n-k}$ *independent of* $z_1, z_2, z_3, z_4, z_6, z_7, \xi_1, \xi_2, (z_5, \xi_3, \xi_4)$.

The proof of Theorem 1 is given in the appendix. The stochastic representations of Theorem 1 appear to be a very useful tool to investigate the distributional properties of $(\hat{R}_{GMV}, \hat{R}_{GMV,n+1})$ as well as of $(T, \hat{R}_{GMV,n+1})$. Moreover, they show that the distributions of $(\hat{R}_{GMV}, \hat{R}_{GMV,n+1})$ and of $(T, \hat{R}_{GMV,n+1})$ depend on $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ only through the three parameters of the efficient frontier (R_{GMV}, V_{GMV}, s).

An important application of the stochastic representation for $(\hat{R}_{GMV}, \hat{R}_{GMV,n+1})$ and of the stochastic representation for $(T, \hat{R}_{GMV,n+1})$ is that they allow for computation of the conditional probabilities P_1 and P_2 from (11) and (12) in a simple and efficient way. It is remarkable that a high numerical precision of the approximations of the conditional probabilities can be obtained by increasing the size of the drawn samples.

In the case of $(\hat{R}_{GMV}, \hat{R}_{GMV,n+1})$, the following algorithm can be used to evaluate P_1:

Algorithm 1 : Computing P_1 from (11)

(i) fix the values of r_f and (R_{GMV}, V_{GMV}, s);

(ii) generate independently $z_1^b, z_2^b, z_3^b, z_4^b, z_5^b, z_6^b, z_7^b \sim \mathcal{N}(0,1)$, $\xi_1^b \sim \chi^2_{n-k+1}$, $\xi_2^b \sim \chi^2_{n-k+2}$, $\xi_3^b \sim \chi^2_{k-2}$;

(iii) generate $\xi_4^b \sim \chi^2_{k-2;\delta^2(s,\xi_3^b,z_5^b)}$ with $\delta^2(s,\xi_3^b,z_5^b) = \frac{s\xi_3^b}{\xi_3^b+\left(\sqrt{ns}+z_5^b\right)^2}$;

(iv) compute $(\hat{R}_{GMV}^b, \hat{R}_{GMV,n+1}^b)$ as in (13) and (14) by using $z_1^b, z_2^b, z_3^b, z_4^b, z_5^b, z_6^b, z_7^b$, $\xi_1^b, \xi_2^b, \xi_3^b, \xi_4^b$;

(v) determine

$$c_1^b = \mathbb{1}_{\{\hat{R}_{GMV}^b > r_f, \hat{R}_{GMV,n+1}^b > r_f\}} \quad \text{and} \quad c_2^b = \mathbb{1}_{\{\hat{R}_{GMV}^b > r_f\}},$$

where $\mathbb{1}_{\{\mathcal{A}\}}$ is the indicator function of set $\mathcal{A}$;

(vi) repeat steps (i)-(v) for $b = 1, ..., B$ and approximate P_1 by

$$\hat{P}_1 = \frac{\sum_{b=1}^B c_1^b}{\sum_{b=1}^B c_2^b}$$

For $(T, \hat{R}_{GMV,n+1})$, the above algorithm is slightly modified and it is given by

Algorithm 2 : Computing P_2 from (12)

(i) fix the values of r_f and (R_{GMV}, V_{GMV}, s);

(ii) generate independently $z_1^b, z_2^b, z_3^b, z_4^b, z_5^b, z_6^b, z_7^b \sim \mathcal{N}(0,1)$, $\xi_1^b \sim \chi^2_{n-k+1}$, $\xi_2^b \sim \chi^2_{n-k+2}$, $\xi_3^b \sim \chi^2_{k-2}$, $\xi_5^b \sim \chi^2_{n-k}$;

(iii) generate $\xi_4^b \sim \chi^2_{k-2;\delta^2(s,\xi_3^b,z_5^b)}$ with $\delta^2(s,\xi_3^b,z_5^b) = \frac{s\xi_3^b}{\xi_3^b + \left(\sqrt{ns}+z_5^b\right)^2}$;

(iv) compute $(T^b, \hat{R}^b_{GMV,n+1})$ as in (13) and (14) by using $z_1^b, z_2^b, z_3^b, z_4^b, z_5^b, z_6^b, z_7^b$, $\xi_1^b, \xi_2^b, \xi_3^b, \xi_4^b$;

(v) determine

$$c_1^b = \mathbb{1}_{\{T^b > t_{n-k,1-\alpha}, \hat{R}^b_{GMV,n+1} > r_f\}} \quad \text{and} \quad c_2^b = \mathbb{1}_{\{T^b > t_{n-k,1-\alpha}\}};$$

(vi) repeat steps (i)-(v) for $b = 1, ..., B$ and approximate P_2 by

$$\hat{P}_2 = \frac{\sum_{b=1}^B c_1^b}{\sum_{b=1}^B c_2^b}$$

In Figure 4 we present the approximated conditional probabilities $\hat{P}_1$ and $\hat{P}_2$ as a function of $R_{GMV} - r_f$ for $r_f = 0.001$, $V_{GMV} = 0.001$, and $s = 0.22$. The values of r_f, V_{GMV}, and s corresponds to the considered data sets of the empirical illustration of Section 5.1 in Bodnar and Schmid (2009). We also put $n = 50$ (Figure 4) and consider $k \in \{5, 10, 15, 20\}$. We observe that the probability $\hat{P}_2$ is always larger than $\hat{P}_1$ and, consequently, the realized expected return of the GMVP at time $(n+1)$ is larger than the risk-free rate with a higher probability when the decision about this investment opportunity is based on the test (5). Furthermore, we note that the distance between the two curves in the figures is larger for smaller values of $R_{GMV} - r_f$ and for larger values of k.

4 Robustness to the assumption of normality

In this section we investigate the robustness of the test procedure presented in Section 2 when the assumption of normality is violated. The empirical power of the test is computed via simulations by generating samples from the multivariate normal distribution and the standardized multivariate t-distribution with 5 and 10 degrees of freedom, where the standardization of the t-distribution is done in order to have samples with the same mean vector and covariance matrices. Recall that as a result of Proposition 2 it is seen that the power function of the test depends on the mean vector and the covariance matrix only through the slope parameter s of the efficient frontier and the Sharpe ratio S_{GMV} of the GMVP. Due to this, we set $\mathbf{\Sigma} = \boldsymbol{I}_k$, an identity matrix of appropriate dimension k in the simulation study, and consider several values of $\boldsymbol{\mu}$ given by[2]

- $\boldsymbol{\mu}_1 = (0.1, 0, \dots, 0)'$;
- $\boldsymbol{\mu}_2 = (0.1, 0.1, 0, \dots, 0)'$;
- $\boldsymbol{\mu}_3 = (0.1, 0.1, 0.1, 0 \dots, 0)'$;
- $\boldsymbol{\mu}_4 = (0.1, 0.1, 0.1, 0.1, 0, \dots, 0)'$;
- $\boldsymbol{\mu}_5 = (0.1, 0.1, 0.1, 0.1, 0.1, 0, \dots, 0)'$.

The resulting values of s and S_{GMV} are summarized in Table 1. The values with $S_{GMV} \leq 0$ corresponds to the null hypothesis in (5), while $S_{GMV} > 0$ favours the alternative hypothesis. For the cases $S_{GMV} = 0$ we expected the empirical significance level of the test obtained via simulations to be at the nominal significance level $\alpha = 0.05$. Further, the risk-free rate is set to be equal to 0.01 and the portfolio size is $k \in \{5, 10, 15, 20\}$. Moreover, we observe that the slope parameter s becomes larger as k increases, while the

[2]From Proposition 2 it is seen that the power of test (5) only depends on $\boldsymbol{\mu}$ and $\mathbf{\Sigma}$ through S_{GMV} and s, hence any choice of $\boldsymbol{\mu}$ and $\mathbf{\Sigma}$ with the same values of S_{GMV} and s will not affect the power of the test if the asset returns are multivariate normally distributed.

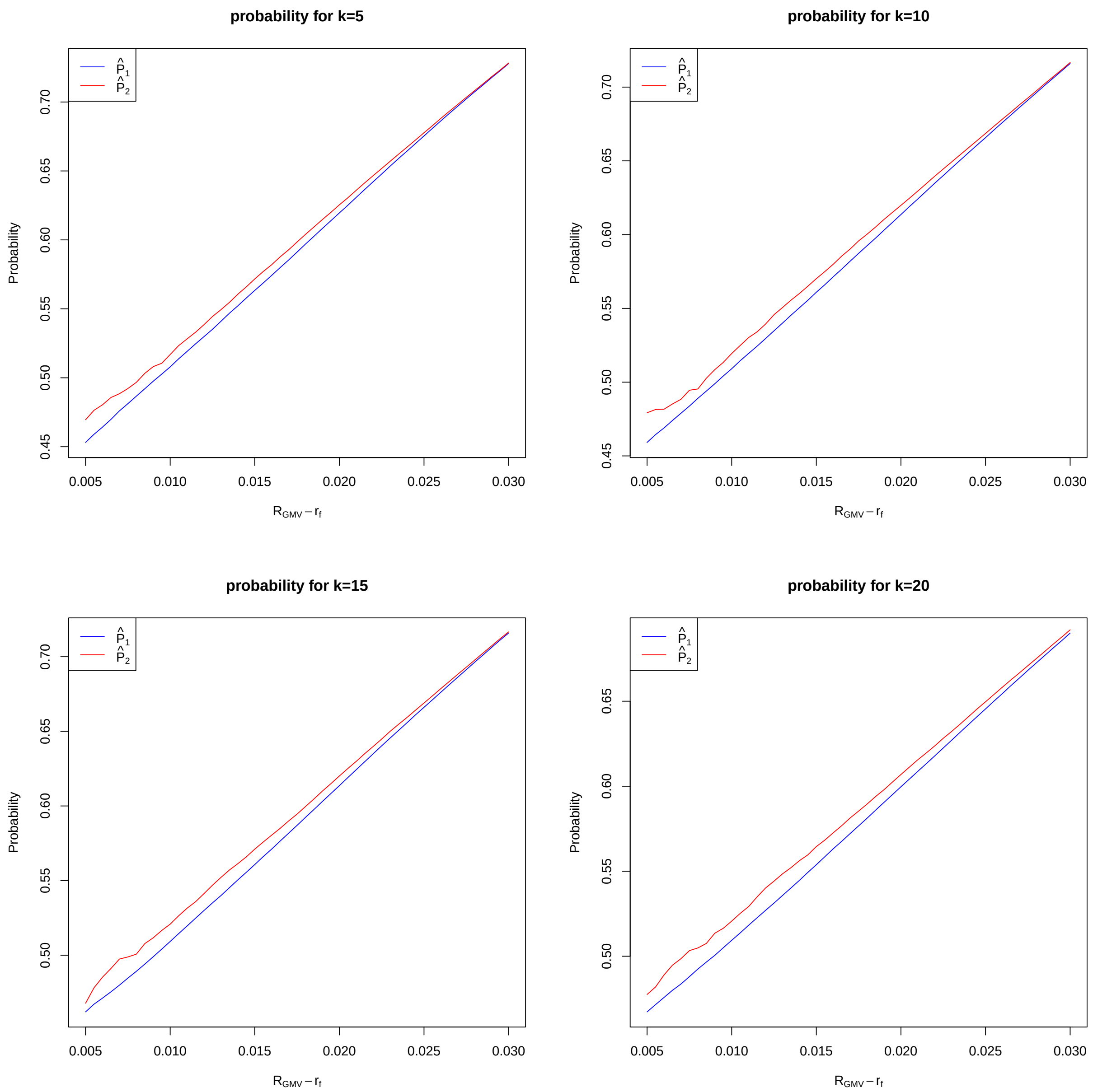


Figure 4: Probabilities $\hat{P}_1$ and $\hat{P}_2$ for portfolio dimension $k \in \{5, 10, 15, 20\}$ and sample size $n = 50$

Sharpe ratio S_{GMV} increases when the number of non-zero elements in the mean vector becomes larger.

k	s&S_{GMV}	$\boldsymbol{\mu}_1$	$\boldsymbol{\mu}_2$	$\boldsymbol{\mu}_3$	$\boldsymbol{\mu}_4$	$\boldsymbol{\mu}_5$
	s	0.0080	0.0120	0.0120	0.0080	0.0000
5	S_{GMV}	0.0224	0.0671	0.1118	0.1565	0.2012
	s	0.0090	0.0160	0.0210	0.0240	0.0250
10	S_{GMV}	0.0000	0.0316	0.0632	0.0949	0.1265
	s	0.0093	0.0173	0.0240	0.0293	0.0333
15	S_{GMV}	-0.0129	0.0129	0.0387	0.0645	0.0904
	s	0.0095	0.018	0.0255	0.0320	0.0375
20	S_{GMV}	-0.0224	0.0000	0.0224	0.0447	0.0671

Table 1: Slope parameter s and Sharpe ratio S_{GMV} for the portfolio dimension $k \in \{5, 10, 15, 20\}$ and several values of $\boldsymbol{\mu}$

In Tables 2, 3 and 4 the results of the simulation study are presented for $k \in \{5, 10, 15, 20\}$ and $n \in \{50, 100, 250\}$. Each value of the power function presented in the tables is obtained by drawing $B = 10^6$ independent samples from the corresponding model. The simulation study suggests that even though data are generated using a heavy tailed t-distribution the tests are performing well. This observation remains true independently of the considered sample size n and portfolio dimension k. Furthermore, we observe that the power grows as the number of non-zero elements in the mean vector becomes larger and decreases for larger values of k. The power is not larger than the nominal significance level of the test, namely 5% in all cases where S_{GMV} is non-positive and it is always larger than 5% for $S_{GMV} > 0$. This statements remains valid independently if data are generated from the normal distribution or from the t-distribution. Finally, we note that the empirical power obtained under the t-distribution is always smaller than the one obtained for the normal distribution and, thus, the test becomes slightly conservative when data are drawn from a heavy-tailed distribution, but it always keeps the nominal significance level.

5 Empirical Study

In order to get a better understanding of the findings obtained in the previous sections, we apply the derived theoretical results to real data. Weekly returns on 29 stocks listed on Dow Jones Industrial (DJI) index are considered for the period from 0.1.01.2006 to 31.12.2015.[3] The 13 weeks US treasury bill covering the aforementioned period is considered as a risk-free asset. The results are obtained for different portfolio dimension

[3]In comparison to daily returns Fama (1976) showed that the distribution of monthly returns is approximately normal. On the other hand, the application of monthly data may result to the bias due to time-varying dynamics in model parameters. For this reason, weekly returns are used as a trade-off between daily and monthly returns.

k	Distribution	$\boldsymbol{\mu}_1$	$\boldsymbol{\mu}_2$	$\boldsymbol{\mu}_3$	$\boldsymbol{\mu}_4$	$\boldsymbol{\mu}_5$
	Normal	0.0669	0.1151	0.1833	0.2731	0.3827
5	t_5	0.0638	0.1061	0.1670	0.2473	0.3438
	t_{10}	0.0660	0.1110	0.1767	0.2622	0.3655
	Normal	0.0497	0.0738	0.1055	0.1456	0.1952
10	t_5	0.0464	0.0683	0.0952	0.1316	0.1752
	t_{10}	0.0485	0.0710	0.1012	0.1402	0.1866
	Normal	0.0426	0.0582	0.0781	0.1020	0.1310
15	t_5	0.0392	0.0526	0.0699	0.0912	0.1163
	t_{10}	0.0414	0.0561	0.0745	0.0969	0.1244
	Normal	0.0389	0.0497	0.0634	0.0801	0.0991
20	t_5	0.0348	0.0447	0.0567	0.0707	0.0873
	t_{10}	0.0371	0.0480	0.0607	0.0762	0.0941

Table 2: Power function for the portfolio dimension $k \in \{5, 10, 15, 20\}$ and the sample size $n = 50$. The nominal significance level of the test is $\alpha = 0.05$.

k	Distribution	$\boldsymbol{\mu}_1$	$\boldsymbol{\mu}_2$	$\boldsymbol{\mu}_3$	$\boldsymbol{\mu}_4$	$\boldsymbol{\mu}_5$
	Normal	0.0771	0.1592	0.2869	0.4492	0.6234
5	t_5	0.0715	0.1443	0.2546	0.4004	0.5631
	t_{10}	0.0744	0.1533	0.2733	0.4285	0.5985
	Normal	0.0500	0.0889	0.1464	0.2239	0.3220
10	t_3	0.0463	0.0801	0.1301	0.1976	0.2833
	t_5	0.0484	0.0856	0.1395	0.2129	0.3058
	Normal	0.0388	0.0636	0.0981	0.1447	0.2041
15	t_5	0.0352	0.0563	0.0864	0.1261	0.1772
	t_{10}	0.0374	0.0609	0.0934	0.1372	0.1928
	Normal	0.0327	0.0499	0.0735	0.1053	0.1453
20	t_5	0.0293	0.0442	0.0646	0.0909	0.1241
	t_{10}	0.0312	0.0479	0.0700	0.0990	0.1360

Table 3: Power function for the portfolio dimension $k \in \{5, 10, 15, 20\}$ and the sample size $n = 100$. The nominal significance level of the test is $\alpha = 0.05$.

k	Distribution	$\boldsymbol{\mu}_1$	$\boldsymbol{\mu}_2$	$\boldsymbol{\mu}_3$	$\boldsymbol{\mu}_4$	$\boldsymbol{\mu}_5$
	Normal	0.0966	0.2735	0.5372	0.7859	0.9339
5	t_5	0.0877	0.2394	0.4732	0.7204	0.8921
	t_{10}	0.0939	0.2596	0.5114	0.7608	0.9185
	Normal	0.0502	0.1231	0.2496	0.4240	0.6136
10	t_5	0.0454	0.1075	0.2145	0.3650	0.5412
	t_{10}	0.0482	0.1167	0.2346	0.3988	0.5837
	Normal	0.0326	0.0736	0.1445	0.2517	0.3892
15	t_5	0.0295	0.0639	0.1234	0.2119	0.3295
	t_{10}	0.0316	0.0700	0.1356	0.2345	0.3637
	Normal	0.0236	0.0498	0.0945	0.1643	0.2587
20	t_5	0.0211	0.0428	0.0797	0.1360	0.2145
	t_{10}	0.0226	0.0471	0.0881	0.1530	0.2401

Table 4: Power function for the portfolio dimension $k \in \{5, 10, 15, 20\}$ and the sample size $n = 250$. The nominal significance level of the test is $\alpha = 0.05$.

$k \in \{5, 10, 15, 20\}$ and sample size $n \in \{50, 100, 250\}$. The chosen values of n roughly correspond to one year, two years, and five years of weekly data.

5.1 Empirical distribution of p-values

In order to provide some general statements about the location of the TP on the efficient frontier independently of the chosen stocks, we perform the test (5) for 1000 randomly selected sets of stocks listed in the DJI index for each $k \in \{5, 10, 15, 20\}$ and $n \in \{50, 100, 250\}$. Namely, for all pairs of k and n we choose randomly k stocks listed in DJI and their n most recent returns. Then, using these data we perform the test on the hypothesis (5) and calculate the corresponding p-value. The procedure is repeated 1000 times resulting in a sample of p-values calculated from different sets of stocks with fixed k and n. From these samples the histograms are constructed which are shown in Figure 5 for $n = 50$, in Figure 6 for $n = 100$, and in Figure 7 for $n = 250$.

We observe that the number of rejection of the null hypothesis depends crucially on the sample size. For $n = 50$, we are not able to reject the null hypothesis at 5% significance level in most of the considered cases. That is, it is not possible to conclude that the TP is a suitable alternative to both the GMVP and the investment into the risk-free asset as it might be located on the lower part of the feasible set of optimal portfolio. However, when n increases, the p-values become smaller and, in particular, they are almost all below 10% for $n = 250$. Table 5 provides further insight into the behavior of the p-values. Here, the number of rejections of the null hypothesis (5) for the significance levels of 1%, 5%, and 10% are present. The number of rejections dramatically increases when n becomes larger. Also, we observe an increase when k is larger. To this end, we conclude that the decision about the location of the tangency portfolio on the feasible set of portfolios

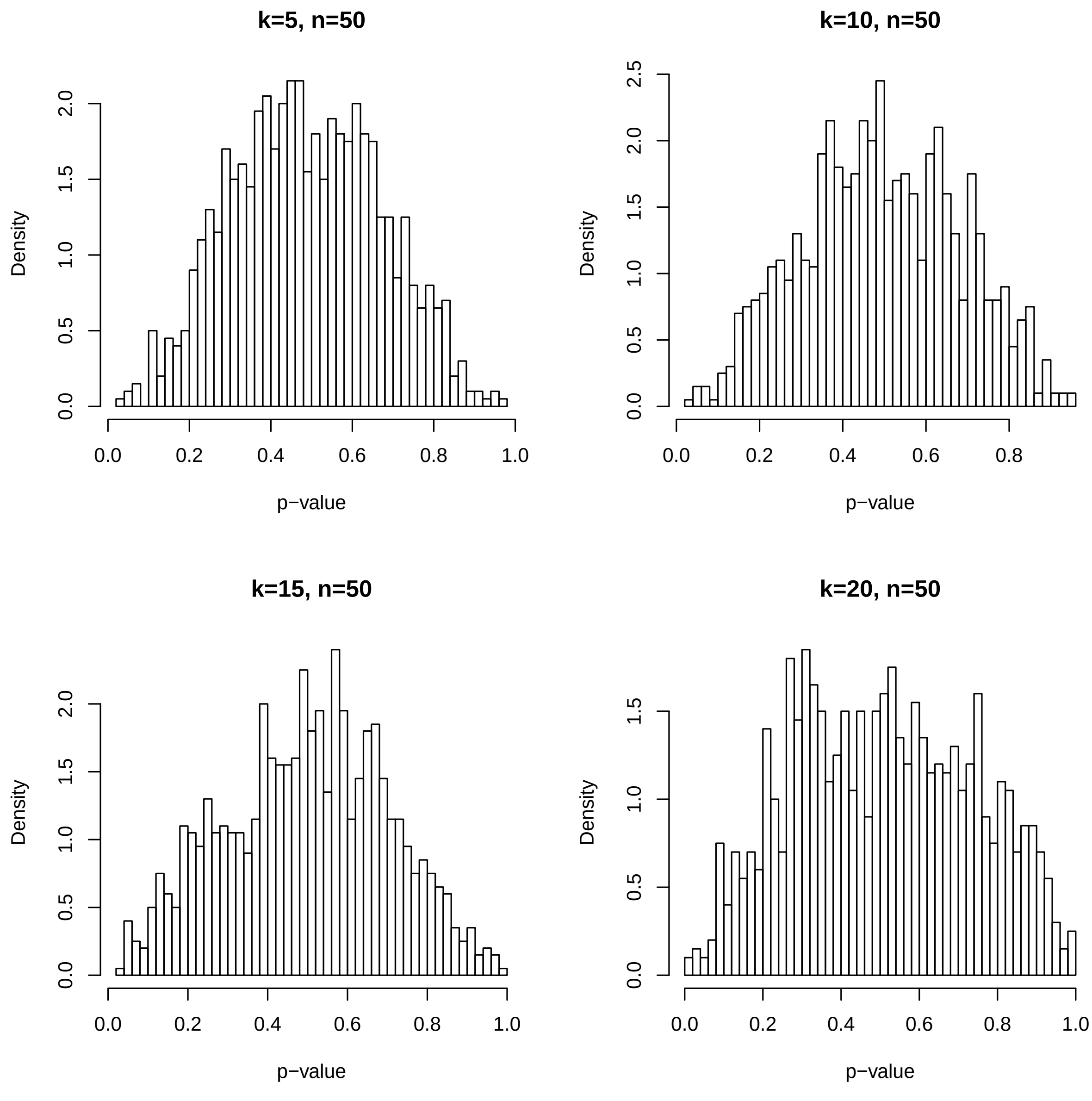


Figure 5: Histograms of p-values for 1000 randomly sampled sets of stocks listed in the DJI index in the case of $k = 5$ (top left), $k = 10$ (top right), $k = 15$ (bottom left), and $k = 20$ (bottom right). For each chosen set of stocks $n = 50$ most recent returns are used.

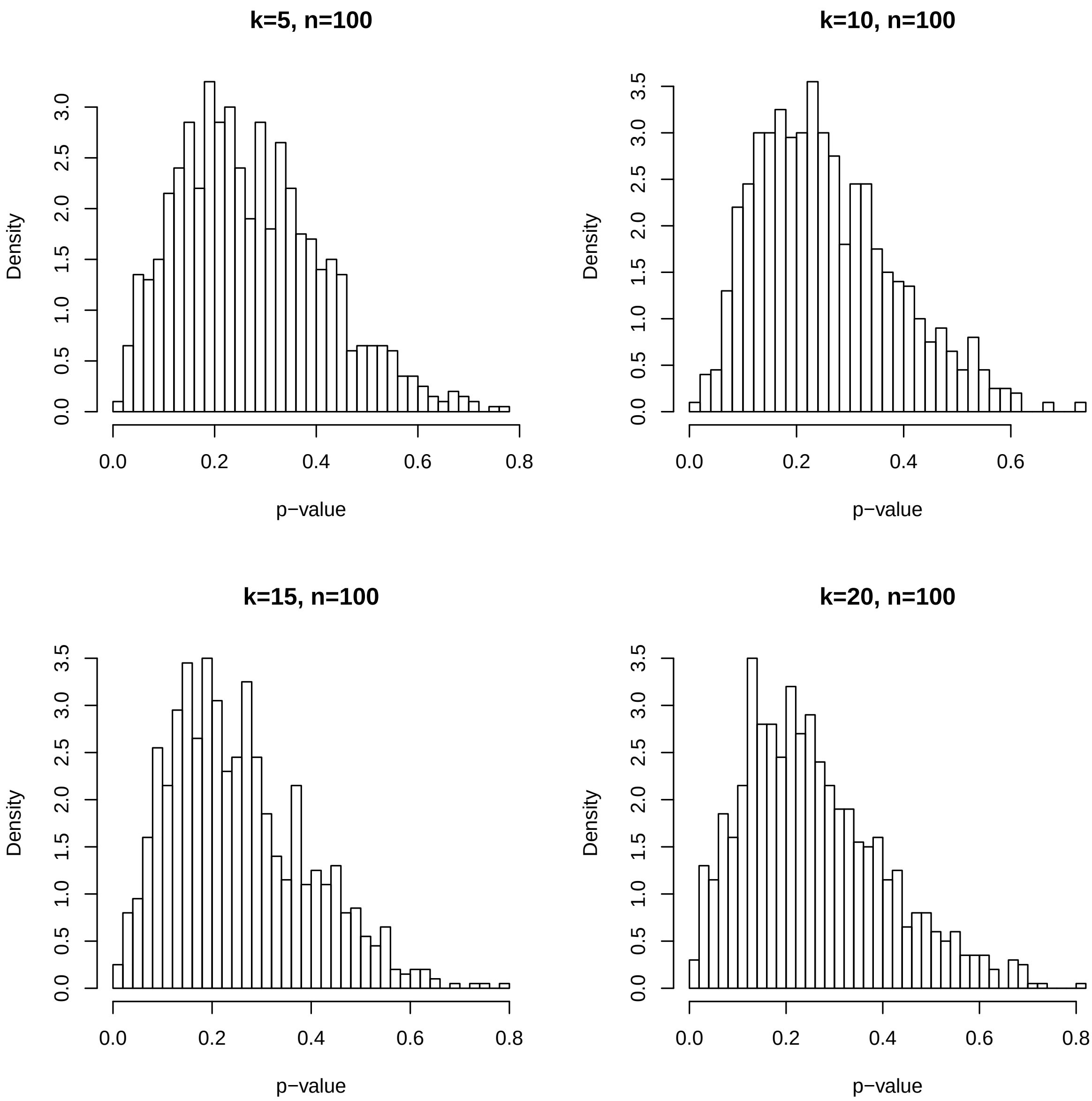


Figure 6: Histograms of p-values for 1000 randomly sampled sets of stocks listed in the DJI index in the case of $k = 5$ (top left), $k = 10$ (top right), $k = 15$ (bottom left), and $k = 20$ (bottom right). For each chosen set of stocks $n = 100$ most recent returns are used.

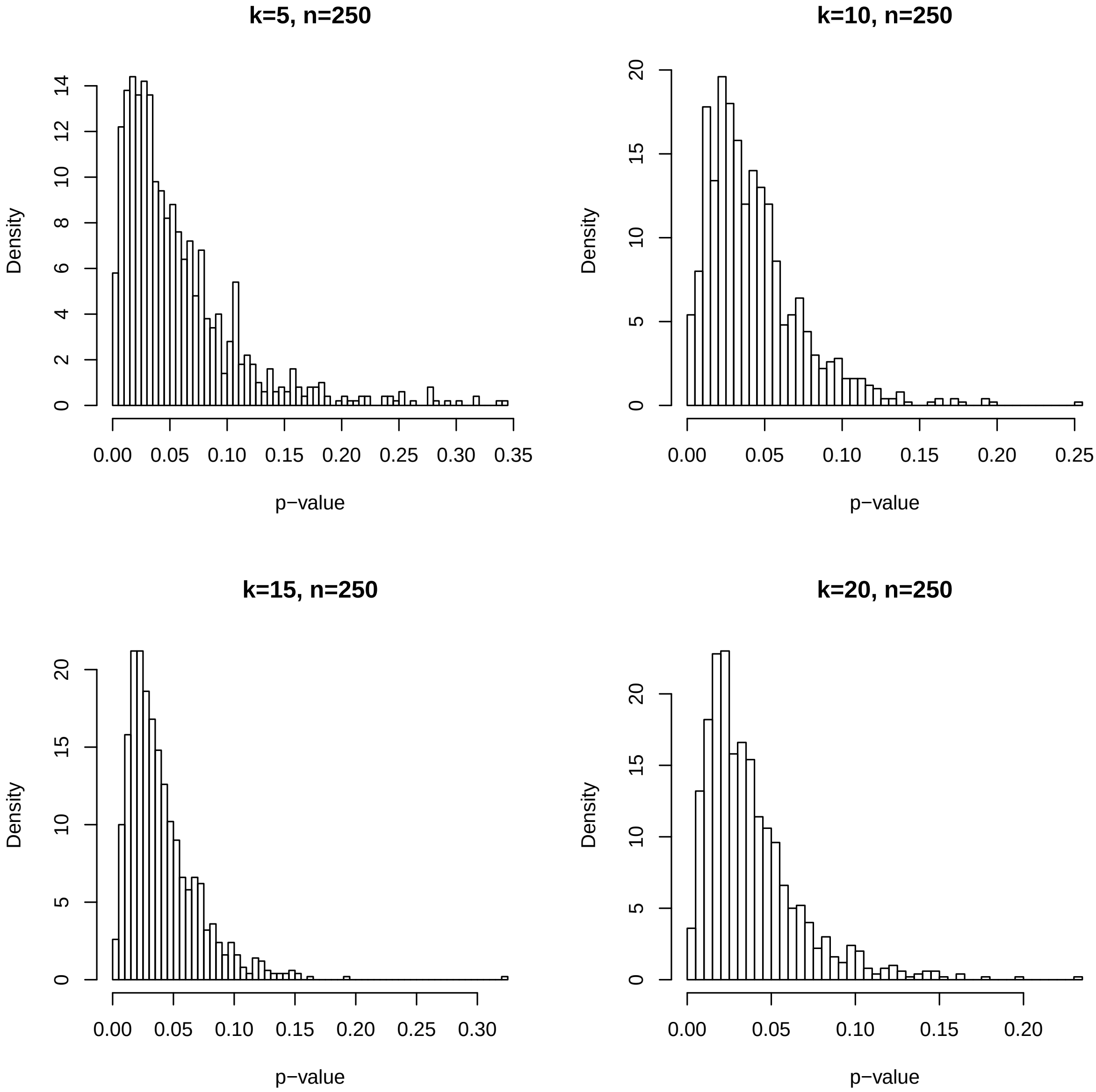


Figure 7: Histograms of p-values for 1000 randomly sampled sets of stocks listed in the DJI index in the case of $k = 5$ (top left), $k = 10$ (top right), $k = 15$ (bottom left), and $k = 20$ (bottom right). For each chosen set of stocks $n = 250$ most recent returns are used.

depends crucially on the amount of information used to make a decision. If the sample size is small, then the test (5) is not powerful enough to reject the null hypothesis and to be able to draw a conclusion about investing in the TP. This finding is in line with the results of the simulation study presented in Section 4.

k/n	5			10			15			20		
	0.01	0.05	0.1	0.01	0.05	0.1	0.01	0.05	0.1	0.01	0.05	0.1
50	0	2	6	0	2	8	0	6	18	1	7	26
100	0	25	98	0	13	89	2	29	123	3	44	124
250	90	575	846	67	685	946	63	719	956	84	753	957

Table 5: Number of rejections of the null hypothesis in (5) for 1000 randomly sampled sets of stocks listed in the DJI index in the case of $k \in \{5, 10, 15, 20\}$ and $n \in \{50, 100, 250\}$. The significance level of the test is set to $\alpha \in \{0.01, 0.05, 0.1\}$

5.2 Time series behavior of the p-values

In order to investigate the performance of the suggested test on the location of the tangency portfolio at several time points, we apply the rolling window estimation (testing) technique with sample size (window length) of $n \in \{50, 100, 250\}$. In all cases we choose $k = \{5, 10, 15, 20\}$ stocks listed in the DJI index following their alphabetical order.

In Figure 8 we present the values of the Sharpe ratio calculated for the estimated GMVP. A very volatile behavior is present, especially when the window length is small. If k increases, then the values of the calculated Sharpe ratio become larger showing a positive effect of diversification, a well-known result in portfolio theory. Finally, larger values of the Sharpe ratio are present at the end of the considered time period leading to the conclusion that the capital market recovers after the financial crisis in 2008, while negative values of the Sharpe ratio are present around the period of the financial crisis. Finally, we point out, that larger values red of the Sharpe ratio can be obtained for smaller sample sizes when most recent data are used in the construction of the GMVP. However, in this case we also see more volatile behaviour of the estimated characteristics of the GMVP which leads to higher risk.

In Figures 9, 10, and 11, the p-values (blue lines) are shown for the test (5) in the case of $k \in \{5, 10, 15, 20\}$ and $n \in \{50, 100, 250\}$. In addition, we also present the p-values (red line) of the test (10) (see Remark 1 in Section 2) where we test if the TP lies on the lower part of the efficient frontier under H_1, i.e. we check if the TP is not mean-variance efficient. Similarly to the Sharpe ratio, the p-values show high fluctuation over time when using smaller sample size, while they are quite stable for larger sample sizes. For $n = 50$ the p-values of both tests are larger than the nominal significance level of 5% and, hence, no decision about the investment into the TP could be done since both null hypotheses cannot be rejected. This point is fully related to the power properties of the tests, i.e. the window length is too small for drawing a conclusive decision. By increasing the value

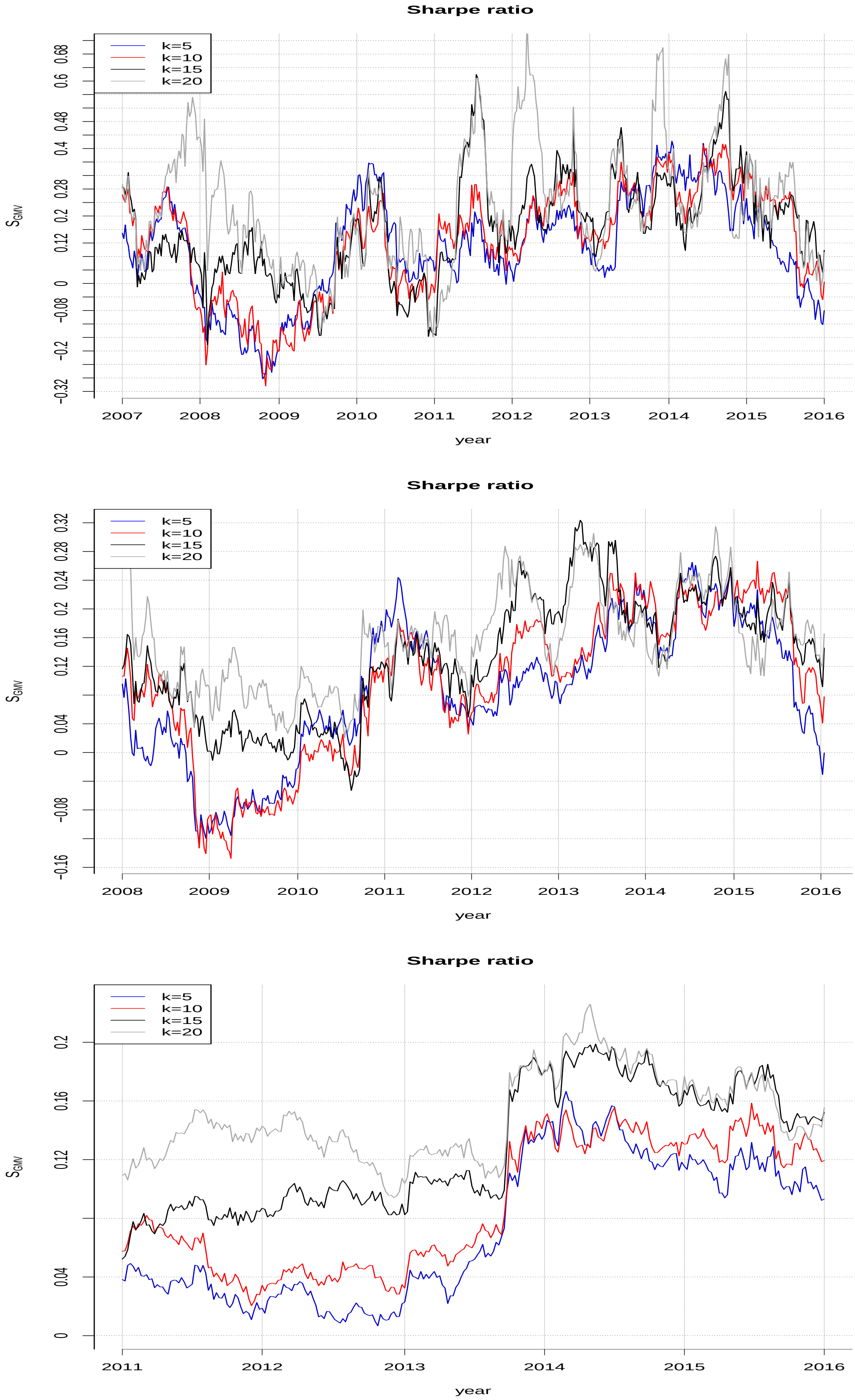


Figure 8: Empirical Sharpe ratio calculated for the GMVP constructed by using first (in alphabet order) $k \in \{5, 10, 15, 20\}$ stocks listed in the DJI index. The size of the rolling window is $n = 50$ (upper), $n = 100$ (middle), and $n = 250$ (button).

of n, the situation improves and we may draw conclusions concerning the mean-variance efficiency of the TP in almost the whole period starting at the end of 2013 for $n = 250$. In contrast, the decision about the inefficiency of the TP can be drawn at the end of 2008 when $n = 50$ and $k \in \{5, 10\}$. We also note that in all cases where the empirical Sharpe ratio is negative, we are not able to reject the null hypothesis of the test (5).

Finally, we present the values of the conditional probabilities $\hat{P}_1$ and $\hat{P}_2$ in Table 6 which are defined in Section 3 as the probabilities that the realized expected return of the GMVP is larger than the risk-free rate in the consequent period provided that the estimated expected return of this portfolio is larger than the risk-free rate (for $\hat{P}_1$) or the test (5) at significance level 5% rejects the null hypothesis (for $\hat{P}_2$). Note that the number of cases used in the computation of $\hat{P}_1$ and $\hat{P}_2$ depends on the occurrence of the events $\{\hat{R}_{GMV} > r_f\}$ and $\{T > t_{n-k,1-\alpha}\}$, respectively. The number of rejections of the null hypothesis by test (5) are summarized in Table 6. In these cases, $\hat{P}_2$ were computed, while slightly larger samples were used for the calculation of $\hat{P}_1$. Table 6 documents that $\hat{P}_2$ outperforms $\hat{P}_1$ for $n = 50$ and $n = 100$, while they have the same performance for $n = 250$. Hence, the best strategy to forecast the efficiency of the TP is to use the statistical approach developed in Section 2. Furthermore, the results of Table 6 are in line with the findings of the simulation study of Section 3 where similar performance is documented.

k/n	5			10			15			20		
	Rej	$\hat{P}_1$	$\hat{P}_2$	Rej	$\hat{P}_1$	$\hat{P}_2$	Rej	$\hat{P}_1$	$\hat{P}_2$	Rej	$\hat{P}_1$	$\hat{P}_2$
50	87	0.9861	1	61	0.9744	1	70	0.9702	1	83	0.9862	1
100	96	0.9853	1	99	0.9823	1	124	0.9875	1	103	1	1
250	107	1	1	118	1	1	130	1	1	248	1	1

Table 6: Empirical probabilities $\hat{P}_1$ and $\hat{P}_2$ of the realized return of the GMVP to be positive calculated for the first $k \in \{5, 10, 15, 20\}$ stocks listed in the DJI index in the alphabetical order. Rolling window estimation is used with the window length equal to $n \in \{50, 100, 250\}$. The nominal significance level of the test (5) used in the calculations of $\hat{P}_2$ is $\alpha = 0.05$.

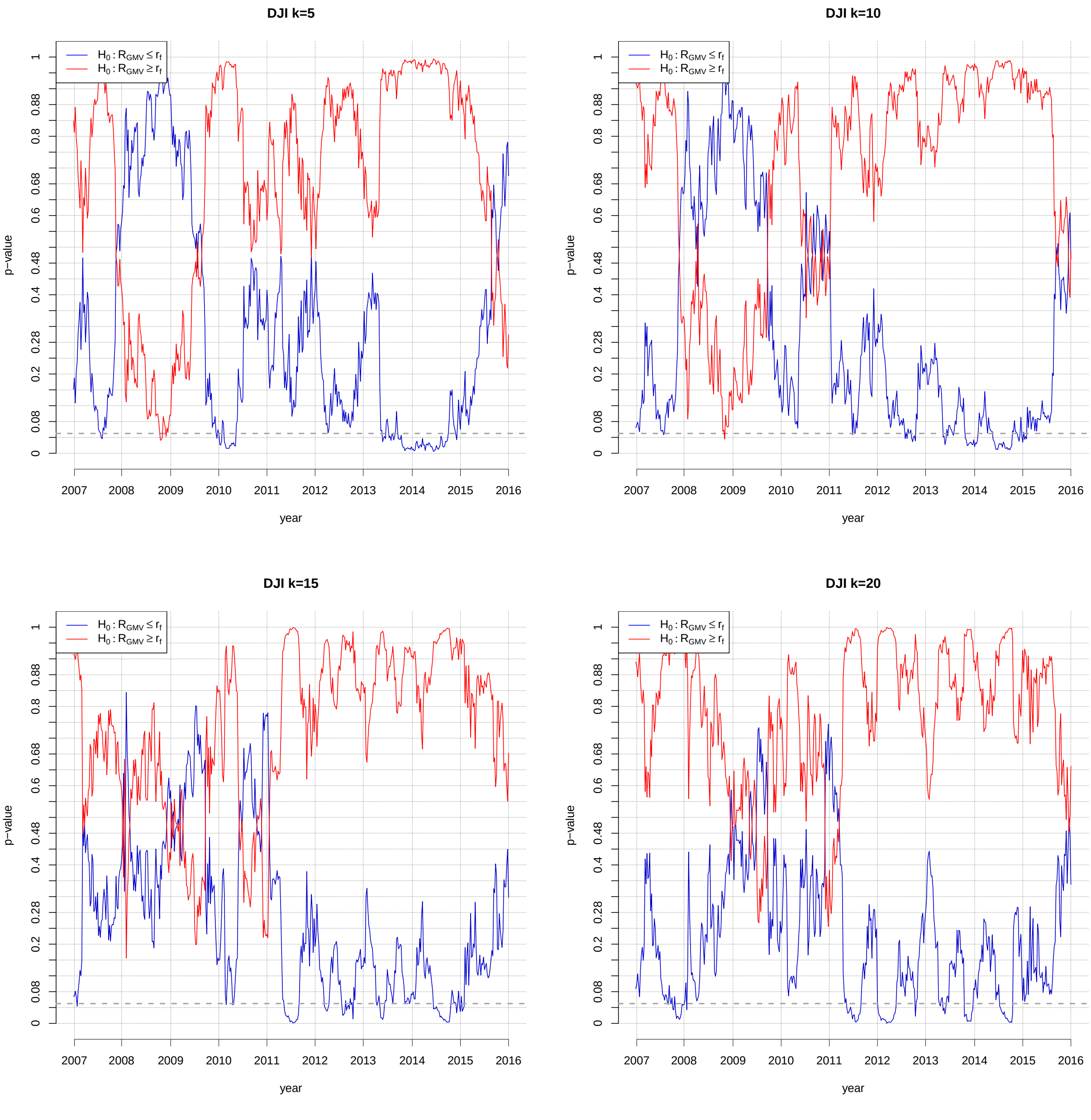


Figure 9: p-values calculated for the test (5) (blue line) and for the test (10) (red line) for the first $k \in \{5, 10, 15, 20\}$ stocks listed in the DJI index in the alphabetical order. Rolling window estimation is used with the window length equal to $n = 50$.

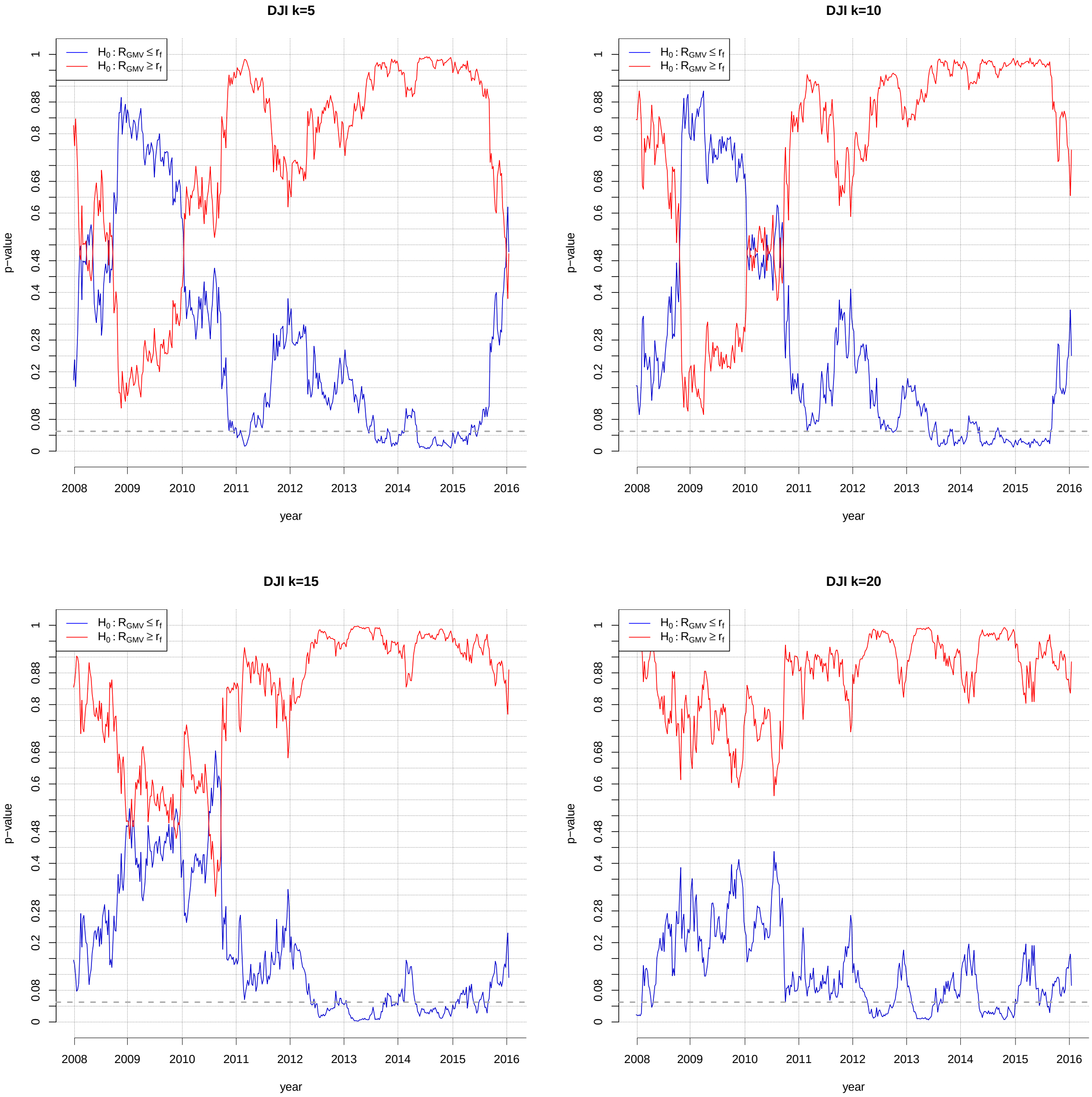


Figure 10: p-values calculated for the test (5) (blue line) and for the test (10) (red line) for the first $k \in \{5, 10, 15, 20\}$ stocks listed in the DJI index in the alphabetical order. Rolling window estimation is used with the window length equal to $n = 100$.

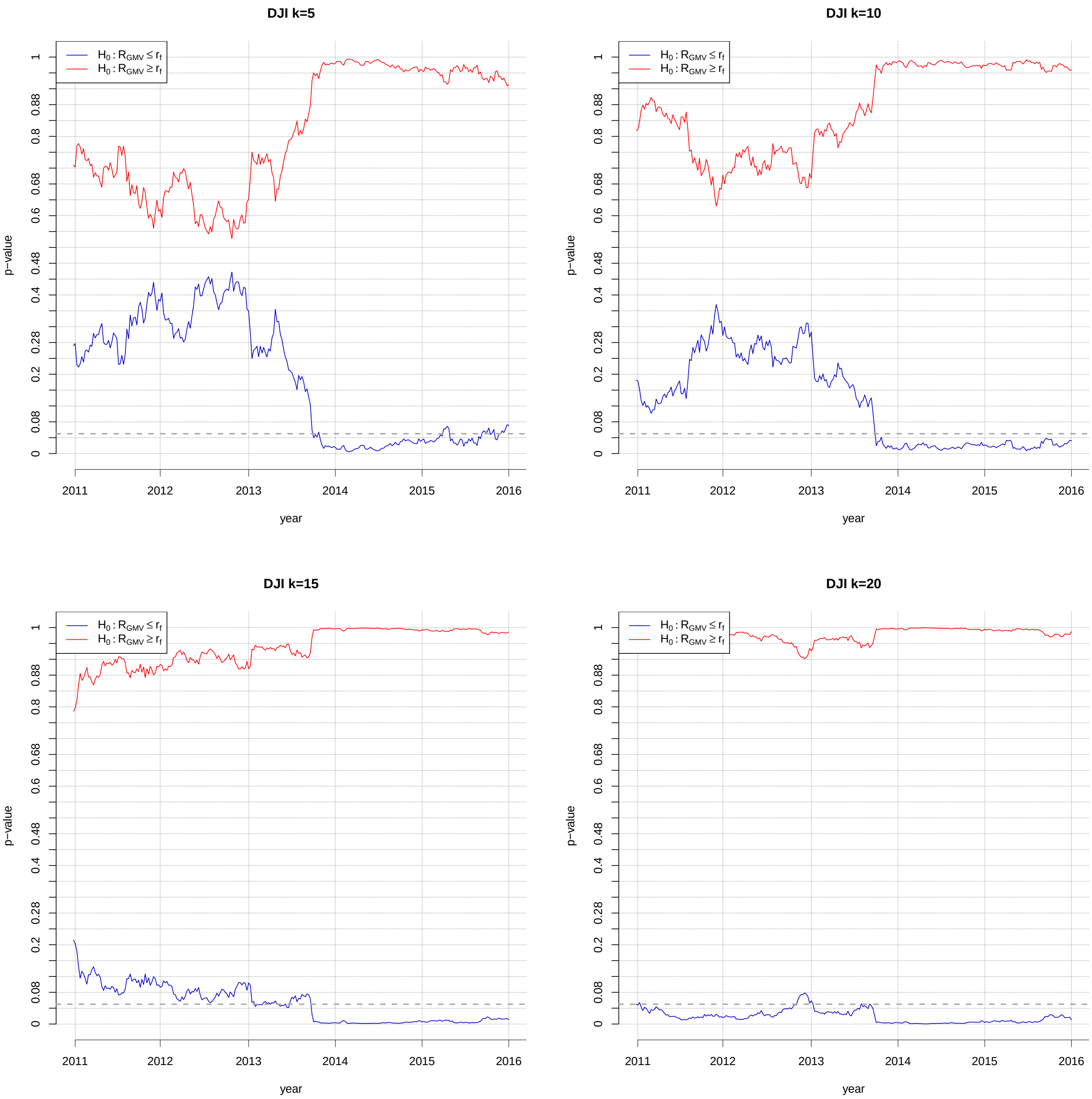


Figure 11: p-values calculated for the test (5) (blue line) and for the test (10) (red line) for the first $k \in \{5, 10, 15, 20\}$ stocks listed in the DJI index in the alphabetical order. Rolling window estimation is used with the window length equal to $n = 250$.

6 Summary

The tangency portfolio plays an important role in the financial literature and is usually used as a market portfolio in the capital asset pricing model. However, due to the way how the TP is constructed together with the large amount of uncertainty that is present in financial markets, the TP might not be mean-variance efficient at all. Although a number of studies is devoted to the estimation of the TP weights and investigating the distributional properties of the tangency portfolio (see, Ingersoll (1987); Britten-Jones (1999); Okhrin and Schmid (2006); Schmid and Zabolotskyy (2008); Bodnar and Zabolotskyy (2017)), the problem of the location of the TP on the set of feasible portfolios has not been treated in the literature to the best of our knowledge.

In this paper we introduce a finite-sample test on the mean-variance efficiency of the tangency portfolio. The distribution of the test statistic is also derived under both hypotheses. Further, it is shown that the suggested test is easily performed in practice by comparing the value of the test statistic with the quantile of a t-distribution. Moreover, the result under the alternative hypothesis is used to investigate the test power. Within an extensive simulation study, we show that the new test is robust to the violation of the normality assumption and can also be used for heavy-tailed stochastic models. Finally, the theoretical results are applied to recent data based on the returns on the stocks included into the DJI index. We conclude, empirically, that the TP is not mean-variance efficient during some parts of the financial crisis. On the other hand, we are not able to accept the efficiency of the TP when the sample size is small because of a large amount of uncertainty present in the financial markets. However, if the sample size is relatively large and a stable period is present on market, then the mean-variance efficiency of the TP can be statistically justified.

7 Appendix

Proof of Proposition 2. For a given constant c, we get that

$$\begin{aligned}
&G_{T,\alpha}(S_{GMV}, s) = \mathbb{P}(T > c) = \int_c^\infty f_T(x)dx \\
&= \frac{n(n-k+1)}{(k-1)(n-1)} \int_c^\infty \int_0^\infty f_{t_{n-k,\delta(y)}}(x) f_{F_{k-1,n-k+1,ns}} \left(\frac{n(n-k+1)}{(k-1)(n-1)} y \right) dydx \\
&= \frac{n(n-k+1)}{(k-1)(n-1)} \int_0^\infty \left(\int_c^\infty f_{t_{n-k,\delta(y)}}(x)dx \right) f_{F_{k-1,n-k+1,ns}} \left(\frac{n(n-k+1)}{(k-1)(n-1)} y \right) dy \\
&= \frac{n(n-k+1)}{(k-1)(n-1)} \int_0^\infty \left(1 - F_{t_{n-k,\delta(y)}}(c) \right) f_{F_{k-1,n-k+1,ns}} \left(\frac{n(n-k+1)}{(k-1)(n-1)} y \right) dy.
\end{aligned}$$

In using that $1 - F_{t_{n-k,\delta(y)}}(c) > 1 - F_{t_{n-k,0}}(c)$ for all $y \geq 0$ and $R_{GMV} < r_f$, we get

$$\begin{aligned} G_{T,\alpha}(S_{GMV}, s) &\leq \frac{n(n-k+1)}{(k-1)(n-1)} \int_0^\infty \left(1 - F_{t_{n-k,0}}(c)\right) f_{F_{k-1,n-k+1,ns}}\left(\frac{n(n-k+1)}{(k-1)(n-1)} y\right) dy \\ &= \left(1 - F_{t_{n-k,0}}(c)\right) \underbrace{\frac{n(n-k+1)}{(k-1)(n-1)} \int_0^\infty f_{F_{k-1,n-k+1,ns}}\left(\frac{n(n-k+1)}{(k-1)(n-1)} y\right) dy}_{1} \\ &= 1 - F_{t_{n-k}}(c) = \alpha. \end{aligned}$$

with $c = t_{n-k,1-\alpha}$ where $t_{n-k,1-\alpha}$ denotes the $(1-\alpha)$ quantile of the t-distribution with $n-k$ degrees of freedom. □

Proof of Theorem 1. From Theorem 3.1.2 and Corollary 3.2.2 in Muirhead (1982), we get $\hat{\boldsymbol{\mu}} \sim \mathcal{N}_k(\boldsymbol{\mu}, \boldsymbol{\Sigma}/n)$, $(n-1)\hat{\boldsymbol{\Sigma}} \sim \mathcal{W}_k(n-1, \boldsymbol{\Sigma})$ (k-dimensional Wishart distribution with $n-1$ degrees of freedom and the parameter matrix $\boldsymbol{\Sigma}$); $\hat{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\Sigma}}$ are independently distributed. Moreover, we get $\mathbf{X}_{n+1}$ is independent of both $\hat{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\Sigma}}$ by the assumptions of the theorem.

Let

$$\hat{\boldsymbol{\Omega}} = \begin{bmatrix} \hat{\boldsymbol{\mu}}' \\ \mathbf{X}'_{n+1} \\ \mathbf{1}'_k \end{bmatrix} \hat{\boldsymbol{\Sigma}}^{-1} \begin{bmatrix} \hat{\boldsymbol{\mu}} & \mathbf{X}_{n+1} & \mathbf{1}_k \end{bmatrix}$$

Since $\hat{\boldsymbol{\Sigma}}$ is independent of $\hat{\boldsymbol{\mu}}$ and $\mathbf{X}_{n+1}$, the conditional distribution of $\hat{\boldsymbol{\Omega}}$ given $\hat{\boldsymbol{\mu}} = \boldsymbol{\mu}_0$ and $\mathbf{X}_{n+1} = \mathbf{X}_0$ is equal to $\tilde{\boldsymbol{\Omega}}$ expressed as

$$\tilde{\boldsymbol{\Omega}} = \begin{bmatrix} \boldsymbol{\mu}'_0 \\ \mathbf{X}'_0 \\ \mathbf{1}'_k \end{bmatrix} \hat{\boldsymbol{\Sigma}}^{-1} \begin{bmatrix} \boldsymbol{\mu}_0 & \mathbf{X}_0 & \mathbf{1}_k \end{bmatrix} = \begin{bmatrix} \boldsymbol{\mu}'_0\hat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{\mu}_0 & \boldsymbol{\mu}'_0\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{X}_0 & \boldsymbol{\mu}'_0\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}_k \\ \mathbf{X}'_0\hat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{\mu}_0 & \mathbf{X}'_0\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{X}_0 & \mathbf{X}'_0\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}_k \\ \mathbf{1}'_k\hat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{\mu}_0 & \mathbf{1}'_k\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{X}_0 & \mathbf{1}'_k\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}_k \end{bmatrix}$$

Defining

$$\boldsymbol{\Omega} = \begin{bmatrix} \boldsymbol{\mu}'_0\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_0 & \boldsymbol{\mu}'_0\boldsymbol{\Sigma}^{-1}\mathbf{X}_0 & \boldsymbol{\mu}'_0\boldsymbol{\Sigma}^{-1}\mathbf{1}_k \\ \mathbf{X}'_0\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_0 & \mathbf{X}'_0\boldsymbol{\Sigma}^{-1}\mathbf{X}_0 & \mathbf{X}'_0\boldsymbol{\Sigma}^{-1}\mathbf{1}_k \\ \mathbf{1}'_k\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_0 & \mathbf{1}'_k\boldsymbol{\Sigma}^{-1}\mathbf{X}_0 & \mathbf{1}'_k\boldsymbol{\Sigma}^{-1}\mathbf{1}_k \end{bmatrix}$$

and using Theorem 3.2.11 by Muirhead (1982), we get that $(n-1)^{-1}\tilde{\boldsymbol{\Omega}}^{-1} \sim \mathcal{W}_3(n-k+2, \boldsymbol{\Omega}^{-1})$. Hence, $(n-1)\tilde{\boldsymbol{\Omega}} \sim \mathcal{W}_3^{-1}(n-k+6, \boldsymbol{\Omega})$.

Let

$$s_0 = \boldsymbol{\mu}'_0\hat{\mathbf{R}}\boldsymbol{\mu}_0, \quad h_0 = \mathbf{X}'_0\hat{\mathbf{R}}\boldsymbol{\mu}_0, \quad v_0 = \mathbf{X}'_0\hat{\mathbf{R}}\mathbf{X}_0.$$

From Theorem 3.(b) in Bodnar and Okhrin (2008) we get

$$\begin{pmatrix} \frac{\boldsymbol{\mu}_0'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}_k}{\mathbf{1}_k'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}_k} \\ \frac{\mathbf{X}_0'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}_k}{\mathbf{1}_k'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}_k} \end{pmatrix} \Bigg| s_0, h_0, v_0 \sim \mathcal{N}_2\left(\begin{pmatrix} \frac{\boldsymbol{\mu}_0'\boldsymbol{\Sigma}^{-1}\mathbf{1}_k}{\mathbf{1}_k'\boldsymbol{\Sigma}^{-1}\mathbf{1}_k} \\ \frac{\mathbf{X}_0'\boldsymbol{\Sigma}^{-1}\mathbf{1}_k}{\mathbf{1}_k'\boldsymbol{\Sigma}^{-1}\mathbf{1}_k} \end{pmatrix}, \frac{(n-1)^{-1}}{\mathbf{1}_k'\boldsymbol{\Sigma}^{-1}\mathbf{1}_k}\begin{pmatrix} s_0 & h_0 \\ h_0 & v_0 \end{pmatrix}\right), \tag{16}$$

where $\mathbf{1}_k'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}_k = \hat{V}_{GMV}^{-1}$ is independent of $\left(\frac{\boldsymbol{\mu}_0'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}_k}{\mathbf{1}_k'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}_k}, \frac{\mathbf{X}_0'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}_k}{\mathbf{1}_k'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}_k}, s_0, h_0, v_0\right)$ and (see, e.g., Lemma A1 in Bodnar and Schmid (2009))

$$(n-1)\frac{\hat{V}_{GMV}}{V_{GMV}} \sim \chi^2_{n-k}. \tag{17}$$

Moreover, we get that (Theorem 3.(b) in Bodnar and Okhrin (2008))

$$(n-1)^{-1}\begin{pmatrix} s_0 & h_0 \\ h_0 & v_0 \end{pmatrix} \sim \mathcal{W}_2^{-1}\left(n-k+5, \begin{pmatrix} \boldsymbol{\mu}_0'\mathbf{R}\boldsymbol{\mu}_0 & \boldsymbol{\mu}_0'\mathbf{R}\mathbf{X}_0 \\ \mathbf{X}_0'\mathbf{R}\boldsymbol{\mu}_0 & \mathbf{X}_0'\mathbf{R}\mathbf{X}_0 \end{pmatrix}\right)$$

and, consequently,

$$(n-1)\frac{\boldsymbol{\mu}_0'\mathbf{R}\boldsymbol{\mu}_0}{s_0} \sim \chi^2_{n-k+1}, \tag{18}$$

$$\frac{h_0}{s_0}|v_0 - h_0^2/s_0 \sim \mathcal{N}\left(\frac{\mathbf{X}_0'\mathbf{R}\boldsymbol{\mu}_0}{\boldsymbol{\mu}_0'\mathbf{R}\boldsymbol{\mu}_0}, \frac{(n-1)^{-1}}{\boldsymbol{\mu}_0'\mathbf{R}\boldsymbol{\mu}_0}\left(v_0 - h_0^2/s_0\right)\right), \tag{19}$$

$$(n-1)\frac{\mathbf{X}_0'\mathbf{R}\mathbf{X}_0 - (\mathbf{X}_0'\mathbf{R}\boldsymbol{\mu}_0)^2/\boldsymbol{\mu}_0'\mathbf{R}\boldsymbol{\mu}_0}{v_0 - h_0^2/s_0} \sim \chi^2_{n-k+2} \tag{20}$$

as well as s_0 is independent of h_0/s_0 and $v_0 - h_0^2/s_0$. Let

$$\hat{s} = \hat{\boldsymbol{\mu}}'\hat{\mathbf{R}}\hat{\boldsymbol{\mu}}, \quad \hat{h} = \mathbf{X}_{n+1}'\hat{\mathbf{R}}\hat{\boldsymbol{\mu}}, \quad \hat{v} = \mathbf{X}_{n+1}'\hat{\mathbf{R}}\mathbf{X}_{n+1}.$$

Then the unconditional distributions of

$$\xi_1 = (n-1)\frac{\hat{\boldsymbol{\mu}}'\mathbf{R}\hat{\boldsymbol{\mu}}}{\hat{s}} \quad \text{and} \quad \xi_2 = (n-1)\frac{\mathbf{X}_{n+1}'\mathbf{R}\mathbf{X}_{n+1} - (\mathbf{X}_{n+1}'\mathbf{R}\hat{\boldsymbol{\mu}})^2/\hat{\boldsymbol{\mu}}'\mathbf{R}\hat{\boldsymbol{\mu}}}{\hat{v} - \hat{h}^2/\hat{s}}$$

coincide with the corresponding conditional ones as given in (18) and (20) as well as ξ_1 is independent of $\hat{h}/\hat{s}$ and ξ_2.

(a) The application of (16)-(20) leads to the stochastic representation for $\left(\hat{R}_{GMV}, \hat{R}_{GMV,n+1}\right)$ given by

$$\begin{aligned}
\hat{R}_{GMV} &\stackrel{d}{=} \frac{\hat{\boldsymbol{\mu}}'\boldsymbol{\Sigma}^{-1}\mathbf{1}_k}{\mathbf{1}_k'\boldsymbol{\Sigma}^{-1}\mathbf{1}_k} + \sqrt{\frac{(n-1)^{-1}}{\mathbf{1}_k'\boldsymbol{\Sigma}^{-1}\mathbf{1}_k}}\sqrt{s}z_1 \\
&\stackrel{d}{=} \frac{\hat{\boldsymbol{\mu}}'\boldsymbol{\Sigma}^{-1}\mathbf{1}_k}{\mathbf{1}_k'\boldsymbol{\Sigma}^{-1}\mathbf{1}_k} + \sqrt{\frac{\hat{\boldsymbol{\mu}}'\mathbf{R}\hat{\boldsymbol{\mu}}}{\mathbf{1}_k'\boldsymbol{\Sigma}^{-1}\mathbf{1}_k}}\frac{z_1}{\sqrt{\xi_1}}
\end{aligned}$$

and

$$
\begin{aligned}
\hat{R}_{GMV,n+1} &\overset{d}{=} \frac{\mathbf{X}'_{n+1}\mathbf{\Sigma}^{-1}\mathbf{1}_k}{\mathbf{1}'_k\mathbf{\Sigma}^{-1}\mathbf{1}_k} + \sqrt{\frac{(n-1)^{-1}}{\mathbf{1}'_k\mathbf{\Sigma}^{-1}\mathbf{1}_k}}\left(\frac{h}{\sqrt{s}}z_1 + \sqrt{v - h^2/s}z_2\right) \\
&\overset{d}{=} \frac{\mathbf{X}'_{n+1}\mathbf{\Sigma}^{-1}\mathbf{1}_k}{\mathbf{1}'_k\mathbf{\Sigma}^{-1}\mathbf{1}_k} + \sqrt{\frac{\hat{\boldsymbol{\mu}}'\mathbf{R}\hat{\boldsymbol{\mu}}}{\mathbf{1}'_k\mathbf{\Sigma}^{-1}\mathbf{1}_k}}\frac{\mathbf{X}'_{n+1}\mathbf{R}\hat{\boldsymbol{\mu}}}{\hat{\boldsymbol{\mu}}'\mathbf{R}\hat{\boldsymbol{\mu}}}\frac{z_1}{\sqrt{\xi_1}} \\
&+ \sqrt{\frac{\mathbf{X}'_{n+1}\mathbf{R}\mathbf{X}_{n+1} - \frac{(\mathbf{X}'_{n+1}\mathbf{R}\hat{\boldsymbol{\mu}})^2}{\hat{\boldsymbol{\mu}}'\mathbf{R}\hat{\boldsymbol{\mu}}}}{\mathbf{1}'_k\mathbf{\Sigma}^{-1}\mathbf{1}_k}}\left(\frac{z_3}{\sqrt{\xi_2}}\frac{z_1}{\sqrt{\xi_1}} + \frac{z_2}{\sqrt{\xi_2}}\right)
\end{aligned}
$$

where $z_1, z_2, z_3 \sim \mathcal{N}(0,1)$, $\xi_1 \sim \chi^2_{n-k+1}$, $\xi_2 \sim \chi^2_{n-k+2}$; $z_1, z_2, z_3, \xi_1, \xi_2$ are mutually independent.

Since

$$
\mathbf{R}\mathbf{\Sigma}\frac{\mathbf{\Sigma}^{-1}\mathbf{1}_k}{\mathbf{1}'_k\mathbf{\Sigma}^{-1}\mathbf{1}_k} = \frac{\mathbf{R}\mathbf{1}_k}{\mathbf{1}'_k\mathbf{\Sigma}^{-1}\mathbf{1}_k} = \mathbf{0},
$$

we get that (see Corollary 7.8.6.1 in Gupta and Nagar (2000))

$$
\begin{pmatrix} \frac{\hat{\boldsymbol{\mu}}'\mathbf{\Sigma}^{-1}\mathbf{1}_k}{\mathbf{1}'_k\mathbf{\Sigma}^{-1}\mathbf{1}_k} \\ \frac{\mathbf{X}'_{n+1}\mathbf{\Sigma}^{-1}\mathbf{1}_k}{\mathbf{1}'_k\mathbf{\Sigma}^{-1}\mathbf{1}_k} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \hat{\boldsymbol{\mu}}'\mathbf{R}\hat{\boldsymbol{\mu}} & \hat{\boldsymbol{\mu}}'\mathbf{R}\mathbf{X}_{n+1} \\ \mathbf{X}'_{n+1}\mathbf{R}\hat{\boldsymbol{\mu}} & \mathbf{X}'_{n+1}\mathbf{R}\mathbf{X}_{n+1} \end{pmatrix}
$$

are independently distributed with

$$
\begin{pmatrix} \frac{\hat{\boldsymbol{\mu}}'\mathbf{\Sigma}^{-1}\mathbf{1}_k}{\mathbf{1}'_k\mathbf{\Sigma}^{-1}\mathbf{1}_k} \\ \frac{\mathbf{X}'_{n+1}\mathbf{\Sigma}^{-1}\mathbf{1}_k}{\mathbf{1}'_k\mathbf{\Sigma}^{-1}\mathbf{1}_k} \end{pmatrix} \sim \mathcal{N}_2\left(R_{GMV}\mathbf{1}_2, \begin{pmatrix} V_{GMV}/n & 0 \\ 0 & V_{GMV} \end{pmatrix}\right).
$$

Moreover, using that $\hat{\boldsymbol{\mu}}$ and $\mathbf{X}_{n+1}$ are independent, we get that

$$
\begin{aligned}
&\frac{\mathbf{X}'_{n+1}\mathbf{R}\hat{\boldsymbol{\mu}}}{\hat{\boldsymbol{\mu}}'\mathbf{R}\hat{\boldsymbol{\mu}}}|\hat{\boldsymbol{\mu}} \sim \mathcal{N}\left(\frac{\boldsymbol{\mu}'\mathbf{R}\hat{\boldsymbol{\mu}}}{\hat{\boldsymbol{\mu}}'\mathbf{R}\hat{\boldsymbol{\mu}}}, \frac{1}{\hat{\boldsymbol{\mu}}'\mathbf{R}\hat{\boldsymbol{\mu}}}\right) \\
&\mathbf{X}'_{n+1}\mathbf{R}\mathbf{X}_{n+1} - \frac{(\mathbf{X}'_{n+1}\mathbf{R}\hat{\boldsymbol{\mu}})^2}{\hat{\boldsymbol{\mu}}'\mathbf{R}\hat{\boldsymbol{\mu}}}|\hat{\boldsymbol{\mu}} \sim \chi^2_{k-2;\delta^2(\hat{\boldsymbol{\mu}})} \quad \text{with} \\
&\delta^2(\hat{\boldsymbol{\mu}}) = \boldsymbol{\mu}\mathbf{R}\boldsymbol{\mu} - \frac{(\boldsymbol{\mu}'\mathbf{R}\hat{\boldsymbol{\mu}})^2}{\hat{\boldsymbol{\mu}}'\mathbf{R}\hat{\boldsymbol{\mu}}} = \frac{\boldsymbol{\mu}'\mathbf{R}\boldsymbol{\mu}}{\hat{\boldsymbol{\mu}}'\mathbf{R}\hat{\boldsymbol{\mu}}}\hat{\boldsymbol{\mu}}'\left(\mathbf{R} - \frac{\mathbf{R}\boldsymbol{\mu}\boldsymbol{\mu}'\mathbf{R}}{\boldsymbol{\mu}'\mathbf{R}\boldsymbol{\mu}}\right)\hat{\boldsymbol{\mu}},
\end{aligned}
$$

and the two quantities given $\hat{\boldsymbol{\mu}}$ are independently distributed. These results follow from Corollary 5.1.3a and Theorem 5.5.1 of Mathai and Provost (1992) since

$$
\left(\mathbf{R} - \frac{\mathbf{R}\hat{\boldsymbol{\mu}}\hat{\boldsymbol{\mu}}'\mathbf{R}}{\hat{\boldsymbol{\mu}}'\mathbf{R}\hat{\boldsymbol{\mu}}}\right)\mathbf{\Sigma}\frac{\mathbf{R}\hat{\boldsymbol{\mu}}}{\hat{\boldsymbol{\mu}}'\mathbf{R}\hat{\boldsymbol{\mu}}} = \mathbf{0}
$$

and

$$\left(\mathbf{R}-\frac{\mathbf{R}\hat{\boldsymbol{\mu}}\hat{\boldsymbol{\mu}}'\mathbf{R}}{\hat{\boldsymbol{\mu}}'\mathbf{R}\hat{\boldsymbol{\mu}}}\right)\boldsymbol{\Sigma}\left(\mathbf{R}-\frac{\mathbf{R}\hat{\boldsymbol{\mu}}\hat{\boldsymbol{\mu}}\mathbf{R}}{\hat{\boldsymbol{\mu}}'\mathbf{R}\hat{\boldsymbol{\mu}}}\right)=\mathbf{R}-\frac{\mathbf{R}\hat{\boldsymbol{\mu}}\hat{\boldsymbol{\mu}}'\mathbf{R}}{\hat{\boldsymbol{\mu}}'\mathbf{R}\hat{\boldsymbol{\mu}}}$$

with $rank\left(\left(\mathbf{R}-\frac{\mathbf{R}\hat{\boldsymbol{\mu}}\hat{\boldsymbol{\mu}}'\mathbf{R}}{\hat{\boldsymbol{\mu}}'\mathbf{R}\hat{\boldsymbol{\mu}}}\right)\boldsymbol{\Sigma}\right)=k-2$.

In using that

$$\hat{\boldsymbol{\mu}}'\mathbf{R}\hat{\boldsymbol{\mu}}=\hat{\boldsymbol{\mu}}'\left(\mathbf{R}-\frac{\mathbf{R}\boldsymbol{\mu}\boldsymbol{\mu}'\mathbf{R}}{\boldsymbol{\mu}'\mathbf{R}\boldsymbol{\mu}}\right)\hat{\boldsymbol{\mu}}+\boldsymbol{\mu}'\mathbf{R}\boldsymbol{\mu}\left(\frac{\hat{\boldsymbol{\mu}}'\mathbf{R}\boldsymbol{\mu}}{\boldsymbol{\mu}'\mathbf{R}\boldsymbol{\mu}}\right)^2$$

and applying Corollary 5.1.3a and Theorem 5.5.1 of Mathai and Provost (1992), we get that

$$\hat{\boldsymbol{\mu}}'\left(\mathbf{R}-\frac{\mathbf{R}\boldsymbol{\mu}\boldsymbol{\mu}'\mathbf{R}}{\boldsymbol{\mu}'\mathbf{R}\boldsymbol{\mu}}\right)\hat{\boldsymbol{\mu}}\ \text{ and }\ \frac{\hat{\boldsymbol{\mu}}'\mathbf{R}\boldsymbol{\mu}}{\boldsymbol{\mu}'\mathbf{R}\boldsymbol{\mu}}$$

are independent with

$$\frac{\hat{\boldsymbol{\mu}}'\mathbf{R}\boldsymbol{\mu}}{\boldsymbol{\mu}'\mathbf{R}\boldsymbol{\mu}}\sim\mathcal{N}\left(1,\frac{n^{-1}}{\boldsymbol{\mu}'\mathbf{R}\boldsymbol{\mu}}\right)$$

and

$$n\hat{\boldsymbol{\mu}}'\left(\mathbf{R}-\frac{\mathbf{R}\boldsymbol{\mu}\boldsymbol{\mu}'\mathbf{R}}{\boldsymbol{\mu}'\mathbf{R}\boldsymbol{\mu}}\right)\hat{\boldsymbol{\mu}}\sim\chi^2_{k-2}.$$

Hence, the stochastic representation for $\left(\hat{R}_{GMV},\hat{R}_{GMV,n+1}\right)$ expressed as

$$\hat{R}_{GMV}\overset{d}{=}R_{GMV}+\frac{\sqrt{V_{GMV}}}{\sqrt{n}}z_4+\sqrt{\frac{1}{n}\xi_3+\frac{1}{n}\left(\sqrt{ns}+z_5\right)^2}\sqrt{V_{GMV}}\frac{z_1}{\sqrt{\xi_1}}$$

and

$$\begin{aligned}\hat{R}_{GMV,n+1}&\overset{d}{=}R_{GMV}+\sqrt{V_{GMV}}z_6+\sqrt{V_{GMV}}\left(\frac{\sqrt{s}(\sqrt{ns}+z_5)}{\sqrt{\xi_3+(\sqrt{ns}+z_5)^2}}+z_7\right)\frac{z_1}{\sqrt{\xi_1}}\\&+\sqrt{V_{GMV}}\sqrt{\xi_4}\left(\frac{z_3}{\sqrt{\xi_2}}\frac{z_1}{\sqrt{\xi_1}}+\frac{z_2}{\sqrt{\xi_2}}\right)\end{aligned}$$

where $z_1,z_2,z_3,z_4,z_5,z_6,z_7\sim\mathcal{N}(0,1)$, $\xi_1\sim\chi^2_{n-k+1}$, $\xi_2\sim\chi^2_{n-k+2}$, $\xi_3\sim\chi^2_{k-2}$, $\xi_4|z_5,\xi_3\sim\chi^2_{k-2;\delta^2(s,\xi_3,z_5)}$ with $\delta^2(s,\xi_3,z_5)=\frac{s\xi_3}{\xi_3+\left(\sqrt{ns}+z_5\right)^2}$; $z_1,z_2,z_3,z_4,z_6,z_7,\xi_1,\xi_2,(z_5,\xi_3,\xi_4)$ are mutually independent.

(b) Let

$$a=\frac{\sqrt{n-k}}{\sqrt{n-1}}\frac{1}{\sqrt{1+\frac{n}{n-1}\hat{s}_0\sqrt{\frac{\hat{V}_{GMV}}{n}}}}.$$

Given $\hat{\boldsymbol{\mu}} = \boldsymbol{\mu}_0$ and $\mathbf{X}_{n+1} = \mathbf{X}_0$, we get

$$\left(\begin{array}{c} a \left(\frac{\mathbf{X}_0' \hat{\boldsymbol{\Sigma}}^{-1} \mathbf{1}_k}{\mathbf{1}_k' \hat{\boldsymbol{\Sigma}}^{-1} \mathbf{1}_k} - r_f \right) \\ \frac{\mathbf{X}_0' \hat{\boldsymbol{\Sigma}}^{-1} \mathbf{1}_k}{\mathbf{1}_k' \hat{\boldsymbol{\Sigma}}^{-1} \mathbf{1}_k} \end{array} \right) \Bigg| s_0, h_0, v_0$$
$$\sim \mathcal{N}_2 \left(\left(\begin{array}{c} a \left(\frac{\boldsymbol{\mu}_0' \boldsymbol{\Sigma}^{-1} \mathbf{1}_k}{\mathbf{1}_k' \boldsymbol{\Sigma}^{-1} \mathbf{1}_k} - r_f \right) \\ \frac{\mathbf{X}_0' \boldsymbol{\Sigma}^{-1} \mathbf{1}_k}{\mathbf{1}_k' \boldsymbol{\Sigma}^{-1} \mathbf{1}_k} \end{array} \right), \frac{(n-1)^{-1}}{\mathbf{1}_k' \boldsymbol{\Sigma}^{-1} \mathbf{1}_k} \left(\begin{array}{cc} a^2 s_0 & a h_0 \\ a h_0 & v_0 \end{array} \right) \right),$$

Then, using the derivation of part (a) and (17) we get a stochastic representation for $\hat{R}_{GMV,n+1}$ as in part (a) and a stochastic representation of T given by

$$\begin{aligned} T &\stackrel{d}{=} \frac{\sqrt{n-k}}{\sqrt{n-1}} \frac{1}{\sqrt{1 + \frac{\xi_3 + \left(\sqrt{ns} + z_5\right)^2}{\xi_1}} \sqrt{\xi_5 \frac{V_{GMV}}{(n-1)n}}} \\ &\times \left(R_{GMV} - r_f + \frac{\sqrt{V_{GMV}}}{\sqrt{n}} z_4 + \sqrt{\frac{1}{n}\xi_3 + \frac{1}{n}\left(\sqrt{ns} + z_5\right)^2} \sqrt{V_{GMV}} \frac{z_1}{\sqrt{\xi_1}} \right) \\ &= \frac{\sqrt{n-k}}{\sqrt{\xi_5}} \frac{1}{\sqrt{1 + \frac{\xi_3 + \left(\sqrt{ns} + z_5\right)^2}{\xi_1}}} \left(\sqrt{n} \frac{R_{GMV} - r_f}{\sqrt{V_{GMV}}} + z_4 + \sqrt{\frac{\xi_3 + \left(\sqrt{ns} + z_5\right)^2}{\xi_1}} z_1 \right), \end{aligned}$$

where $\xi_5 \sim \chi^2_{n-k}$ independent of $z_1, z_2, z_3, z_4, z_6, z_7, \xi_1, \xi_2, (z_5, \xi_3, \xi_4)$.

□

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