Mathematical Statistics
Stockholm University

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Research Report 2017:3

ISSN 1650-0377

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Mathematical Statistics
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Research Report **2017:3**,
http://www.math.su.se

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April 2017

Abstract

A multi-type preferential attachment model is introduced, and studied using general multi-type branching processes. For the p-type case we derive a framework for studying the model where a type i vertex generates new type j vertices with rate $w_{ij}(n_1, n_2, \dots, n_p)$ where n_k is the number of type k vertices previously generated by the type i vertex, and w_{ij} is a function from $\mathbb{N}^p$ to $\mathbb{R}$. The framework is then used to derive results for models with more specific attachment rates.

In the case with linear preferential attachment—where type i vertices generate new type j vertices with rate $w_{ij}(n_1, n_2, \dots, n_p) = \gamma_{ij}(n_1 + n_2 + \dots + n_p) + \beta_{ij}$, where γ_{ij} and β_{ij} are positive constants—we show that under mild regularity conditions on the parameters $\{\gamma_{ij}\}, \{\beta_{ij}\}$ the asymptotic degree distribution of a vertex is a power law distribution. The asymptotic composition of the vertex population is also studied.

Keywords: Multi-type preferential attachment; multi-type general branching process; power law degree distribution; asymptotic composition.

1 Introduction

The preferential attachment model is a well-studied model of random network growth where, traditionally, new vertices arrive according to some process (often at integer times $\{1, 2, \ldots\}$) and upon arrival attach randomly to an existing vertex with probability proportional to that vertex's degree. This may also be interpreted as the existing vertex giving birth to a new vertex. Preferential attachment is particularly interesting for modeling empirical networks since it gives rise to power law degree distributions—something that is often observed in empirical networks, see [2]. The preferential attachment model was first introduced in [2] and the degree sequence was rigorously analyzed in [4].

In this paper we extend the preferential attachment model by allowing vertices to be of one of p different types, where vertices of different types give birth to new vertices at different rates. One can think of applications of this in male-female network growth; networks evolving according to political view, or level of education; Internet networks with websites of different types; or more generally, networks exhibiting some level of homophily or heterophily.

Traditionally, preferential attachment is studied in discrete time, but we will develop the framework for studying the model in continuous time. The reason for this is that the model can then be analyzed as a general branching process in continuous time and existing results on such processes can then be applied.

In studying the model we shall mainly be interested in the asymptotic degree distribution of a vertex. Usually, in one-type preferential attachment this is analyzed by first deriving a recursion for the *expected* fraction of vertices having degree k and showing that this converges. Stronger convergence results follow with an additional martingale argument, see [4] for a rigorous treatment. The drawback of this method is that it depends heavily on the linear structure of the attachment dynamic (new vertices attach proportional to degree, and not to an arbitrary function of the degree), and it tends to be difficult to apply the method to more general preferential attachment models, e.g. multi-type preferential attachment.

An alternative way of studying preferential attachment models is to embed the process in continuous time and interpret it as a general branching process. This was first done by Rudas et al. [14] and later extended by Deijfen [6] who also allowed for vertex death. In this article we build on the methods used in [14], [6] and extend them to the multi-type case. Deijfen and Fitzner [7] have studied two-type preferential attachment heuristically, using different methods. In the context of multi-type graphs we also mention the seminal paper [3], which treats a very general model that includes an instance resembling preferential attachment.

1.1 The Model

We begin by defining the multi-type degree-based attachment model, of which the multi-type preferential attachment model is a special case. For notational simplicity we will let $\vec{n} = (n_1, \ldots, n_p) \in \mathbb{N}^p$ throughout. In the multi-type degree-based attachment model a vertex may be of one of p different types. The vertex population evolves in continuous time where new vertices are born, but may not die. A type i vertex currently having n_k type k children, $k = 1, 2, \ldots, p$ give birth to a new type j child at rate $w_{ij}(\vec{n})$—i.e. given the number of children $\vec{n}$ of respective types of a type i vertex the time until the next birth is exponentially distributed with rate $w_{i1}(\vec{n})+\cdots+w_{ip}(\vec{n})$ and the birth is of type j with probability $\frac{w_{ij}(\vec{n})}{w_{i1}(\vec{n})+\cdots+w_{ip}(\vec{n})}$. This process is then turned into a graph by letting the relation mother-child be represented by a directed edge from child to mother. The number of children of a vertex is then the vertex's in-degree, or total degree minus 1. The graph starts at time $t = 0$ with *one* vertex of any type present.

For a formal definition we let $\{\xi_{ij}(\cdot)\}$ $i, j = 1, \ldots, p$ be a point process on $\mathbb{R}_+$, and $\xi_{ij}(t) = \xi_{ij}([0, t])$ the counting process associated with it, see [13] for more details on point processes. The process $\{\xi_{ij}(\cdot)\}$ is to be interpreted as the number of type j vertices that a type i vertex gives birth to in some time interval. Now, define the multi-type degree-based attachment model as the stochastic process evolving according to the following dynamics (where births are represented as directed edges),

(i) The weight functions satisfy $w_{ij} : \mathbb{N}^p \to (0, \infty)$, $i, j = 1, 2, \ldots, p$.

(ii) $\{\xi_{ij}(t),\ t \geq 0\}$ is a counting process with $\xi_{ij}(0) = 0$, $i, j = 1, 2 \ldots, p$.

(iii) For $i = 1, 2, \ldots, p$ the process $\vec{\xi_i}(t) = (\xi_{i1}(t), \xi_{i2}(t), \ldots, \xi_{ip}(t))$ is a continuous-time Markov chain on $\mathbb{N}^p$ with transition rates $w_i(\vec{n}) = w_{i1}(\vec{n}) + \cdots + w_{ip}(\vec{n})$ and transition probabilities
$p(\vec{n} \to (n_1, \ldots, n_j + 1, \ldots, n_p)) = \frac{w_{ij}(\vec{n})}{w_{i1}(\vec{n})+\cdots+w_{ip}(\vec{n})}$.

This means that the vertex population evolves according to a multi-type general branching process, see [12, Ch. 6] or [8], and the graph can therefore be analyzed within that framework. When the weight functions are increasing in the type specific degrees we shall instead refer to the model as the multi-type preferential attachment model, and reserve the pronoun degree-based for the case when the weight functions are more arbitrary. The preferential attachment model is hence a special case of the degree-based attachment model.

By looking at the graph at the times when new vertices arrive we can connect it to a discrete time degree-based attachment model. Let $\mathcal{G}(t)$ denote the σ-field generated by the graph up until time t; and σ_n the birth time of the n:th vertex. Also let $w^v(t)$ be the weight of the vertex v at time t, i.e. $w^v(t) = \sum_{j=1}^{p} w_{ij}(\vec{\xi_i}(t))$ if v is of type i. Similarly, let $w_k^v(t) = w_{ik}(\vec{\xi_i}(t))$ if v is of type i. Then the probability that the $(n+1)$:th new vertex arriving is of type i given the current state of the graph is given by

$$p(i|\mathcal{G}(\sigma_n)) = \frac{\sum_{v=0}^{n} w_i^v(\sigma_n)}{\sum_{v=0}^{n} w^v(\sigma_n)},$$

where we have numbered vertices in the order they arrived. Given the current state of the graph the $(n+1)$:th vertex v attaches to an old vertex u with probability

$$p(v \to u|\mathcal{G}(\sigma_n)) = \frac{w^u(\sigma_n)}{\sum_{v=0}^{n} w^v(\sigma_n)}.$$

For the one-type case $p = 1$ with weight function $w(k) = k + 1$ the graph model that arises by inspecting the continuous time model at the birth times coincides with the standard discrete time preferential attachment model. This connection was noted already in [14].

Remark. Using the connection between the continuous time model and the discrete time model we see that limit results for the continuous time case are valid for the discrete time case as long as $\sigma_n \to \infty$ when $n \to \infty$, i.e. the continuous time model does not explode in finite time.

1.2 Notation

Finally, we set some notation. Again, $\vec{n} = (n_1, \ldots, n_p) \in \mathbb{N}^p$ and $\vec{\xi_i}(t) = (\xi_{i1}(t), \xi_{i2}(t), \ldots, \xi_{ip}(t)) \in \mathbb{N}^p$; the letters i and j will always refer to a vertex's type; the indexed letters n_j and m_j will always refer to the number of type j children of a vertex or, equivalently, its in-degree from type j vertices. We will use the star notation * to denote the Laplace transform of a function or a matrix. Throughout we let $Z(t)$ denote the number of vertices in the graph at time t; $Z_i(t)$ the number of type i vertices in the graph at time t; $Z_i^{\vec{n}}(t)$ the number of type i vertices in the graph at time t with n_j type j children, $j = 1, \ldots, p$; $Z_i^k(t)$ the number of type i vertices in the graph at time t with k children in total. Also, let $f \sim g$ denote that $\lim\limits_{t\to\infty} \frac{f(t)}{g(t)} = 1$.

1.3 Results for Degree-based Preferential Attachment

The main results concern asymptotic composition of the population and degree distribution: we shall formulate conditions on $\{w_{ij}(\vec{n})\}$ so that the ratios $p_i(t) = \frac{Z_i(t)}{Z(t)}$, $p_i(\vec{n}, t) = \frac{Z_i^{\vec{n}}(t)}{Z(t)}$, and $p_i(k, t) = \frac{Z_i^k(t)}{Z(t)}$ converges almost surely as $t \to \infty$ and identify the limits.

For each Borel measurable set A in $\mathbb{R}_+$ let $\mu_{ij}(A) = \mathbb{E}(\xi_{ij}(A))$—i.e. the expected number of type j vertices born by a type j vertex in the time set A. It follows that $\mu_{ij}(A)$ is a measure (see e.g. [13, Lemma 1.1.1]) and we define for each $\theta > 0$ the new measure $\mu_{ij}^{\theta}(A) = \mu_{ij}(A, \theta)$ on $\mathcal{B}(\mathbb{R}_+)$ through the distribution function

$$\mu_{ij}(t, \theta) = \int_0^t e^{-\theta s} \mu_{ij}(ds), \qquad \theta > 0.$$

Furthermore, let

$$\mu(t, \theta) = \begin{pmatrix} \mu_{11}(t, \theta) & \dots & \mu_{1p}(t, \theta) \\ \vdots & \ddots & \vdots \\ \mu_{p1}(t, \theta) & \dots & \mu_{pp}(t, \theta) \end{pmatrix}.$$

Define $\mu_{ij}^*(\theta) = \mu_{ij}(\infty, \theta)$ and $\mu^*(\theta) = \mu(\infty, \theta)$, and let $\rho(\mu^*(\theta))$ denote the largest eigenvalue of $\mu^*(\theta)$ (also known as the Perron-Frobenius root). Throughout we shall assume the existence of a *Malthusian* parameter $\alpha \in (0, \infty)$ such that $\rho(\mu^*(\alpha)) = 1$ In fact we shall assume that

$$\exists \theta_0 \in (0, \infty): \qquad \rho(\mu^*(\theta_0)) \in (1, \infty). \tag{A1}$$

The Perron root $\rho(\mu^*(\theta))$, is continuous and strictly decreasing in θ, see [11, Lemma 9.1], and as the Perron root is assumed to be finite the same is true for the entries of $\mu^*(\theta)$, $\theta \geq \theta_0$. Furthermore, it is a standard result (see e.g. [10, Ch. 8]) that

$$\min_i \sum_{j=1}^{p} \mu_{ij}^*(\theta) \leq \rho(\mu^*(\theta)) \leq \max_i \sum_{j=1}^{p} \mu_{ij}^*(\theta).$$

Since, by monotone convergence, $\lim_{\theta \to \infty} \mu_{ij}^*(\theta) = 0$ the row sums of $\mu^*(\theta)$ also converges to 0 as $\theta \to \infty$. Hence, (A1) implies that $\lim_{\theta \to \infty} \rho(\mu^*(\theta)) = 0$, and $\rho(\mu^*(\theta_0)) > 1$, as well as that there exists an $\alpha > \theta_0 > 0$ such that the Perron root equals 1. There are two ways for a branching process to explode in finite time: either $\sum_{i=1}^{p} \mu_{ij}(0) > 1$ or $\mu_{ij}(t) = \infty$ for some t. It follows from the model definition that $\sum_{i=1}^{p} \mu_{ij}(0) = 0$, and (A1) implies that

$\mu_{ij}(t) < e^{\alpha t}\mu^*_{ij}(\alpha) < \infty$ for all $t \geq 0$, implying that the multi-type degree based attachment model satisfying (A1) does not explode in finite time.

The Malthusian parameter α is, even for simple models, given by a very complicated expression. However, our assumptions ensure that it always exists and can be calculated numerically. For $\mu^*(\alpha)$ we denote the corresponding *left* eigenvector by $u = (u_1, \ldots, u_p)$ and the *right* eigenvector by $v = (v_1, \ldots, v_p)^t$. By the Perron-Frobenius theorem, both these exists, are positive, and can be normed so that

$$u_1v_1 + u_2v_2 + \cdots + u_pv_p = v_1 + v_2 + \cdots + v_p = 1.$$

We will throughout assume that the eigenvectors are normed in this way.

We can now state our main results. The first one concerns the asymptotic composition of the vertex population.

Theorem 1. *For the multi-type degree-based attachment model starting with one vertex of any type, with weight functions $\{w_{ij}(\vec{n})\}$ satisfying condition* (A1)*, the asymptotic proportion of type i vertices satisfies*

$$p_i(t) = \frac{Z_i(t)}{Z(t)} \to \frac{u_i}{u_1 + u_2 + \cdots + u_p} \text{ almost surely as } t \to \infty.$$

The next theorem asserts that the empirical degree distribution converges almost surely and identifies the limit.

Theorem 2. *For the multi-type degree-based attachment model starting with one vertex of any type, with weight functions $\{w_{ij}(\vec{n})\}$ satisfying condition* (A1)*, we have that*

$$\frac{Z_i^{\vec{n}}(t)}{Z(t)} \to \alpha\frac{u_i}{u_1 + \cdots + u_p} I_i(\vec{n}) \text{ almost surely as } t \to \infty$$

where $I_i(\vec{n})$ satisfies the recursion

(i) $I_i(\vec{0}) = \frac{1}{\alpha + w_{i1}(\vec{0}) + \cdots + w_{ip}(\vec{0})}$;

(ii) if $|\vec{n}| > 0$ *then* $I_i(\vec{n}) = \sum_{j=1}^{p} \frac{w_{ij}(n_1,\ldots,n_j-1,\ldots,n_p) I_i(n_1,\ldots,n_j-1,\ldots,n_p)}{\alpha + w_{i1}(\vec{n}) + \cdots + w_{ip}(\vec{n})}$

with $w_{ij}(\vec{n}) = I_i(\vec{n}) = 0$ *if* $\min(n_1, \ldots, n_p) < 0$, $j = 1, \ldots, p$.

1.4 Results on Multi-type Linear Preferential Attachment

A particularly interesting choice of weight functions is $w_{ij}(\vec{n}) = \gamma_{ij}(n_1 + \cdots + n_p) + \beta_{ij}$ where γ_{ij} and β_{ij} are positive constants. This is an extension of one-type linear preferential attachment. We call this model multi-type linear preferential attachment based on total in-degree. As with its one-type counterpart the linear multi-type model exhibits a power law degree distribution as soon as $\gamma_{i1} + \cdots + \gamma_{ip} > 0$.

Theorem 3. *For the multi-type preferential attachment model starting with one vertex of any type, with weight functions $w_{ij}(\vec{n}) = \gamma_{ij}(n_1+\cdots+n_p)+\beta_{ij}$ where $\gamma_{ij} \geq 0$ and $\beta_{ij} > 0$, $p_i(k) = \lim_{t\to\infty} \frac{Z_i^k(t)}{Z(t)}$ exists almost surely. Furthermore,*

$$p_i(k) \sim \begin{cases} C_1 \cdot k^{-(1+\frac{\alpha}{\gamma_{i1}+\cdots+\gamma_{ip}})} & \text{if } \gamma_{i1} + \cdots + \gamma_{ip} > 0, \\ C_2 \cdot e^{-k\log(1+\frac{\alpha}{\beta_{i1}+\cdots+\beta_{ip}})} & \text{if } \gamma_{i1} + \cdots + \gamma_{ip} = 0. \end{cases}$$

By summing over all type i vertices it is easy to see that the total asymptotic proportion of vertices with degree k also follows a power law distribution, i.e.

$$p(k) = \lim_{t\to\infty} \frac{Z^k(t)}{Z(t)} = \sum_{i=1}^{p} \lim_{t\to\infty} \frac{Z_i^k(t)}{Z(t)} \sim C \cdot k^{-(1+\frac{\alpha}{\max_i\{\gamma_{i1}+\cdots+\gamma_{ip}\}})}.$$

The rest of the paper is organized as follows. In the next section we introduce the theory for multi-type branching processes needed to prove Theorem 1, 2, and 3. In Section 3 we prove Theorem 1 and 2, and in Section 4 we prove Theorem 3. Finally, in Section 5, we investigate the results numerically.

2 General Multi-type Branching Processes

General branching processes has been extensively studied, and it is not our intention to summarize the results here, for this see e.g. the book by Jagers [9] for the single-type case and [8, 12] for multi-type generalizations. We shall, however, explain some of the concepts and results needed for proving Theorems 1, 2, and 3. When defining general multi-type branching processes we will follow the terminology of [12], and in applying the theory we will mainly use the results of [8]. The main result needed from [8] is stated below as Theorem 4.

A p-type general branching process is a process where individuals can be of one of p different types, and i-type individuals live for a random time

$\lambda_i \in [0, \infty]$ during which they give birth to j-type individuals according to the points of a point process ξ_{ij} defined on $\mathbb{R}_+$. Individuals live and reproduce independently of each other, but there is no restriction on the dependence between an individual's life time and reproduction process. Individuals of the same type have the same reproduction and life-time law.

We denote individuals by x and their type by $\tau(x) \in \{1, 2, \ldots, p\}$. If $x = (0, \tau_0; i_1, \tau_1; \ldots; i_n, \tau_n)$ then x is the i_n:th child of type τ_n of $\ldots$ of the i_1:th child of type τ_1 of the ancestor 0, which is of type τ_0. The space of possible individuals is denoted $\mathcal{J}$ and is defined by

$$\mathcal{J} = \bigcup_{k=0}^{\infty} \mathcal{J}_k$$
$$\mathcal{J}_0 = \{(0,1), (0,2) \ldots, (0,p)\}$$
$$\mathcal{J}_k = \{(0, \tau_0; i_1, \tau_1, \ldots, i_k, \tau_k);\ i_j \in \{1, 2, \ldots\}, \tau_j \in \{1, 2, \ldots, p\}, j \in \{0, 1, \ldots\}\}, \qquad k > 0.$$

To each individual x we assume there is a probability space $(\Omega_x, \mathbb{B}_x, \mathbb{P}_x)$ associated, on which x's life-length λ_x, a characteristic ϕ_x (defined below and more stringent in [8]), and x's reproduction $\xi_x = (\xi_x^1, \ldots, \xi_x^p)$ are defined. A characteristic is a product-measurable, separable (random) process $\phi : \Omega \times \mathbb{R} \to \mathbb{R}$ with $\phi(\omega, t) = 0$ if $t < 0$: Let $\phi_x(t) = \phi(\omega_{\downarrow x}, t)$ where $\omega_{\downarrow x}$ is the outcome of the branching process starting with individual x as ancestor. Hence, $\phi_x(t) = \phi(\omega_{\downarrow x}, t)$ is the score given to the individual x of age t. Note that $\phi_x(t)$ is allowed to depend on x and its whole progeny, a fact which is a major strength of random characteristics. However, we shall only use characteristics that depends only on the life history of x, not its entire progeny.

We can now define the p-type process with ancestor x_0 on the probability space

$$(\Omega, \mathbb{B}, \mathbb{P}) = \prod_{x \in \mathcal{J}} (\Omega_x, \mathbb{B}_x, \mathbb{P}_x)$$

through the birth times $\{\sigma_x\}$ defined by induction

$$\sigma_{x_0} = 0, \text{ and if } x = (x'; j_k, \tau_k)$$
$$\sigma_x = \sigma_{x'} + \inf\{t \geq 0,\ \xi_{x'}^{j_k}([0,t]) \geq j_k\}.$$

We note that an individual x who is never born will have $\sigma_x = \infty$. Also note that $\mathbb{P}_x = \mathbb{P}_y$ if x and y are of the same type. Now, let $Z^\phi(t)$ denote the total score of the population at time t, that is,

$$Z^\phi(t) = \sum_{x \in \mathcal{J}} \phi_x(t - \sigma_x).$$

We call $\{Z^{\phi}(t),\ t \geq 0\}$ the general multi-type ϕ-counted branching process. When necessary we use the notation ${}_iZ^{\phi}(t)$ to emphasize that the process starts with one type i individual. Different choices of ϕ give rise to different processes, e.g. if $\phi_x(t) = 1, t \geq 0$ then $Z^{\phi}(t)$ represents the number of individuals that have been born up to time t; and if $\phi_x(t) = 1\{0 \leq t \leq \lambda_x\}$ then $Z^{\phi}(t)$ represents the number of individuals alive at time t.

Remark. Note that the process always starts with a single individual x_0 of type τ_0 and that we omit this in the notation to simplify expressions.

For all Borel measurable sets A in $\mathbb{R}_+$ let $\mu_{ij}(A) = \mathbb{E}(\xi_x^j(A))$ if x is of type i. It follows that μ_{ij} is a measure, see [13, Lemma 1.1.1]. For each $\theta > 0$, define the new measure $\mu_{ij}^{\theta}(A) = \mu_{ij}(A, \theta)$, $A \in \mathcal{B}(\mathbb{R}_+)$, through the distribution function

$$\mu_{ij}(t, \theta) = \int_0^t e^{-\theta s} \mu_{ij}(ds), \qquad \theta > 0.$$

Also, define

$$M(\theta) = \mu^*(\infty, \theta) = [\mu_{ij}(\infty, \theta)]_{i,j}.$$

Following [8] we shall assume throughout this section that:

(C1) For $i, j = 1, 2, \ldots, p$, the measure μ_{ij} is non-lattice, i.e. not concentrated on any set $\{b + \lambda \cdot \mathbb{Z},\ \lambda \in \mathbb{R}\}$

(C2) Either $M(0)$ has at least one infinite entry, or only finite entries and Perron root $\rho > 1$. Also $M(0)^n$ is assumed to have all positive entries (possibly infinite) for some $n \geq 1$.

(C3) There exists an $\alpha > 0$ such that $M(\alpha)$ has only finite entries and Perron root $\rho = 1$, with corresponding left and right positive eigenvalues u and v normed such that $uv^T = \mathbf{1}v^t = 1$.

(C4) For $i, j = 1, 2, \ldots, p$ we have that $\int_0^{\infty} ue^{-\alpha u} \mu_{ij}(du) < \infty$.

The following assumption will only be in force when explicitly stated:

(C5) There is some $\theta \in (0, \alpha)$ such that $M(\theta)$ has finite entries only.

Remark. In the proof of Theorem 1 we shall see that, for multi-type degree-based attachment, condition (A1) implies conditions (C1)-(C5). Condition (C3) implies that the process does not explode in finite time, also know as regularity. For regularity to hold, we need $\mu_{ij}(t) < \infty$ and $\sum_{i=1}^p \mu_{ij}(0) \leq 1$. That $\mu_{ij}(t) < \infty$ follows by the same argument as for (A1) (see page 4), and $\sum_{i=1}^p \mu_{ij}(0) \leq 1$ follows from that $M(\theta) \to [\mu_{ij}(0)]_{i,j}$ as $\theta \to \infty$ and since the Perron root is decreasing we get $\rho([\mu_{ij}(0)]_{i,j}) < 1$ and regularity follows from [11, p. 148].

Most limit results on general branching processes are first proved with methods from renewal theory, often by solving a renewal type equation. This tends to yield convergence results in a weak sense, e.g. convergence of expectation. For stronger convergence results (e.g. almost sure convergence), a martingale argument is needed in addition to results from renewal theory. Unfortunately, the proofs of the stronger results are more complicated. In Theorem 4, we state the main result from branching process theory needed in order to prove Theorem 1, 2, and 3. Intuition on the theorem and how the Malthusian parameter, the Perron root and the corresponding eigenvectors determine the growth of the branching process can be found in the Appendix.

We shall mainly be interested in results regarding the ratio of the process counted in different ways, i.e. $\lim_{t\to\infty} \frac{Z^{\phi}(t)}{Z^{\psi}(t)}$. However, one needs to put some restrictions on the random characteristics. Let ψ be a random characteristic not 0 a.e. and with paths in the Skorohod space $D(\mathbb{R})$ of right-continuous functions with finite left limits. Also assume that there exists a $\theta < \alpha$ such that, for $i = 1, 2, \ldots, p$,

$$\mathbb{E}(\sup_{t\geq 0} e^{-\theta t}\phi_i(t)) < \infty. \tag{C6}$$

Remark. In the degree-based attachment setting we will work with bounded random characteristics and hence (C6) is trivially satisfied.

Finally, we can quote the result we need.

Theorem 4. *[8, Theorem 2.7] Assume that $\{Z(t)\}$ is a branching process with intensity measures $\{\mu_{ij}\}$ satisfying conditions (C1)-(C5). Furthermore assume that ϕ and ψ are both random characteristics satisfying condition (C6). Then on the event that $\{Z(t) \to \infty\}$*

$$\frac{Z^{\phi}(t)}{Z^{\psi}(t)} \to \frac{\sum_{j=1}^{p} u_j \int_0^{\infty} \mathbb{E}(e^{-\alpha s}\phi_j(s))ds}{\sum_{j=1}^{p} u_j \int_0^{\infty} \mathbb{E}(e^{-\alpha s}\psi_j(s))ds} \text{ a.s. as } t \to \infty.$$

3 General Degree-Based Attachment

In this section we will provide a general framework for deriving asymptotic ratio results on the multi-type degree-based attachment model as defined in Section 1.1. It is clear from the model definition that the vertex population of the multi-type degree-based attachment model evolves as a general multi-type branching process. Recall that, in addition to being a non-lattice process (which follows from model definition), the model is assumed to satisfy (A1) throughout.

Clearly of much importance is the matrix

$$\mu^*(\theta) = \begin{pmatrix} \mu^*_{11}(\theta) & \dots & \mu^*_{1p}(\theta) \\ \vdots & \ddots & \vdots \\ \mu^*_{p1}(\theta) & \dots & \mu^*_{pp}(\theta) \end{pmatrix}.$$

as it is part of the condition (A1). Hence, we need a way of calculating the integrals

$$\mu^*_{ij}(\theta) = \int_0^\infty e^{-\theta t}\mu_{ij}(dt).$$

One way of doing this is to calculate the Radon-Nikodym derivate of μ_{ij} with respect to the Lebesgue measure. An application of the fundamental theorem of calculus gives us the next useful result.

Proposition 1. *The intensity measure μ_{ij} has a Radon-Nikodym derivative $h_{ij}(t)$ with respect to the Lebesgue measure (dt) on $\mathbb{R}_+$ satisfying*

$$h_{ij}(t) = \mathbb{E}(w_{ij}(\vec{\xi_i}(t))), \qquad i, j = 1, 2, \dots, p.$$

Proof. Let $\vec{\xi_i}(t) = (\xi_{i1}(t), \dots, \xi_{ip}(t))$. It follows from the model definition that

$$\begin{aligned} &\mathbb{P}(\xi_{ij}(t+dt) - \xi_{ij}(t) = 1 | \vec{\xi_i}(t) = \vec{n}) = w_{ij}(\vec{n})dt + o(dt) \\ &\mathbb{P}(\xi_{ij}(t+dt) - \xi_{ij}(t) > 1 | \vec{\xi_i}(t) = \vec{n}) = o(dt). \end{aligned}$$

Condition (A1) implies that the process does not explode in finite time, since $\mu^*_{ij}(\theta)$ is finite implying that the same must hold true for $\mu_{ij}(t)$. We get

$$\begin{aligned} \mu_{ij}(dt) = \mathbb{E}\left(\xi_{ij}([t, t+dt])\right) &= \sum_{\vec{n}\in\mathbb{N}^p} \mathbb{E}\left(\xi_{ij}([t, t+dt]) | \vec{\xi_i}(t) = \vec{n}\right) \cdot \mathbb{P}(\vec{\xi_i}(t) = \vec{n}) \\ &= \sum_{\vec{n}\in\mathbb{N}^p} w_{ij}(\vec{n})dt \cdot \mathbb{P}(\vec{\xi_i}(t) = \vec{n}) + K \cdot o(dt) \\ &= \sum_{\vec{n}\in\mathbb{N}^p} w_{ij}(\vec{n}) \cdot \mathbb{P}(\vec{\xi_i}(t) = \vec{n})dt + o(dt) = \mathbb{E}(w_{ij}(\vec{\xi_i}(t))dt + o(dt). \end{aligned} \tag{1}$$

As every null set can be covered by a countable union of open intervals with arbitrary small total length it follows that μ_{ij} is absolutely continuous with respect to the Lebesgue measure. By the Radon-Nikodym theorem there exists a measurable function $h_{ij}(t) : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\mu_{ij}(A) = \int_A h_{ij}(t)dt, \qquad A \in \mathcal{B}(\mathbb{R}_+).$$

Taking $A = [0, t]$ and differentiating we see that $h_{ij}(t) = \mu_{ij}(dt)$. Dividing with dt in (1) and taking $dt \to 0$ shows that $h_{ij}(t) = \mathbb{E}(w_{ij}(\vec{\xi_i}(t)))$. $\square$

Using Proposition 1 we can calculate $\mu_{ij}(t,\alpha)$ through

$$\mu_{ij}(t,\alpha) = \int_0^t e^{-\alpha s} h_{ij}(s)ds = \int_0^t e^{-\alpha s}\mathbb{E}(w_{ij}(\vec{\xi}_i(s)))ds.$$

3.1 A Useful Integral

In what follows we shall see that the integral

$$I_i(\vec{n},\theta) = \int_0^\infty e^{-\theta s}\mathbb{P}(\vec{\xi}_i(t) = \vec{n})ds, \qquad \vec{n} \in \mathbb{N}^p$$

is very useful in deriving Theorem 1, 2, and 3. This is because many quantities of interest can be written as a sum of $I_i(\vec{n},\theta)$ over some set. For instance,

$$\mu^*_{ij}(\theta) = \int_0^\infty e^{-\theta s} h_{ij}(s)ds = \sum_{\vec{n}\in\mathbb{N}^p} w_{ij}(\vec{n})I_i(\vec{n},\theta).$$

We shall therefore spend some time investigating $I_i(\vec{n}) \equiv I_i(\vec{n},\alpha)$, and deriving a recursion for it.

Proposition 2. *Let $\vec{\xi}_i(t) = (\xi_{i1}(t),\dots,\xi_{ip}(t))$ where $\xi_{ij}(t)$ are the counting processes of a multi-type degree-based attachment model. For $\vec{n} = (n_1,\dots,n_p) \in \mathbb{N}^p$ define*

$$I_i(\vec{n}) = \int_0^\infty e^{-\alpha s}\mathbb{P}(\vec{\xi}_i(t) = \vec{n})ds.$$

Then

(i) $I_i(\vec{0}) = \frac{1}{\alpha + w_{i1}(\vec{0}) + \dots + w_{ip}(\vec{0})}$;

(ii) if $|\vec{n}| > 0$ *then* $I_i(\vec{n}) = \sum_{j=1}^p \frac{w_{ij}(n_1,\dots,n_j-1,\dots,n_p)I_i(n_1,\dots,n_j-1,\dots,n_p)}{\alpha + w_{i1}(\vec{n}) + \dots + w_{ip}(\vec{n})}$.

where $w_{i1}(\vec{n}) = w_{ip}(\vec{n}) = I_i(\vec{n}) = 0$ *if* $\min(n_1,\dots,n_p) < 0$, $i,j = 1,2,\dots,p$.

Proof. By definition, $\vec{\xi}_i(t)$ is a Markov process on $\mathbb{N}^p$ and, by condition (A1), it does not explode in finite time. Hence, it satisfies the Kolmogorov forward equations.

Assuming that that $n_1,\dots,n_p > 1$, we get

$$\frac{d}{dt}\mathbb{P}(\vec{\xi}_i(t) = \vec{n}) = \sum_{j=1}^p w_{ij}(n_1,\dots,n_j{-}1,\dots,n_p)\mathbb{P}(\vec{\xi}_i(t) = (n_1,\dots,n_j{-}1,\dots,n_p)) \\ - (w_{i1}(\vec{n}) + \dots + w_{ip}(\vec{n}))\mathbb{P}(\vec{\xi}_i(t) = \vec{n}).$$

Using this together with integration by parts yields

$$\begin{aligned}
I_i(\vec{n}) &= \int_0^\infty e^{-\alpha s}\mathbb{P}(\vec{\xi}_i(s) = \vec{n})ds = \frac{1}{\alpha}\int_0^\infty e^{-\alpha s}\frac{d}{ds}\mathbb{P}(\vec{\xi}_i(s) = \vec{n})ds \\
&= \frac{1}{\alpha}\sum_{j=1}^{p} w_{ij}(n_1, \ldots, n_j - 1, \ldots, n_p) I_i(n_1, \ldots, n_j - 1, \ldots, n_p) \\
&\qquad - \frac{1}{\alpha}(w_{i1}(\vec{n}) + \cdots + w_{ip}(\vec{n})) I_i(\vec{n}).
\end{aligned}$$

Solving for $I_i(\vec{n})$ we get

$$I_i(\vec{n}) = \sum_{j=1}^{p} \frac{w_{ij}(n_1, \ldots, n_j - 1, \ldots, n_p) I_i(n_1, \ldots, n_j - 1, \ldots, n_p)}{\alpha + w_{i1}(\vec{n}) + \cdots + w_{ip}(\vec{n})}.$$

Special cases for $|\vec{n}| \leq 1$ follows by the same method. □

We can now prove Theorem 1 and 2.

Proof of Theorem 1. Let $\phi_x(t) = 1\{\tau(x) = i\}$ be the random characteristic assigning type i vertices score 1 and type $j \neq i$ vertices score 0. The branching process $Z_i(t) = Z^{\phi}(t)$ starting with 1 vertex of *any type* represents the number of type i vertices at time t.

Let $\psi_x(t) \equiv 1$ and put $Z(t) = Z^{\psi}(t)$—this is the original branching process counting the number of vertices alive at time t.

We want to apply Theorem 4 to the ratio $\frac{Z_i(t)}{Z(t)}$ and need to check that conditions (C1)-(C6) are satisfied. The random characteristics trivially satisfy the condition (C6) of Theorem 4 and, since $\mu_{ij}(t)$ are non-lattice measures by design, also condition (C1) is fulfilled.

It remains to prove that condition (A1) implies conditions (C2)-(C5). By (A1), the Perron root exists for all $\lambda \geq \theta_0$. Since $\rho(\mu^*(\theta_0)) > 1$ for some $\theta_0 > 0$ and as $\rho(\mu^*(\theta))$ is a decreasing function, $\rho(\mu^*(0))$ is larger than 1 (or an entry is infinite) and condition (C2) is satisfied. We have previously seen (in Section 1.3) that conditions (C3) and (C5) follows from (A1). Left to show is that (C4) is satisfied, i.e. that

$$\int_0^\infty u e^{-\alpha u} \mu_{ij}(du) < \infty.$$

It follows from (A1) that $\alpha > \theta_0$ and $\int_0^\infty e^{-\theta_0 u}\mu_{ij}(du) < \infty$. For large enough u, we have that $ue^{-\alpha u} < e^{-\theta_0 u}$ and therefore that

$$\int_0^\infty u e^{-\alpha u}\mu_{ij}(du) < \int_0^{u_0} u e^{-\alpha u}\mu_{ij}(du) + \int_{u_0}^\infty e^{-\theta_0 u}\mu_{ij}(du) < \infty.$$

All conditions of Theorem 4 are satisfied and applying it to $\lim_{t\to\infty} \frac{Z_i(t)}{Z(t)}$ yields

$$\lim_{t\to\infty} \frac{Z_i(t)}{Z(t)} = \frac{u_i}{u_1 + \cdots + u_p}$$

$\square$

The proof of Theorem 2 is similar.

Proof of Theorem 2. Again we wish to apply Theorem 4. Let $\phi_x(t) = 1\{\tau(x) = i,\ \xi(t) = \vec{n}\}$ be the random characteristic assigning score 1 to type i vertices with n_k children of type k, $k = 1, \ldots, p$. Let $Z_i^{\vec{n}}(t) = Z^{\phi}(t)$ be the branching process associated with this characteristic. Similarly let $\psi_x(t) = 1$ and $Z(t) = Z^{\psi}(t)$ be the branching process counting the number of vertices born/alive at time t.

The proof of Theorem 1 shows that we can apply Theorem 4 and we get

$$\lim_{t\to\infty} \frac{Z_i^{\vec{n}}(t)}{Z(t)} = \frac{u_i \int_0^\infty e^{-\alpha s}\mathbb{P}(\xi_i(t) = \vec{n})}{(u_1 + \ldots + u_p)/\alpha} = \alpha \frac{u_i}{u_1 + \ldots + u_p} I_i(\vec{n}).$$

The latter part of the theorem follows directly from Proposition 2. $\square$

4 Multi-type Linear Preferential Attachment

In this section the theory from Section 3 is applied to investigate the limiting behavior of the degree distribution, and asymptotic composition of the vertex population, for a specific family of weight functions $w_{ij}(\vec{n})$. We consider the case when the weight functions are given by $w_{ij}(\vec{n}) = \gamma_{ij}(n_1 + \cdots + n_p) + \beta_{ij}$, with γ_{ij} and β_{ij} being positive constants. We call this model multi-type linear preferential attachment based on total in-degree. The main purpose of this section is to prove Theorem 3. Most of the results in this section are for the slightly more general case when $w_{ij}(\vec{n}) = w_{ij}(n_1 + \cdots + n_p)$, but the rate function is not necessarily linear.

First we will need to investigate if condition (A1) is satisfied. Hence, we will need to calculate $\mu_{ij}^*(\alpha)$. First note that the density (Radon-Nikodym derivative) in Proposition 1 is given by

$$h_{ij}(t) = \mathbb{E}(w_{ij}(\vec{\xi_i}(t))) = \mathbb{E}(w_{ij}(\xi_{i1}(t) + \cdots + \xi_{ip}(t))) = \sum_{k=0}^{\infty} w_{ij}(k)\mathbb{P}(\xi_i^{\Sigma}(t) = k)$$

where $\xi_i^{\Sigma}(t) = \xi_{i1}(t) + \cdots + \xi_{ip}(t)$. In deriving an expression for $\mu_{ij}^*(\alpha)$ it will first be useful to study $I_i(k) = \sum_{n_1+\cdots+n_p=k} I_i(\vec{n}) = \int_0^\infty e^{-\lambda s}\mathbb{P}(\xi_i^{\Sigma}(s) = k)ds$.

Proposition 3. *If the weight functions of the multi-type preferential attachment model satisfy $w_{ij}(\vec{n}) = w_{ij}(n_1 + \cdots + n_p)$ then*

$$I_i(k) = \frac{1}{\alpha + w_{i1}(k) + \cdots + w_{ip}(k)} \prod_{n=0}^{k-1} \frac{w_{i1}(n) + \cdots + w_{ip}(n)}{\alpha + w_{i1}(n) + \cdots + w_{ip}(n)}$$

where an empty product is defined to equal 1.

Proof. We prove the formula by induction. First note that it holds $k = 0$. Assume that it holds for $k > 0$. By Proposition 2 and the induction assumption

$$I_i(k+1) = \sum_{n_1+\cdots+n_p=k+1} I_i(\vec{n}) = \sum_{n_1+\cdots+n_p=k+1} \sum_{j=1}^{p} \frac{w_{ij}(k) I_i(n_1, \ldots, n_j - 1, \ldots, n_p)}{\alpha + w_{i1}(k+1) + \cdots + w_{ip}(k+1)}$$

$$= \sum_{j=1}^{p} \frac{w_{ij}(k)}{\alpha + w_{i1}(k+1) + \cdots + w_{ip}(k+1)} \sum_{n_1+\cdots+n_p=k} I_i(\vec{n})$$

$$= \sum_{j=1}^{p} \frac{w_{ij}(k)}{\alpha + w_{i1}(k+1) + \cdots + w_{ip}(1+k)} I_i(k) = \frac{(\sum_{j=1}^{p} w_{ij}(k)) I_i(k)}{\alpha + w_{i1}(k+1) + \cdots + w_{ip}(k+1)}$$

which proves the formula.

□

This result is easily extended to an expression for $\mu^*_{ij}(\theta)$ given that the weight functions depend only on the total in-degree.

Corollary 1. *If the weight functions of the multi-type preferential attachment model satisfy $w_{ij}(\vec{n}) = w_{ij}(n_1 + \cdots + n_p)$, then for $\theta > 0$*

$$\mu^*_{ij}(\theta) = \sum_{k=0}^{\infty} \frac{w_{i1}(k)}{\theta + w_{i1}(k) + \cdots + w_{ip}(k)} \prod_{n=0}^{k-1} \frac{w_{i1}(n) + \cdots + w_{ip}(n)}{\theta + w_{i1}(n) + \cdots + w_{ip}(n)}$$

Proof. Let $\xi_i^{\Sigma}(t) = \xi_{i1}(t) + \cdots + \xi_{ip}(t)$ then

$$\mu^*_{ij}(\theta) = \int_0^{\infty} e^{-\theta s} \mu_{ij}(ds) = \int_0^{\infty} e^{-\theta s} h_{ij}(s) ds$$

$$= \int_0^{\infty} e^{-\theta s} \sum_{k=0}^{\infty} w_{ij}(k) \mathbb{P}(\xi_i^{\Sigma}(s) = k) ds = \sum_{k=0}^{\infty} w_{ij}(k) \int_0^{\infty} e^{-\theta s} \mathbb{P}(\xi_i^{\Sigma}(s) = k) ds = \sum_{k=0}^{\infty} w_{ij}(k) I_i(k, \theta)$$

where the second to last equality follows from monotone convergence. An application of Proposition 3 yields the desired result. □

Remark. The above expression for $\mu^*_{ij}(\theta)$ is in accordance with the results of [14] and [6] when $p=1$. Also, note that $\mu^*_{ij}(\lambda)$ may well be infinite if $\theta<\theta_0$, with θ_0 as in (A1).

Given that the weight functions are linear in total degree we can derive an even more explicit result for the Laplace transform $\mu^*_{ij}(\theta)$.

Corollary 2. *If the weight functions of the multi-type preferential attachment model satisfy* $w_{ij}(\vec{n}) = \gamma_{ij}(n_1+\cdots+n_p)+\beta_{ij}$*, with* $\gamma_{ij}\geq 0, \beta_{ij}>0$ *and* $\gamma_{i1}+\cdots+\gamma_{ip}>0$*, then*

$$\mu^*_{ij}(\theta) = \int_0^\infty e^{-\theta s}\mu_{ij}(ds) = \begin{cases} \infty & \text{if } 0<\theta\leq \gamma_{i1}+\cdots+\gamma_{ip}, \\ \frac{\beta_{ij}}{\theta} + \frac{\gamma_{ij}(\beta_{i1}+\cdots+\beta_{ip})}{\theta(\theta-(\gamma_{i1}+\cdots+\gamma_{ip}))} & \text{if } \theta > \gamma_{i1}+\cdots+\gamma_{ip}\end{cases}$$

Proof. For convenience set $\gamma=\gamma_{i1}+\cdots+\gamma_{ip}$ and $\beta=\beta_{i1}+\cdots+\beta_{ip}$. First assume that $\gamma_{ij}>0$. By Corollary 1, whenever the Laplace transform exists, we have

$$\begin{aligned}\mu^*_{ij}(\theta) &= \sum_{k=0}^{\infty}\frac{w_{i1}(k)}{\theta+w_{i1}(k)+\cdots+w_{ip}(k)}\prod_{n=0}^{k-1}\frac{w_{i1}(n)+\cdots+w_{ip}(n)}{\theta+w_{i1}(n)+\cdots+w_{ip}(n)} \\ &= \sum_{k=0}^{\infty}\frac{\gamma_{ij}k+\beta_{ij}}{\gamma k+\alpha+\beta}\prod_{n=0}^{k-1}\frac{\gamma n+\beta}{\gamma n+\alpha+\beta}.\end{aligned}$$

We want to use the relation $\prod_{i=0}^{k}(i+c) = \frac{\Gamma(k+1+c)}{\Gamma(c)}$ and therefore re-write the above expression as

$$\begin{aligned}\mu^*_{ij}(\theta) &= \frac{\gamma_{ij}}{\gamma}\sum_{k=0}^{\infty}\frac{k+\beta_{ij}/\gamma_{ij}}{k+\alpha/\gamma+\beta/\gamma}\prod_{n=0}^{k-1}\frac{n+\beta/\gamma}{n+\alpha/\gamma+\beta/\gamma} \\ &= \frac{\gamma_{ij}}{\gamma}\sum_{k=0}^{\infty}\frac{k+\beta/\gamma-\beta/\gamma+\beta_{ij}/\gamma_{ij}}{k+\alpha/\gamma+\beta/\gamma}\prod_{n=0}^{k-1}\frac{n+\beta/\gamma}{n+\alpha/\gamma+\beta/\gamma} \\ &= \frac{\gamma_{ij}}{\gamma}\left(\sum_{k=0}^{\infty}\prod_{n=0}^{k}\frac{n+\beta/\gamma}{n+\alpha/\gamma+\beta/\gamma} + (\beta_{ij}/\gamma_{ij}-\beta/\gamma)\sum_{k=0}^{\infty}\frac{\prod_{n=0}^{k-1}n+\beta/\gamma}{\prod_{n=0}^{k}n+\alpha/\gamma+\beta\gamma}\right) \\ &= \frac{\gamma_{ij}\Gamma(\alpha/\gamma+\beta/\gamma)}{\gamma\Gamma(\beta/\gamma)}\left(\sum_{k=0}^{\infty}\frac{\Gamma(k+1+\beta/\gamma)}{\Gamma(k+1+\alpha/\gamma+\beta/\gamma)} + (\beta_{ij}/\gamma_{ij}-\beta/\gamma)\sum_{k=0}^{\infty}\frac{\Gamma(k+\beta/\gamma)}{\Gamma(k+1+\alpha\gamma+\beta/\gamma)}\right).\end{aligned}$$

Using the relation $\sum_{k=0}^{n}\frac{\Gamma(k+a)}{\Gamma(k+c)} = \frac{1}{1+a-c}\left(\frac{\Gamma(n+1+a)}{\Gamma(n+c)}-\frac{\Gamma(a)}{\Gamma(c-1)}\right)$ valid for all real numbers a,c (see [6]) together with Stirling's formula $\Gamma(k+c)/\Gamma(k)\sim k^c$ we get

$$\frac{\Gamma(n+1+a)}{\Gamma(n+c)}\to 0 \text{ as } n\to\infty \text{ if } 1+a<c \implies \sum_{k=0}^{\infty}\frac{\Gamma(k+a)}{\Gamma(k+c)} = \frac{1}{c-a-1}\frac{\Gamma(a)}{\Gamma(c-1)}, \text{ if } 1+a<c.$$

We note that the sum diverges if $1 + a \geq c$. Hence, $\mu^*_{ij}(\theta)$ converges if and only if $\theta > \gamma = \gamma_{i1} + \cdots + \gamma_{ip}$. For $\theta > \gamma$, we get

$$\mu^*_{ij}(\theta) = \frac{\gamma_{ij}\Gamma(\alpha/\gamma + \beta/\gamma)}{\gamma\Gamma(\beta/\gamma)} \left(\frac{\Gamma(1 + \beta/\gamma)}{(\theta/\gamma - 1)\Gamma(\theta/\gamma + \beta/\gamma)} + (\beta_{ij}/\gamma_{ij} - \beta/\gamma) \frac{\Gamma(\beta/\gamma)}{\theta/\gamma\Gamma(\theta/\gamma + \beta/\gamma)} \right).$$

All together we then have (again using $\prod_{i=0}^{k}(i + c) = \frac{\Gamma(k+1+c)}{\Gamma(c)}$),

$$\mu^*_{ij}(\theta) = \begin{cases} \infty, & \text{if } \theta \leq \gamma_{i1} + \cdots + \gamma_{ip} \\ \frac{\beta_{ij}}{\theta} + \frac{\gamma_{ij}(\beta_{i1} + \cdots + \beta_{i2})}{\theta(\theta - (\gamma_{i1} + \cdots + \gamma_{ip}))}, & \text{if } \theta > \gamma_{i1} + \cdots + \gamma_{ip}. \end{cases}$$

Next assume that $\gamma_{ij} = 0$. Then $\mu^*_{ij}(\theta)$ is just the Laplace transform of the intensity measure of a Poisson process which is in accordance with the formula, i.e. $\mu^*_{ij}(\theta) = \frac{\beta_{ij}}{\theta}$. □

Corollary 2 immediately implies that condition (A1) is satisfied for multi-type linear preferential attachment since

$$\min_i \sum_{j=1}^{p} \mu^*_{ij}(\theta) \leq \rho(\mu^*(\theta))$$

and $\lim_{\theta \downarrow \gamma_{i1} + \cdots + \gamma_{ip}} \mu^*_{ij}(\theta) = \infty$.

Using Corollary 2, it is possible to calculate the Perron root of $\mu^*(\theta)$ and Malthusian parameter α as well as the corresponding eigenvectors. However, already for $p = 2$ these expressions are rather complicated and are better calculated numerically.

We are finally ready to prove Theorem 3.

Proof of Theorem 3. Given that the model starts with one vertex we have by Theorem 4 that

$$\frac{Z_i^k(t)}{Z_i(t)} \to \alpha \frac{u_i}{u_1 + \cdots + u_p} I_i(k) \text{ as } t \to \infty$$

i.e. the proportion of type i vertices with k children in total converges to the expression on the right-hand side above. Again let $\gamma = \gamma_{i1} + \cdots + \gamma_{ip}$ and $\beta = \beta_{i1} + \cdots + \beta_{ip}$. First assume that $\gamma > 0$. Then by Proposition 3

$$I_i(k) = \frac{\prod_{n=0}^{k-1} \gamma n + \beta}{\prod_{n=0}^{k} \alpha + \gamma n + \beta} = \frac{\Gamma(\frac{\alpha+\beta}{\gamma})}{\gamma\Gamma(\frac{\beta}{\gamma})} \frac{\Gamma(k + \frac{\beta}{\gamma})}{\Gamma(k + 1 + \frac{\alpha+\beta}{\gamma})}.$$

By the same methods as in the proof of Corollary 2, we get

$$I_i(k) \sim C \cdot k^{-(1+\frac{\alpha}{\gamma})}$$

and the first part of the theorem is proved.

Secondly, assume that $\gamma_{i1} + \cdots + \gamma_{ip} = 0$. Then

$$I_i(k) = \frac{1}{\alpha+\beta}\left(\frac{\beta}{\alpha+\beta}\right)^k = \frac{1}{\alpha+\beta}e^{-k(\log(1+\frac{\alpha}{\beta}))}$$

and the second part of the theorem is proved. □

5 Numerical Examples

We now numerically investigate the behavior of the asymptotic composition of the vertex population as well as the exponent of the empirical degree distribution for some natural examples.

Consider first the two-type linear preferential attachment model with $w_{11}(k) = \gamma_{11}k + 1$ and $w_{12}(k) = w_{21}(k) = w_{22}(k) = k + 1$. We now vary the rate at which type 1 vertices generate new type 1 vertices, i.e. γ_{11}, while keeping everything else fixed. For $\gamma_{11} < 1$ we expect fewer type 1 than type 2 vertices in the graph and the opposite for $\gamma_{11} > 1$. This is indeed true, see Figure 1. Recall from Theorem 3 that the asymptotic behavior of the empirical degree distribution is given by

$$\lim_{t\to\infty} \frac{Z_i^k(t)}{Z(t)} \sim C_1 \cdot k^{-(1+\frac{\alpha}{\gamma_{i1}+\gamma_{i2}})} \qquad \text{if } \gamma_{i1} + \gamma_{i2} > 0. \tag{2}$$

Hence, a lower absolute value of the power law exponent corresponds to a heavier tail of the degree distribution. Clearly, for large values of γ_{11}, type 1 will have a heavier tail, and this can be observed in Figure 1.

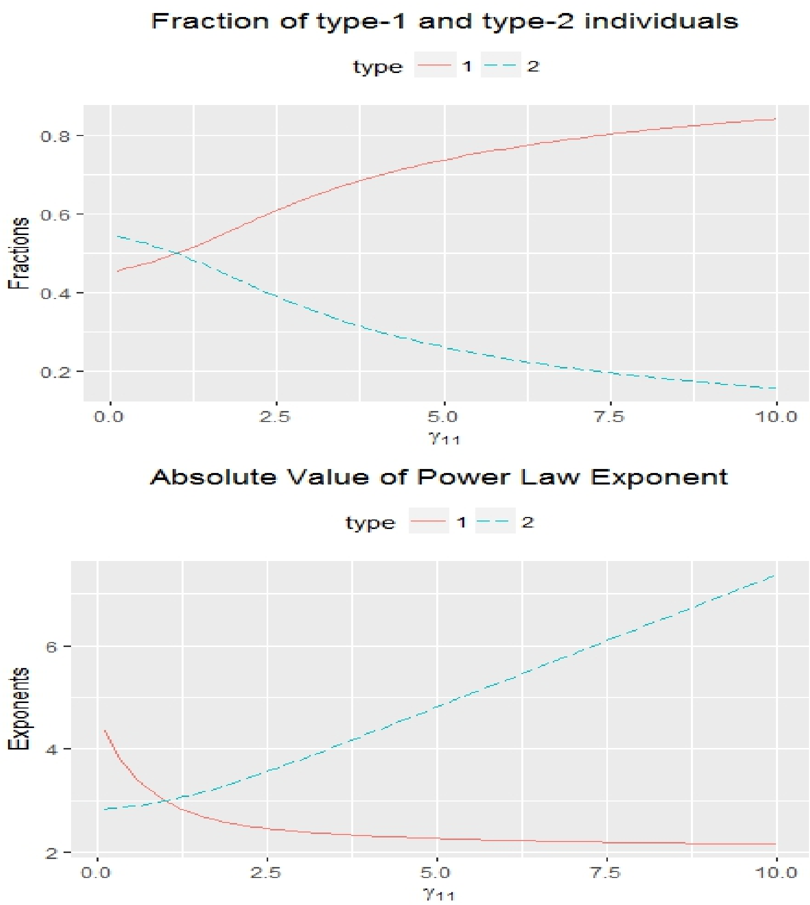

Figure 1: $w_{11}(k) = \gamma_{11}k + 1$ and $w_{12}(k) = w_{21}(k) = w_{22}(k) = k + 1$

Next consider the case when $w_{12}(k) = \gamma_{12}k + 1$ and $w_{11}(k) = w_{21}(k) = w_{22}(k) = k + 1$. We now vary the rate at which type 1 vertices generate new type 2 vertices while keeping everything else fixed. In Figure 2 we can see that for $\gamma_{12} < 1$ there is a majority of type 1 vertices, while for $\gamma_{12} > 1$ there is a majority of type 2 vertices. In fact, the qualitative behavior of the asymptotic composition is the opposite of previous model, compare Figure 1 and 2. Although there are more type 2 vertices for values of $\gamma_{12} > 1$ we note that it is type 1 vertices that generate them. Hence, there should still be more type 1 vertices with high total degree. This is indeed true, and can be observed in Figure 2. We note that power law exponents are the same as for the previous model—this follows from (2).

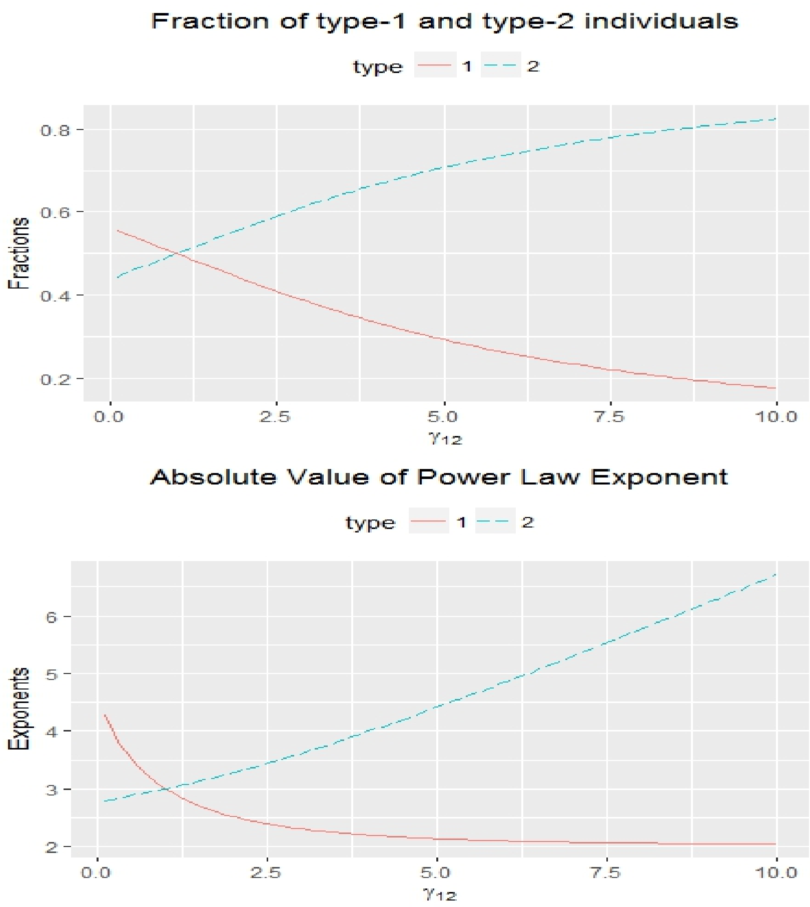

Figure 2: $w_{12}(k) = \gamma_{12}k + 1$ and $w_{11}(k) = w_{21}(k) = w_{22}(k) = k + 1$

The value of the parameters $\{\beta_{ij}\}$ influences the power law exponents through the Malthusian parameter, and the asymptotic composition through the left eigenvector u. For the last example we consider the model where $w_{11}(k) = \gamma_{11}k + 1$, $w_{12}(k) = k + 10$, $= w_{21}(k) = k + 1$ and $w_{22}(k) = k + 10$. Hence, the model has larger constants for generating type 2 vertices. Even for large values of γ_{11} there are still more type 2 than type 1 vertices in the graph, i.e. the constants $\beta_{12} = 10$ and $\beta_{22} = 10$ have a large influence on the asymptotic composition of the vertex population. Comparing Figure 3 with Figure 1 and 2 we see that degree distributions have thinner tails. This is because the larger values of the constants β_{12} and β_{21} weaken the preferential attachment mechanic in that it puts more weight on vertices with lower degree, e.g. degree 0 vertices have rate 10 instead of 1 as in the previous two examples.

Comparing the figures above we conclude that $\{\beta_{ij}\}$ has a large influence on asymptotic composition of the vertex population, and less influence on the degree distributions.

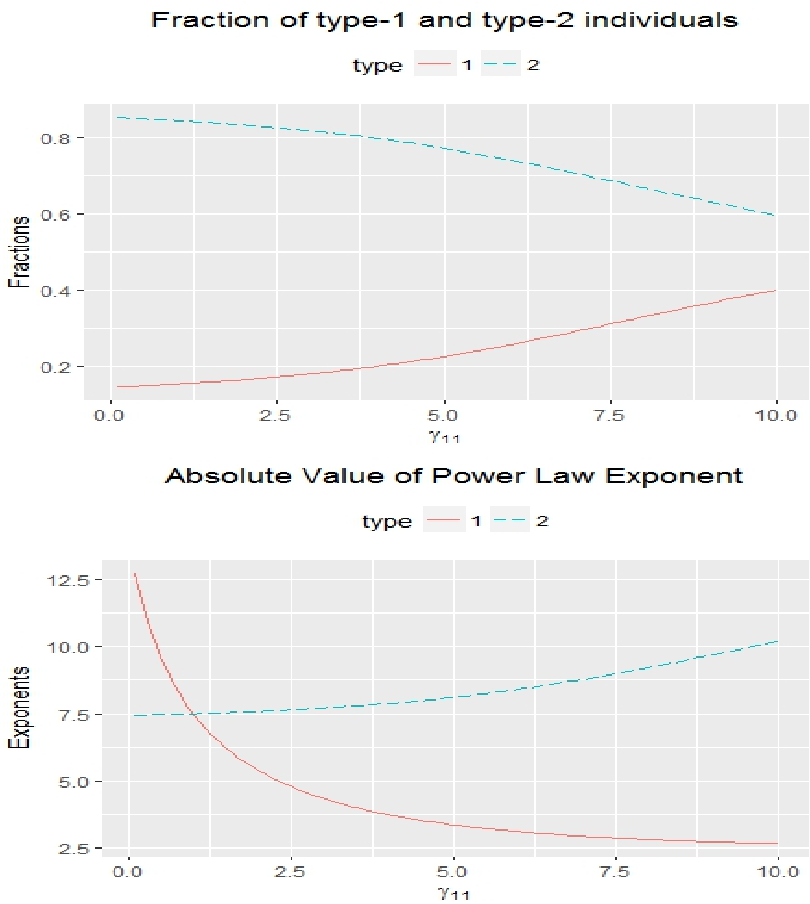

Figure 3: $w_{11}(k) = \gamma_{11}k + 1$, $w_{12}(k) = k + 10$, $= w_{21}(k) = k + 1$ and $w_{22}(k) = k + 10$.

6 Further Work

There are special cases of the model which can be studied further. For instance, the framework can be applied to the case when the rate functions are given by $w_{ij}(\vec{n}) = w_{ij}(n_j)$, i.e. when the reproduction processes of a vertex are independent. Using the framework one can identify the limit of $\lim_{t\to\infty} \frac{Z_i^{\phi^j}(t)}{Z(t)}$, where $Z_i^{\phi^j}(t)$ is the number of vertices at time t of type i with k type j children. For instance, if $w_{ij}(\vec{n}) = \gamma_{ij}n_j + \beta_{ij}$ then the asymptotic behavior in k of this fraction is given by

$$\lim_{t\to\infty}\frac{Z_i^{\phi^j}(t)}{Z(t)} \sim C\cdot k^{-(1+\frac{\alpha}{\gamma_{ij}})}, \qquad C\in\mathbb{R}.$$

This follow by noting that, by Theorem 4, we have

$$\lim_{t\to\infty}\frac{Z_i^{\phi^j}(t)}{Z(t)} = C_1\alpha\int_0^\infty e^{-\alpha s}\mathbb{P}(\xi_{ij}(s)=k)ds = C\sum_{\vec{n}:\ n_j=k} I_i(\vec{n})$$

and, by Proposition 2 and the proof of Corollary 2, we get

$$\sum_{\vec{n}:\ n_j=k} I_i(\vec{n}) = \frac{1}{\alpha+w_{ij}(k)}\prod_{n=0}^{k-1}\frac{w_{ij}(k)}{\alpha+w_{ij}(k)}$$

$$=\frac{1}{\alpha+\gamma_{ij}k+\beta_{ij}}\prod_{n=0}^{k-1}\frac{\gamma_{ij}n+\beta_{ij}}{\alpha+\gamma_{ij}n+\beta_{ij}} = \frac{\Gamma(\frac{\alpha+\beta_{ij}}{\gamma_{ij}})}{\gamma_{ij}\Gamma(\frac{\beta_{ij}}{\gamma_{ij}})}\frac{\Gamma(k+\frac{\beta_{ij}}{\gamma_{ij}})}{\Gamma(k+1+\frac{\alpha+\beta_{ij}}{\gamma_{ij}})} \sim C\cdot k^{-(1+\frac{\alpha}{\gamma_{ij}})}.$$

However, an expression for how the fraction of type i vertices with k children in total behaves as k grows large does not follow easily from the framework, and is left as an open problem.

There are also extensions of the model which can be studied. Following [6] we could allow for vertex death. The framework developed here can not be directly applied to this situation, but it should be possible to extend to allow for vertex death. With no vertex death the preferential attachment graph is a tree, and questions about the largest component are not interesting. However, with vertex death the graph becomes a forest, and questions about the largest component arise. Will a large component emerge? If so, how large is it?

7 Acknowledgments

I would like to extend my gratitude towards my supervisor Mia Deijfen for introducing me to the model, and for helpful comments on the manuscript. I would also like to thank KaYin Leung and the Journal Club at Stockholm University for helpful feedback, as well as Professor Olle Nerman for pointing out the newest results on general multi-type branching processes.

References

[1] K. B. Athreya and K. , Rama Murthy, *Feller's Renewal Theorem For Systems Of Renewal Equations*, Indian Institute of Science, (1975).

[2] A.-L. Barabási and R. Albert, *Emergence of scaling in random networks*, Science, 286 (1999), pp. 509–512.

[3] B. Bollobás, S. Janson, and O. Riordan, *The phase transition in inhomogeneous random graphs*, Random Structures and Algorithms, 31 (2007), pp. 3–122.

[4] B. Bollobás, O. Riordan, J. Spencer, and G. Tusnády, *The degree sequence of a scale-free random graph process*, Random Structures and Algorithms, 18 (2001), pp. 279–290.

[5] J. Chen, K. H. Lundberg, D. E. Davison, and D. S. Bernstein, *The Final Value Theorem Revisited*, IEEE Control Systems, 27 (2007), pp. 97 – 99.

[6] M. Deijfen, *Random networks with preferential growth and vertex death*, Journal of Applied Probability, 47 (2010), p. 1150–1163.

[7] M. Deijfen and R. Fitzner, *Birds of a feather or opposites attract - effects in network modelling*, ArXiv e-prints, (2016).

[8] A. Iksanov and M. Meiners, *Rate of Convergence in the Law of Large Numbers For Supercritical General Multi-type Branching Processes*, Stochastic Processes and Their Applications, 125 (2015), pp. 708–738.

[9] P. Jagers, *Branching Processes With Biological Applications*, Wiley, New York, 1975.

[10] C. D. Meyer, ed., *Matrix Analysis and Applied Linear Algebra*, Society for Industrial and Applied Mathematics, Philadelphia, 2000.

[11] C. Mode, *Multitype Branching Processes*, American Elsevier, New York, 1971.

[12] O. Nerman, *On The Convergence Of Supercritical General Branching Processes*, PhD thesis, University of Göteborg, 1979.

[13] R.-D. Reiss, *A course on Point Processes*, Springer-Verlag, New York, 1993.

[14] A. Rudas, B. Tóth, and B. Valkó, *Random Trees and General Branching Processes*, Random Structures and Algorithms, 31 (2006), pp. 186–202.

8 Appendix

For the reader interested in how the Malthusian parameter, the Perron root, and the corresponding eigenvectors enters Theorem 4 and how they play a role in the theory more generally we give an indication. Define $m_i^\phi(t) = \mathbb{E}(e^{-\alpha t}{}_iZ^\phi(t))$ where ${}_iZ^\phi(t)$ is the process starting with one type i individual and counted with the random characteristic ϕ. Denote by $\phi_i(t)$ the random score of a type i individual of age t. With this notation we get the standard system of renewal equations for a general multi-type branching process, see e.g. [12].

Lemma 1. *The processes $\{m_i^\phi(t)\}, i = 1, 2, \ldots, p$, satisfies a system of renewal equations*

$$m_i^\phi(t) = \mathbb{E}(e^{-\alpha t}\phi_i(t)) + \sum_{j=1}^{p} \int_0^t m_j^\phi(t-s)e^{-\alpha s}\mu_{ij}(ds), \qquad i = 1, 2, \ldots, p. \quad (3)$$

It is reasonable to assume that $\mathbb{E}(Z(t))$ (and by proxy $\mathbb{E}(Z^\phi(t))$) grows exponentially, as exponential growth is inherent in supercritical branching processes. Indeed, if each individual is, on average, replaced with more than one individual, the process should grow exponentially. For instance, in the single-type Galton-Watson case we have that $\mathbb{E}(Z_n) = m^n$, where m is the expected number of children of an individual. Another example of exponential growth is the two-type Markovian branching process with constant reproduction rates and no deaths; in our multi-type preferential attachment terminology $w_{ij}(m, n) = \gamma_{ij}$. Let $m_i(t) = \mathbb{E}(Z_i(t))$ be the expected number of type i individuals alive at time t. As the process is Markovian, the number of type i individuals alive at time $t + dt$ given the population sizes $Z_1(t)$ and $Z_2(t)$ is the sum of every individual alive at t of type i and all type i offspring produced in the interval $[t, t + dt]$. We get

$$\begin{aligned} m_i(t+dt) &= \mathbb{E}(\mathbb{E}(Z_i(t+dt)|Z_1(t), Z_2(t))) \\ &= \mathbb{E}(Z_i(dt)|\tau(x_0) = 1)\mathbb{E}(Z_1(t)) + \mathbb{E}(Z_i(dt)|\tau(x_0) = 2)\mathbb{E}(Z_2(t)). \end{aligned}$$

Ignoring the infinitesimal probability that two births can occur in a small time interval we have that $\mathbb{E}(Z_1(dt)|\tau(x_0) = 1) = 1 - \gamma_{11}dt + 2\gamma_{11}dt = 1 + \gamma_{11}dt$ and $\mathbb{E}(Z_1(dt)|\tau(x_0) = 2) = \gamma_{21}dt$ (here we do not count type 2 individuals that are alive). We get the system of equations

$$\begin{aligned} m_1(t+dt) &= (1 + \gamma_{11}dt)m_1(t) + \gamma_{21}m_2(t)dt \\ m_2(t+dt) &= \gamma_{12}m_1(t)dt + (1 + \gamma_{22}dt)m_2(t). \end{aligned}$$

Implying that

$$\begin{aligned} m_1'(t) &= \gamma_{11}m_1(t) + \gamma_{21}m_2(t) \\ m_2'(t) &= \gamma_{12}m_1(t) + \gamma_{22}m_2(t) \\ m_1(0) &= m, \qquad m_2(0) = n. \end{aligned}$$

Such a system is solved by $m_i(t) = c_1e^{\lambda_1 t}v_1 + c_2e^{\lambda_2 t}v_2$, where λ_1 and λ_2 are the eigenvalues of the matrix $M = [\gamma_{ij}]$ and v_1 and v_2 corresponding right eigenvectors. Hence, $m_i(t)$ grows as $e^{\lambda t}$ where λ is the largest eigenvalue of M.

We have argued that general branching processes exhibit exponential growth, however at which rate remains to be understood. It turns out that the rate of the growth is determined by the Malthusian parameter. For instance, in the Markov example above, one can easily show that the Malthusian parameter equals the largest eigenvalue of the matrix $M = [\gamma_{ij}]$, and therefore determines the growth of $Z(t)$. In fact, Malthusian growth holds for a large class of multi-type general branching processes, e.g. processes satisfying (C1)-(C5). But why does the Malthusian parameter α — satisfying $\rho(\mu^*(\alpha)) = 1$ — determine the growth? To answer that we first need to set some notation. Let $m_i^*(\theta) = \int_0^\infty e^{-\theta s} m_i^\phi(s)ds$, $\phi_i^*(\theta) = \int_0^\infty e^{-\theta s}\mathbb{E}(\phi_i(s))ds$, and finally $\mu_{ij}^*(\theta) = \int_0^\infty e^{-\theta s}\mu_{ij}(ds)$. Let $m^*(\theta) = [m_i^*(\theta)]_i, \Phi^*(\theta) = [\phi_i^*(\theta)]_i, \mu^*(\theta) = [\mu_{ij}^*(\theta)]_{ij}$, $i, j = 1, \ldots, p$. By Lemma 1, $m_i^\phi(t)$ satisfies

$$m_i^\phi(t) = \mathbb{E}(e^{-\alpha t}\phi_i(t)) + \sum_{j=1}^p \int_0^t m_j^\phi(t-s)e^{-\alpha s}\mu_{ij}(ds), \qquad i = 1, 2, \ldots, p.$$

Taking the Laplace transform of this and expressing it in matrix notation we get

$$m^*(\theta) = \Phi^*(\alpha + \theta) + \mu^*(\alpha + \theta)m^*(\theta), \qquad \theta > 0.$$

By [11, Lemma 8.2], condition (C5) shows that the Laplace transform indeed exists. Solving for $m^*(\theta)$ we get

$$m^*(\theta) = (I - \mu^*(\alpha + \theta))^{-1}\Phi^*(\alpha + \theta).$$

Since $\rho(\mu^*(\alpha)) = 1$ and $\rho(\mu^*(\theta))$ is decreasing we have that $\rho(\mu^*(\alpha+\theta)) < 1$, implying that $\det((I - \mu^*(\alpha + \theta))) \neq 0$. Therefore $(I - \mu^*(\alpha + \theta))$ must be invertible. Let $B(\alpha + \theta)$ be the adjoint matrix of $(I - \mu^*(\alpha + \theta))$ and let $\Delta(\alpha+\theta)$ be the determinant of $(I - \mu^*(\alpha+\theta))$. We note that $\Delta(\alpha) = 0$ since

α is the Malthusian parameter. It is well know that

$$(I - \mu^*(\alpha + \theta))B(\alpha + \theta) = B(\alpha + \theta)(I - \mu^*(\alpha + \theta)) = \Delta(\alpha + \theta)I,$$
$$(I - \mu^*(\alpha + \theta))^{-1} = \frac{B(\alpha + \theta)}{\Delta(\alpha + \theta)}.$$

We can then write $m_i^*(\theta)$ as

$$m_i^*(\theta) = \frac{B(\alpha + \theta)\Phi^*(\alpha + \theta)}{\Delta(\alpha + \theta)}.$$

Readers familiar with complex analysis will realize that $m_i^*(\theta)$ has a pole at $\theta = 0$, as $\Delta(\alpha) = 0$, and $m_i^*(\theta)$'s behavior around $\theta = 0$ will determine the growth of $m_i^\phi(t)$.

Furthermore, $B(\alpha)\mu^*(\alpha) = \mu^*(\alpha)B(\alpha) = B(\alpha)$. Therefore, as the Perron root equals 1, every column vector in $B(\alpha)$ must be a multiple of the right eigenvector v of μ^*, and every row vector in $B(\alpha)$ must be a multiple of the left eigenvector u of $\mu^*(\alpha)$. Hence, $B(\alpha) = c \cdot vu$.

By the standard version of the Laplace transform final value theorem, see [5], $\lim_{t\to\infty} m_i^\phi(t) = \lim_{t\to\infty} e^{-\alpha t}\mathbb{E}(Z^\phi(t))$ exists and equal

$$\lim_{t\to\infty} m_i^\phi(t) = \lim_{\theta\to 0} \theta m_i^*(\theta) = cv_i \left(\lim_{\theta\to 0} \frac{\theta}{\Delta(\alpha + \theta)}\right) \sum_{j=1}^{p} u_j \phi_j^*(\alpha).$$

Is is shown in [1, p. 454] that $\lim_{\theta\to 0} \frac{\theta}{\Delta(\alpha+\theta)}$ exists and is positive (also this is where the non-lattice assumption is used). In conclusion

$$m_i^\phi(t) \to Cv_i \sum_{j=1}^{p} u_j \int_0^\infty e^{-\alpha s}\mathbb{E}(\phi_j(s))ds \text{ as } t \to \infty, \qquad i = 1, 2, \ldots, p$$

where $u = (u_1, \ldots, u_p)$ and $v = (v_1, \ldots, v_p)^t$ are the left and right eigenvectors corresponding to the eigenvalue $\lambda = 1$ of the matrix $\mu^*(\alpha)$.

Hence, as $\lim_{t\to\infty} \frac{m_i^\phi(t)}{m_i^\psi(t)} = \frac{\sum_{j=1}^p u_j \int_0^\infty \mathbb{E}(e^{-\alpha s}\phi_j(s))ds}{\sum_{j=1}^p u_j \int_0^\infty \mathbb{E}(e^{-\alpha s}\psi_j(s))ds}$ we might expect

$$\lim_{t\to\infty} \frac{{}_iZ^\phi(t)}{{}_iZ^\psi(t)} = \frac{\sum_{j=1}^p u_j \int_0^\infty \mathbb{E}(e^{-\alpha s}\phi_j(s))ds}{\sum_{j=1}^p u_j \int_0^\infty \mathbb{E}(e^{-\alpha s}\psi_j(s))ds}$$

to converge in a stronger sense than a ratio of expectations. This is indeed true as shown in Theorem 4, but it is much more difficult to prove.

Finally, a word on the non-lattice assumption (C1). Consider the traditional single-type Galton-Watson branching process but in continuous time.

This process is then governed by a lattice measure—the intensity measure is concentrated on the integer one. Why does not $\mathbb{E}(Z(t))/e^{\alpha t}$ converge? From standard result on the discrete time version of the process we know that $\mathbb{E}(Z([t]))/m^{[t]} \to 1$ as $t \to \infty$, where m is the expected number of children of an individual. It follows that $\alpha = \log(m)$. Hence,

$$\frac{\mathbb{E}(Z(t))}{e^{\alpha t}} = \frac{\mathbb{E}(Z([t]))}{m^{[t]}} e^{-\log(m)([t]-t)}$$

and this does not converge.