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Abstract

We consider multi-period cost-of-capital valuation of a liability cash flow subject to repeated capital requirements that are partly financed by capital injections from capital providers with limited liability. Limited liability means that, in any given period, the capital provider is not liable for further payment in the event that the capital provided at the beginning of the period turns out to be insufficient to cover both the current-period payments and the updated value of the remaining cash flow. The liability cash flow is modeled as a continuous-time stochastic process on $[0, T]$. The multi-period structure is given by a partition of $[0, T]$ into subintervals, and on the corresponding finite set of times a discrete-time value process is defined. Our main objective is the analysis of existence and properties of continuous-time limits of discrete-time value processes corresponding to a sequence of partitions whose meshes tend to zero. Moreover, we provide explicit and interpretable valuation formulas for a wide class of cash flow models.

1 Introduction

1.1 Multi-period cost-of-capital valuation

The paper focuses on the multi-period cost-of-capital valuation of a cumulative liability cash flow $L = \{L_t\}_{t\in[0,T]}$ subject to repeated capital requirements at the beginning of each time period, where the time periods form a partition of $[0, T]$. Here T is a time after which no cash flow occurs. In line with current regulatory frameworks, the time periods may be one-year periods. However, we will here investigate the effects of varying the number and lengths of the periods and in particular consider a sequence of partitions of $[0, T]$ whose meshes tend to 0. That is, we will analyze continuous-time limits of discrete-time cost-of-capital valuations of the liability cash flow L. In what follows, all cash flows and financial values are discounted by a given numéraire, or equivalently, denoted in units of this numéraire. A classical bank account numéraire, a rolling one-period bond, may be a natural choice.

In order to clarify the economic motivation of the valuation setup, let T be a positive integer and consider times $0, 1, \dots, T$. The multi-period cost-of-capital valuation of the liability cash flow $L = \{L_t\}_{t\in[0,T]}$ is based on considering a hypothetical transfer of the liability cash flow at time 0 from the company currently liable for this cash flow to another company whose single purpose is to manage the runoff of the liability cash flow. The company receiving the liability cash flow has no own funds but receives the current value of the liability, V_0, together with the liability. In order to meet the externally imposed capital requirements associated with the liability, according to the regulatory environment, the receiver of the liability cash flow requests external capital injections from a capital provider. The mathematical problem arising is the determination of V_0: this value is not a priori given but rather a value implied by the repeated financing of the capital requirements by a capital provider demanding compensation for providing capital injections.

Let V_t denotes the value of the liability cash flow $\{L_s\}_{s\in(t,T]}$, i.e. beyond time t. In particular, $V_T = 0$. Assume that the amount V_t is available at time t and that the required capital is $\mathrm{VaR}_{t,p}(-\Delta L_{t+1} - V_{t+1}) > V_t$, where $\Delta L_{t+1} := L_{t+1} - L_t$ is the accumulated cash flow during the time period $(t, t+1]$ and $\mathrm{VaR}_{t,p}$ is the risk measure Value-at-Risk conditional on the information available at time t. A capital provider is asked to provide the difference $\mathrm{VaR}_{t,p}(-\Delta L_{t+1} - V_{t+1}) - V_t$ between the required and the available capital. If this capital is provided, then, in return, the capital provider receives the amount $(\mathrm{VaR}_{t,p}(-\Delta L_{t+1} - V_{t+1}) - \Delta L_{t+1} - V_{t+1})_+$ at time $t+1$, where $(x)_+ := \max\{x, 0\}$. The rational for the amount $(\mathrm{VaR}_{t,p}(-\Delta L_{t+1} - V_{t+1}) - \Delta L_{t+1} - V_{t+1})_+$ is the following. The capital provider is entitled to any excess capital at time $t+1$ above what is needed for the one-period payment ΔL_{t+1} plus V_{t+1} that is needed to match the value of the remaining liability cash flow at time $t+1$. A capital provider will accept providing the capital at time t if the expected return is good enough, in the sense that

$$\frac{\mathbb{E}_t[(\mathrm{VaR}_{t,p}(-\Delta L_{t+1} - V_{t+1}) - \Delta L_{t+1} - V_{t+1})_+]}{\mathrm{VaR}_{t,p}(-\Delta L_{t+1} - V_{t+1}) - V_t} = 1 + \eta_t, \qquad (1)$$

where $\mathbb{E}_t$ denotes conditional expectation with respect to the information at time t, and $\eta_t \geqslant 0$ is the excess expected rate of return (above that of the numéraire asset) at time t on the capital provided until time $t+1$. The value of η_t is determined by a combination of factors such as the degree of risk-averseness and competition between potential capital providers, besides properties of the liability cash flow L. In what follows, we will simply consider $\{\eta_t\}_{t=0}^{T-1}$ to be an exogenously given stochastic process.

Given the liability cash flow $\{L_t\}_{t\in[0,T]}$ and a discrete-time stochastic process $\{\eta_t\}_{t=0}^{T-1}$, the acceptability condition (1) immediately gives the following backward recursion for $\{V_t\}_{t=0}^T$:

$$\begin{aligned}V_t &= \mathrm{VaR}_{t,p}(-\Delta L_{t+1} - V_{t+1}) \\ &\quad - \frac{1}{1+\eta_t}\mathbb{E}_t[(\mathrm{VaR}_{t,p}(-\Delta L_{t+1} - V_{t+1}) - \Delta L_{t+1} - V_{t+1})_+], \\ V_T &= 0.\end{aligned}$$

Notice that the values $\{V_t\}_{t=0}^T$ are not a priori given but rather the solution to the above recursion given a model for both $\{L_t\}_{t\in[0,T]}$ and $\{\eta_t\}_{t=0}^{T-1}$.

1.2 Related literature

The approach to multi-period cost-of-capital valuation above was studied in [7] for more general risk measures and acceptability criteria. The choice of one-year periods corresponds to the current regulatory solvency frameworks under which both banks and insurance companies operate, and is in line with accounting practice. However, it is quite reasonable to consider the financing of liability cash flows subject to repeated capital requirements by capital injections at a higher frequency. Moreover, by letting the length of the time periods tend to zero we may derive explicit interpretable continuous-time valuation formulas whereas solutions to discrete-time backward recursions of the above type can often only be obtained numerically. It is also interesting to analyze which features of a liability cash flow vanish and which persist in the limit process from discrete-time valuation to continuous-time valuation as the mesh of the partition of time periods tends to zero.

There are similarities with the our objectives here and works, such as [21], [15] and [13], analyzing continuous-time dynamic risk measures (or risk-adjusted values) which can be represented as limits of discrete-time risk measures in multi-period models. However, there are also major differences. A detailed comparison of our setup with that considered in [15] and [21] is found in Remark 2. The aim in [13] is different from ours: there the objective is the construction and analysis of dynamic risk measures expressed in terms of backward stochastic differential equations (BSDEs) that arise as limits of iterated spectral risk measures. Notice that a spectral risk measure is a form of coherent and convex risk measure.

The vast majority of works on dynamic risk measurement consider coherent or convex risk measures, and many of them aim for a representation of the risk measure in terms of a solution to a BSDE. See [1], [2], [3], [4], [8], [9], [12], [17], [18], [19], and references therein. The discrete-time value process $\{V_t\}_{t=0}^T$ from the multi-period cost-of-capital framework above share most of the properties of the multi-period risk adjusted values in [2], precise statements are found in [7]. In particular, the properties called time-consistency and recursiveness hold. However, the limited liability of the capital provider, which is an essential economic property, makes the value processes based on multi-period cost-of-capital valuation lack the super-additivity property in general. This fact holds irrespectively of whether $\mathrm{VaR}_{t,p}$ is replaced by a coherent alternative. Therefore, the general representation results for dynamic coherent or convex risk measures, or similarly for multi-period risk adjusted values, are not available in our multi-period cost-of-capital valuation setup. This fact makes the mathematical analysis here very different from that in works on dynamic coherent and convex risk measures.

The multi-period cost-of-capital valuation described in the introduction above is not market-consistent if the liability cash flow $\{L_t\}_{t\in[0,T]}$ includes or depends on the values of traded financial instruments. For market-consistency, a set of replication instruments must be considered and dynamic replication of the liability cash flow in these instruments. In this case, the multi-period cost-of-capital valuation applies to the replication error which is always present. These issues are highly relevant for obtaining a conceptually sound valuation framework for liability cash flows of insurance companies. A valuation framework of this kind is presented in [14]. The underlying principles of the current regulatory framework Solvency 2 are similar, although the implementation of these principles into Solvency 2 is not fully satisfactory and has received criticism, see e.g. [14].

1.3 Outline

Section 2 presents basic results for discrete-time value processes for a given continuous-time liability cash flow. In this setting, the value process is defined on a time grid $0 = \tau_0 < \cdots < \tau_m = T$ corresponding to an arbitrary partition τ of $[0, T]$.

Section 3 presents the main results of this paper on existence and properties of continuous-time limits of a sequence of discrete-time value processes for a given continuous-time liability cash flow. The continuous-time limit, defined in Definition 4, arises by considering an arbitrary sequence $\{\tau_m\}_{m=1}^{\infty}$ of partitions whose meshes tend to 0.

Theorem 2 gives mild conditions under which the continuous-time value of a sum of two cash flows decomposes into a sum of the corresponding two continuous-time value processes. Moreover, it gives mild conditions under which the continuous-time value process of a cash flow degenerates into a process of conditional expectations of the remaining cash flow.

Theorem 3 presents a wide class of Itô processes that satisfy the conditions of Theorem 2 under which the continuous-time value process is a process of conditional expectations of the remaining cash flow.

Theorem 4 derives the continuous-time limit of discrete-time value processes when the underlying liability cash flow is given by an additive process with a jump component, with Lévy processes and compound Poisson processes driven by inhomogeneous Poisson processes as special cases.

All proofs of the main results are found in Section 4. Section 5 contains technical lemmas used in the proofs of the main results that may also be of independent interest.

2 Discrete-time value processes

In this section we will present the mathematical framework for multi-period cost-of-capital valuation of a continuous-time cumulative liability cash flow.

Fix $T > 0$ and consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} := \{\mathcal{F}_t\}_{t\in[0,T]}$ satisfies the so-called usual conditions, see e.g. Chapter 1 in [16]. Write $L^0(\mathcal{F}_t)$ $(L^1(\mathcal{F}_t))$ for the set of $\mathcal{F}_t$-measurable (integrable) random variables. Write $L^0(\mathbb{F})$ $(L^1(\mathbb{F}))$ for the set of $\mathbb{F}$-adapted stochastic processes X with $X_t \in L^0(\mathcal{F}_t)$ $(X_t \in L^1(\mathcal{F}_t))$ for every $t \in [0,T]$. For $t < u$ and Y $\mathcal{F}_u$-measurable we use subscript t to denote conditioning on $\mathcal{F}_t$:

$$F_{t,Y}(y) := \mathbb{P}_t(Y \leqslant y) := \mathbb{P}(Y \leqslant y \mid \mathcal{F}_t), \quad \mathbb{E}_t[Y] := \mathbb{E}[Y \mid \mathcal{F}_t].$$

We consider an arbitrary partition of $[0,T]$ into subintervals and discrete-time value processes evaluated at the time points corresponding to the given partition. We will call a set of points $\tau := \{\tau_k\}_{k=0}^m$ with $0 = \tau_0 < \cdots < \tau_m = T$ a partition of the time interval $[0,T]$. For any such partition we denote by $\delta_k := \tau_{k+1} - \tau_k$ the lengths of the subintervals.

For $Y \in L^0(\mathcal{F}_{\tau_k+\delta_k})$, Value-at-Risk of Y at the level $1 - \alpha_{\delta_k} \in (0,1)$ conditional on $\mathcal{F}_{\tau_k}$ is defined as

$$\mathrm{VaR}_{\tau_k,1-\alpha_{\delta_k}}(Y) := F^{-1}_{\tau_k,-Y}(\alpha_\delta).$$

The discrete-time value of a liability cash flow is defined in terms of a backward recursion of the kind presented in Section 1. The one-step valuation mapping presented next enables this definition to be formulated with mathematical rigor.

Definition 1. *For $Y \in L^1(\mathcal{F}_{\tau_k+\delta_k})$, $\alpha_{\delta_k} \in (0,1)$ and a nonnegative $\eta_{\tau_k} \in L^0(\mathcal{F}_{\tau_k})$, the one-step valuation mapping is defined as*

$$W^{\delta_k}_{\tau_k}(Y) := \mathrm{VaR}_{\tau_k,1-\alpha_{\delta_k}}(-Y) - \frac{1}{1+\eta_{\tau_k}}\mathbb{E}_{\tau_k}\big[\big(\mathrm{VaR}_{\tau_k,1-\alpha_{\delta_k}}(-Y) - Y\big)_+\big].$$

Remark 1. *Notice that the backward recursion for the discrete-time value process $\{V_t\}_{t=0}^T$ in Section 1 may be expressed as*

$$V_t = W^1_t(L_{t+1} - L_t + V_{t+1}), \quad V_T = 0,$$

and corresponds to partitioning $[0,T]$ into subintervals of lengths one.

For economically meaningful applications, $1 - \alpha_{\delta_k}, \eta_{\tau_k}$ are both close to 0: η_{τ_k} is the expected excess rate of return, above that for the numéraire asset, for the capital provider on the provided capital ensuring solvency at time τ_k, α_{δ_k} is essentially the probability that the provided solvency capital at time τ_k is found sufficient at time $\tau_{k+1} = \tau_k + \delta_k$.

An alternative expression for $W^{\delta_k}_{\tau_k}(Y)$, see Lemma 2, is

$$W^{\delta_k}_{\tau_k}(Y) = \frac{1}{1+\eta_{\tau_k}}\Big\{\mathbb{E}_{\tau_k}[Y] - (1-\alpha_{\delta_k})\,\mathrm{ES}_{\tau_k,1-\alpha_{\delta_k}}(-Y) \\ + (1-\alpha_{\delta_k}+\eta_{\tau_k})\,\mathrm{VaR}_{\tau_k,1-\alpha_{\delta_k}}(-Y)\Big\},$$

where $\mathrm{ES}_{\tau_k,1-\alpha_{\delta_k}}$ denotes Expected Shortfall conditional on $\mathcal{F}_{\tau_k}$ defined as

$$\mathrm{ES}_{\tau_k,1-\alpha_\delta}(-Y) := \frac{1}{1-\alpha_\delta}\int_0^{1-\alpha_\delta} \mathrm{VaR}_{\tau_k,p}(-Y)\mathrm{d}p.$$

Theorem 1. *Consider a partition τ of $[0,T]$, $0=\tau_0<\cdots<\tau_m=T$. For each k, $W^{\delta_k}_{\tau_k}$ is a mapping from $L^1(\mathcal{F}_{\tau_k+\delta_k})$ to $L^1(\mathcal{F}_{\tau_k})$ satisfying*

if $\lambda \in L^1(\mathcal{F}_{\tau_k})$ and $Y \in L^1(\mathcal{F}_{\tau_{k+1}})$, then $W^{\delta_k}_{\tau_k}(Y+\lambda) = W^{\delta_k}_{\tau_k}(Y)+\lambda$,
if $Y,\widetilde{Y} \in L^1(\mathcal{F}_{\tau_{k+1}})$ and $Y \leqslant \widetilde{Y}$, then $W^{\delta_k}_{\tau_k}(Y) \leqslant W^{\delta_k}_{\tau_k}(\widetilde{Y})$,
$W^{\delta_k}_{\tau_k}(0) = 0$.

The proof of Theorem 1 is an immediate consequence of Propositions 1 and 4 in [7] and therefore omitted.

Based on the statement of Theorem 1 we may define the discrete-time value process $\{V^\tau_t(L)\}_{t\in\tau}$ of a continuous-time cumulative liability cash flow $L \in L^1(\mathbb{F})$. By Theorem 1, $\{V^\tau_t(L)\}_{t\in\tau} \in L^1(\{\mathcal{F}_t\}_{t\in\tau})$.

Definition 2. *Given $L \in L^1(\mathbb{F})$ and a partition τ of $[0,T]$, $0=\tau_0<\cdots<\tau_m=T$, the value process $\{V^\tau_t(L)\}_{t\in\tau}$ of L with respect to the partition τ and filtration $\mathbb{F}$ is defined in terms of a sequence of one-step valuation mappings defined in Definition 1 as follows:*

$$V^\tau_{\tau_k}(L) := W^{\delta_k}_{\tau_k}(\Delta L_{\tau_{k+1}} + V^\tau_{\tau_{k+1}}(L)), \quad V^\tau_T(L) := 0, \tag{2}$$

where $\Delta L_{\tau_{k+1}} := L_{\tau_{k+1}} - L_{\tau_k}$.

In order to analyze continuous-time limits of sequences of discrete-time value processes we will need a stability property with respect to small perturbations of α_{δ_k} in the conditional risk measure $\mathrm{VaR}_{\tau_k,1-\alpha_{\delta_k}}$. Therefore, we introduce the notion of lower and upper one-step valuation mappings and, in Definition 3, lower and upper discrete-time value processes.

For $\beta \in (0,1)$ and $Y \in L^1(\mathcal{F}_{\tau_k+\delta_k})$, the lower and upper one-step valuation mappings are defined as

$$\begin{aligned}\breve{W}^{\delta_k,\beta}_{\tau_k}(Y) &:= W^{\delta_k}_{\tau_k}(Y) \\ &\quad + \frac{1-\alpha_{\delta_k}+\eta_{\tau_k}}{1+\eta_{\tau_k}}\big(\mathrm{VaR}_{\tau_k,1-\alpha_{\delta_k}+\delta_k^{1+\beta}}(-Y) - \mathrm{VaR}_{\tau_k,1-\alpha_{\delta_k}}(-Y)\big),\\ \widehat{W}^{\delta_k,\beta}_{\tau_k}(Y) &:= W^{\delta_k}_{\tau_k}(Y) \\ &\quad + \frac{1-\alpha_{\delta_k}+\eta_{\tau_k}}{1+\eta_{\tau_k}}\big(\mathrm{VaR}_{\tau_k,1-\alpha_{\delta_k}-\delta_k^{1+\beta}}(-Y) - \mathrm{VaR}_{\tau_k,1-\alpha_{\delta_k}}(-Y)\big).\end{aligned}$$

By the same arguments as in the proof of Theorem 1, $\breve{W}^{\delta_k,\beta}_{\tau_k}$ and $\widehat{W}^{\delta_k,\beta}_{\tau_k}$ are mappings from $L^1(\mathcal{F}_{\tau_k+\delta_k})$ to $L^1(\mathcal{F}_{\tau_k})$. In particular, the lower and upper value process $\{\check{V}^{\tau,\beta}_t(L)\}_{t\in\tau}$ and $\{\widehat{V}^{\tau,\beta}_t(L)\}_{t\in\tau}$ may be defined analogously to the definition of $\{V^{\tau}_t(L)\}_{t\in\tau}$.

Definition 3. *Given $L \in L^1(\mathbb{F})$ and a partition τ of $[0,T]$, $0 = \tau_0 < \cdots < \tau_m = T$, the lower and upper value process $\{\check{V}^{\tau,\beta}_t(L)\}_{t\in\tau}$ and $\{\widehat{V}^{\tau,\beta}_t(L)\}_{t\in\tau}$ of L with respect to the partition τ and filtration $\mathbb{F}$ are given by*

$$\begin{aligned}\check{V}^{\tau,\beta}_{\tau_k}(L) &:= \breve{W}^{\delta_k,\beta}_{\tau_k}(\Delta L_{\tau_{k+1}} + \check{V}^{\tau,\beta}_{\tau_{k+1}}(L)), \quad \check{V}^{\tau,\beta}_{T}(L) := 0,\\ \widehat{V}^{\tau,\beta}_{\tau_k}(L) &:= \widehat{W}^{\delta_k,\beta}_{\tau_k}(\Delta L_{\tau_{k+1}} + \widehat{V}^{\tau,\beta}_{\tau_{k+1}}(L)), \quad \widehat{V}^{\tau,\beta}_{T}(L) := 0.\end{aligned}$$

Notice that $\check{V}^{\tau,\beta}_t(L) \leqslant V^{\tau}_t(L) \leqslant \widehat{V}^{\tau,\beta}_t(L)$ for all $t \in \tau$.

The purpose of this paper is to study the behavior of the discrete-time value processes when varying the partition τ of $[0,T]$ and in particular the convergence to and properties of continuous-time value processes when letting the mesh of the partition tend to zero. Serious modeling of the exogenously given random sequence $\{\eta_{\tau_k}\}_{k=0}^{m-1}$ of cost-of-capital rates requires modeling of how the risk aversion of the capital provider varies over time and also mechanisms for competition between capital providers. Moreover, realistic modeling of the random sequences $\{\eta_{\tau_k}\}_{k=0}^{m-1}$ as the partition τ of $[0,T]$ is replaced by a sequence of partitions $\{\tau_m\}_{m=1}^{\infty}$ is not straightforward. Those aspects of the valuation are not within the scope of the current paper. Therefore, we will throughout the remainder of this paper make the simplifying assumption that $\eta_{\tau_k} \equiv \eta_{\delta_k}$ is nonrandom and depends only on the length δ_k of the subinterval $[\tau_k, \tau_{k+1})$ and not on τ_k.

Assumption 1. *$\eta_{\tau_k} \equiv \eta_{\delta_k}$ is nonrandom and depends only on the length δ_k of the subinterval $[\tau_k, \tau_{k+1})$.*

We will consider sequences $\{\tau_m\}_{m=1}^{\infty}$ of partitions of $[0,T]$, $0 = \tau_{m,0} < \cdots < \tau_{m,m} = T$, with $\lim_{m\to\infty} \mathrm{mesh}(\tau_m) = 0$. With $\delta_{m,k} := \tau_{m,k+1} - \tau_{m,k}$ we further assume the existence of sequences $\{\alpha_{\delta_{m,k}}\}_{k=0}^{m-1}$ and $\{\eta_{\delta_{m,k}}\}_{k=0}^{m-1}$, of nonrandom elements $\alpha_{\delta_{m,k}}, \eta_{\delta_{m,k}} \in (0,1)$, for $m \geqslant 1$, such that, for $k = 0,\dots,m-1$,

$$\lim_{m\to\infty} \sup_{k\leqslant m-1} \left|\frac{1-\alpha_{\delta_{m,k}}}{\delta_{m,k}} + \log(\alpha)\right| = \lim_{m\to\infty} \sup_{k\leqslant m-1} \left|\frac{\eta_{\delta_{m,k}}}{\delta_{m,k}} - \log(1+\eta)\right| = 0 \tag{3}$$

for some $\alpha, \eta \in (0,1)$. For the convergence from discrete to continuous time to make economic sense, the limits

$$\lim_{m\to\infty} \prod_{k=0}^{m-1} \alpha_{\delta_{m,k}}, \quad \lim_{m\to\infty} \prod_{k=0}^{m-1} (1+\eta_{\delta_{m,k}})$$

should exists finitely and be strictly positive. For the first limit, the interpretation is that the probability that the repeated capital injections are sufficient throughout the time period $[0,T]$ is some number strictly between zero and one. Similarly, for the second limit the interpretation is that the capital provider's aggregate expected return on the repeated capital injections is finite. It is shown in Lemma 11 that (3) implies

$$\lim_{m\to\infty} \prod_{k=0}^{m-1} \alpha_{\delta_{m,k}} = \alpha^T, \quad \lim_{m\to\infty} \prod_{k=0}^{m-1} (1+\eta_{\delta_{m,k}}) = (1+\eta)^T.$$

Remark 2. *In our setup, as well as in [15] and [21], discrete-time multi-period risk-adjusted values, given a partition $\tau = \{\tau_k\}_{k=0}^m$ of $[0,T]$, may be expressed as*

$$\Phi^\tau_{\tau_k}(X) = \varphi_{\tau_k,\tau_{k+1}}(\Phi^\tau_{\tau_{k+1}}(X)), \quad \Phi^\tau_T(X) = X_T, \tag{4}$$

where the mapping $\varphi_{\tau_k,\tau_{k+1}}$ is a mapping from a subspace of $L^0(\mathcal{F}_{\tau_{k+1}})$ to a subspace of $L^0(\mathcal{F}_{\tau_k})$. In our setting,

$$X = L, \quad \Phi^\tau_{\tau_k}(X) = L_{\tau_k} + V^\tau_{\tau_k}(L), \quad \varphi_{\tau_k,\tau_{k+1}} = W^{\delta_k}_{\tau_k},$$

where $\delta_k = \tau_{k+1} - \tau_k$ and

$$\begin{aligned} W^{\delta_k}_{\tau_k}(Y) &= \mathrm{VaR}_{\tau_k,1-\alpha_{\delta_k}}(-Y) - \frac{1}{1+\eta_{\delta_k}} \mathbb{E}_{\tau_k}\big[(\mathrm{VaR}_{\tau_k,1-\alpha_{\delta_k}}(-Y) - Y)_+\big] \\ &= \frac{1}{1+\eta_{\delta_k}} \Big\{ \mathbb{E}_{\tau_k}[Y] - (1-\alpha_{\delta_k})\,\mathrm{ES}_{\tau_k,1-\alpha_{\delta_k}}(-Y) \\ &\qquad + (1-\alpha_{\delta_k}+\eta_{\delta_k})\,\mathrm{VaR}_{\tau_k,1-\alpha_{\delta_k}}(-Y) \Big\} \end{aligned}$$

with $1-\alpha_\delta \sim \delta\log\alpha$ and $\eta_\delta \sim \delta\log(1+\eta)$ as $\delta \to 0$. Notice from the last expression above for $W^{\delta_k}_{\tau_k}(Y)$ that for very small values η_{δ_k}, $W^{\delta_k}_{\tau_k}(Y) < \mathbb{E}_{\tau_k}[Y]$. This inequality is a consequence of the limited liability for the capital *provider. Notice also that for* $W^{\delta_k}_{\tau_k}(Y) > \mathbb{E}_{\tau_k}[Y]$ *for at least moderately large values* η_{δ_k}*.*

In [15], actuarial valuation rules are extended from discrete to continuous time. Modified to our setting where financial values are expressed in the numéraire, the valuation rule most similar to the one considered here is

$$\begin{aligned}\varphi_{\tau_k,\tau_{k+1}}(Y) &= \mathbb{E}_{\tau_k}[Y] + \eta\sqrt{\delta_k}\,\mathrm{VaR}_{\tau_k,1-\alpha}(-Y - \mathbb{E}_{\tau_k}[-Y]) \\ &= (1-\eta\sqrt{\delta_k})\mathbb{E}_{\tau_k}[Y] + \eta\sqrt{\delta_k}\,\mathrm{VaR}_{\tau_k,1-\alpha}(-Y), \end{aligned} \tag{5}$$

where $\eta, \alpha \in (0,1)$ *are fixed constants. Notice that* $\varphi_{\tau_k,\tau_{k+1}}(Y) > \mathbb{E}_{\tau_k}[Y]$ *if* $\mathrm{VaR}_{\tau_k,1-\alpha}(-Y) > \mathbb{E}_{\tau_k}[Y]$*. Although the expressions for* $\varphi_{\tau_k,\tau_{k+1}}$ *may appear similar, they are fundamentally different. The mapping* $\varphi_{\tau_k,\tau_{k+1}}$ *in [15] is a priori given by an actuarial valuation rule whereas in our case it is the result of the capital providers' acceptability condition for financing the repeated capital requirements, taking the capital providers' limited liability into account.*

In [21], a negative liability value corresponds to a positive value in our setting, and vice versa. The mappings $\varphi_{\tau_k,\tau_{k+1}}$ *in [21], modified to our sign convention, are of the form*

$$\varphi_{\tau_k,\tau_{k+1}}(Y) = (1-\sqrt{\delta_k})\mathbb{E}_{\tau_k}[Y] - \delta_k F_{\tau_k}(-Y/\sqrt{\delta_k})$$

and F_{τ_k} *may be chosen as*

$$F_{\tau_i}(Y) = \mathbb{E}_{\tau_i}[Y] - \eta\,\mathrm{VaR}_{\tau_i,1-\alpha}(Y - \mathbb{E}_{\tau_i}[Y])$$

which gives

$$\varphi_{\tau_k,\tau_{k+1}}(Y) = (1-\eta\sqrt{\delta_k})\mathbb{E}_{\tau_k}[Y] + \eta\sqrt{\delta_k}\,\mathrm{VaR}_{\tau_k,1-\alpha}(-Y)$$

which coincides with (5)*.*

Notice that in [15] and [21] the quantities $\mathbb{E}_{\tau_k}[Y]$ *and* $\mathrm{VaR}_{\tau_k,1-\alpha}(-Y)$ *appearing in the mapping* $\varphi_{\tau_k,\tau_{k+1}}(Y)$ *are scaled appropriately in order to obtain convergence of discrete-time value processes to continuous-time value processes, and* α *and* η *are constants that do not depend on the partition of the time interval* $[0,T]$*. In our setting, the sequences* $\{\alpha_{\delta_k}\}$ *and* $\{\eta_{\delta_k}\}$ *are chosen so that, regardless of the partition of* $[0,T]$*, there is a reasonable non-trivial probability of successful financing of the capital requirements through the entire time period and a reasonable expected excess return to the capital providers for providing capital. We find that our approach is more natural from an economic perspective.*

3 Continuous-time value processes

This section contains the main results of the paper. We first define the continuous-time limit of a sequence of discrete-time value processes of a given continuous-time liability cash flow, where the sequence of discrete-time value processes corresponds to a sequence of partitions of $[0,T]$ with meshes tending to zero.

Definition 4. *Given a sequence $\{\tau_m\}_{m=1}^{\infty}$ of partitions of $[0,T]$, $0=\tau_{m,0}<\dots<\tau_{m,m}=T$, such that $\lim_{m\to\infty}\mathrm{mesh}(\tau_m)=0$, the stochastic process $\{V_t(L)\}_{t\in[0,T]}$ is the continuous-time limit of the sequence of discrete-time value processes $\{V_t^{\tau_m}(L)\}_{t\in\tau_m}$ if*

$$\sup_{t\in\tau_m}\big|V_t^{\tau_m}(L)-V_t(L)\big|\to 0 \text{ a.s. as } m\to\infty.$$

Recall from Section 2 that

$$W_{\tau_{m,k}}^{\delta_{m,k}}(Y)=\frac{1}{1+\eta_{\delta_{m,k}}}\Big\{\mathbb{E}_{\tau_{m,k}}[Y]-(1-\alpha_{\delta_{m,k}})\,\mathrm{ES}_{\tau_{m,k},1-\alpha_{\delta_{m,k}}}(-Y)\\ +(1-\alpha_{\delta_{m,k}}+\eta_{\delta_{m,k}})\,\mathrm{VaR}_{\tau_{m,k},1-\alpha_{\delta_{m,k}}}(-Y)\Big\}$$

and, by (2),

$$V_{\tau_{m,k}}^{\tau}(L)=W_{\tau_{m,k}}^{\delta_{m,k}}\circ\dots\circ W_{\tau_{m,m-1}}^{\delta_{m,m-1}}(L_T-L_{\tau_{m,k}}). \tag{6}$$

Motivated by economic arguments, we have assumed that $1-\alpha_{\delta_{m,k}}$ and $\eta_{\delta_{m,k}}$ are both of order $\delta_{m,k}$. For some stochastic processes, precise details are provided below, the aggregate contribution to the value $V_{\tau_{m,k}}^{\tau}(L)$ from $\mathrm{ES}_{\tau_{m,i},1-\alpha_{\delta_{m,i}}}$ and $\mathrm{VaR}_{\tau_{m,i},1-\alpha_{\delta_{m,i}}}$ for $i>k$ will be asymptotically negligible as $m\to\infty$. In this case, asymptotically as $m\to\infty$, (6) collapses into a composition of conditional expectations which, by the tower property of conditional expectations, is simply a conditional expectation of the remaining cash flow. Heuristically, cash flow models of e.g. diffusion-process type give asymptotically negligible risk (VaR and ES) contributions to the liability value, whereas cash flow models allowing for jumps (with sufficiently high probability) give nonnegligible risk contributions to the liability value. Precise statements are found in Theorems 2, 3 and 4 below.

Notice also from the representation of the one-step valuation mapping $W_{\tau_{m,k}}^{\delta_{m,k}}$ that a discrete-time value is not additive, $V_{\tau_{m,k}}^{\tau}(X+L)\neq V_{\tau_{m,k}}^{\tau}(X)+V_{\tau_{m,k}}^{\tau}(L)$, and not even subadditive in general. Theorem 2 below gives sufficient conditions under which the continuous-time limit is additive, i.e. $V_t(X+L)=V_t(X)+V_t(L)$. This property does not hold in general.

The following technical result, Lemma 1, is a key result for proving convergence of a sequence of discrete-time value process to a continuous-time limit process. Its main feature is that it enables explicit control of error terms appearing in the sequence of recursions leading to the continuous-time limit process. The reason why this result is placed here and not in Section 5 is that it is instructive to highlight the statement of Lemma 1 which provides the induction step that is key to proving Theorem 2 below.

We use the notation $f(\delta) \in o(\delta)$ and $g(\delta) \in o(1)$ for a functions f, g satisfying $\lim_{\delta\to 0} f(\delta)/\delta = 0$ and $\lim_{\delta\to 0} g(\delta) = 0$.

Lemma 1. *Let $L = \{L_t\}_{t\in[0,T]}$ and $X = \{X_t\}_{t\in[0,T]}$ be processes in $L^1(\mathbb{F})$. Suppose that there exist constants $\delta_0 \in (0, 1/2)$, $u \in (0, 2)$, $\varepsilon \in (0, 1)$ and $C_1, C_2 > 0$ such that for $\delta \in (0, \delta_0)$ and for any $y \geqslant \delta^{(u-\varepsilon u)/2}$ and any $t \in [0, T-\delta]$*

$$\mathbb{P}_t\Big(|X_{t+\delta} - X_t| > y(1 + |X_t|)\Big) \leqslant C_1\delta^2 y^{-2/u}, \tag{7}$$

$$\mathbb{P}_t\Big(|X^2_{t+\delta} - X^2_t| > y(1 + |X_t|)^2\Big) \leqslant C_1\delta^2 y^{-2/u}, \tag{8}$$

$$\mathbb{P}_t\Big(\big|\mathbb{E}_{t+\delta}[X_T - X_{t+\delta}] - \mathbb{E}_t[X_T - X_t]\big| > y(1 + |X_t|)\Big) \leqslant C_1\delta^2 y^{-2/u}, \tag{9}$$

$$\big|\mathbb{E}_t[X^2_{t+\delta} - X^2_t]\big| \leqslant C_2\delta(1 + X^2_t). \tag{10}$$

Let τ be a partition of $[0, T]$, $0 = \tau_0 < \cdots < \tau_m = T$, and let $0 < \varepsilon'' < \varepsilon' < \varepsilon$. If, for some $i \in \{0, \ldots, m-1\}$, there exists a constant $A_{\tau_{i+1}} \geqslant 0$ such that

$$\widehat{V}^{\tau,\varepsilon'}_{\tau_{i+1}}(X + L) \leqslant \mathbb{E}_{\tau_{i+1}}[X_T - X_{\tau_{i+1}}] + A_{\tau_{i+1}}(1 + X^2_{\tau_{i+1}}) + \widehat{V}^{\tau,\varepsilon''}_{\tau_{i+1}}(L), \tag{11}$$

$$\check{V}^{\tau,\varepsilon'}_{\tau_{i+1}}(X + L) \geqslant \mathbb{E}_{\tau_{i+1}}[X_T - X_{\tau_{i+1}}] - A_{\tau_{i+1}}(1 + X^2_{\tau_{i+1}}) + \check{V}^{\tau,\varepsilon''}_{\tau_{i+1}}(L), \tag{12}$$

then, for $\tau_{i+1} - \tau_i$ sufficiently small,

$$\widehat{V}^{\tau,\varepsilon'}_{\tau_i}(X + L) \leqslant \mathbb{E}_{\tau_i}[X_T - X_{\tau_i}] + A_{\tau_i}(1 + X^2_{\tau_i}) + \widehat{V}^{\tau,\varepsilon''}_{\tau_i}(L), \tag{13}$$

$$\check{V}^{\tau,\varepsilon'}_{\tau_i}(X + L) \geqslant \mathbb{E}_{\tau_i}[X_T - X_{\tau_i}] - A_{\tau_i}(1 + X^2_{\tau_i}) + \check{V}^{\tau,\varepsilon''}_{\tau_i}(L), \tag{14}$$

where, for some constant $B > 0$ and $f(\delta) \in o(\delta)$,

$$A_{\tau_i} = A_{\tau_{i+1}}(1 + B(\tau_{i+1} - \tau_i)) + f(\tau_{i+1} - \tau_i). \tag{15}$$

The following result, which relies strongly on Lemma 1, gives mild sufficient conditions under which the continuous-time value process $V(X + L)$ of a sum of two cash flows X and L decomposes into a sum $V(X) + V(L)$ of the continuous-time value processes of the two cash flows. Moreover, by considering the special case $L = 0$, it gives sufficient conditions under which the continuous-time value collapses into a conditional expectation of the remaining cash flow: $V_t(X) = \mathbb{E}_t[X_T - X_t]$. Notice that in Theorem 2 below, we make no assumptions about independence or some form of dependence between the processes L and X.

Theorem 2. *Let $L = \{L_t\}_{t\in[0,T]}$ and $X = \{X_t\}_{t\in[0,T]}$ be processes in $L^1(\mathbb{F})$. Suppose that there exist constants $\delta_0 \in (0, 1/2)$, $u \in (0, 2)$, $\varepsilon \in (0, 1)$ and $C_1, C_2 > 0$ such that for $\delta \in (0, \delta_0)$ and for any $y \geqslant \delta^{(u-\varepsilon u)/2}$ and any $t \in [0, T-\delta]$, X satisfies conditions (7)-(10) in Lemma 1. Suppose that,* *for some* $\beta_2 \in (0,\varepsilon)$*, and any sequence* $\{\tau_m\}_{m=1}^{\infty}$ *of partitions of* $[0,T]$*,* $0 = \tau_{m,0} < \cdots < \tau_{m,m} = T$ *with* $\lim_{m\to\infty} \mathrm{mesh}(\tau_m) = 0$*,*

$$\sup_{t\in\tau_m} \left|V_t^{\tau_m}(L) - V_t(L)\right| \to 0 \ a.s. \ as\ m \to \infty,$$
$$\sup_{t\in\tau_m} \left|\widehat{V}_t^{\tau_m,\beta_2}(L) - \check{V}_t^{\tau_m,\beta_2}(L)\right| \to 0 \ a.s. \ as\ m \to \infty.$$

If further $\sup_{t\in[0,T]} X_t^2 < \infty$*, then for any* $\beta_1 \in (\beta_2,\epsilon)$ *and any sequence* $\{\tau_m\}_{m=1}^{\infty}$ *of partitions of* $[0,T]$*,* $0 = \tau_{m,0} < \cdots < \tau_{m,m} = T$*, such that* $\lim_{m\to\infty} \mathrm{mesh}(\tau_m) = 0$*,*

$$\sup_{t\in\tau_m} \left|V_t^{\tau_m}(X+L) - \mathbb{E}_t[X_T - X_t] - V_t(L)\right| \to 0 \ a.s. \ as\ m \to \infty,$$
$$\sup_{t\in\tau_m} \left|\widehat{V}_t^{\tau_m,\beta_1}(X+L) - \check{V}_t^{\tau_m,\beta_1}(X+L)\right| \to 0 \ a.s. \ as\ m \to \infty.$$

Notice the following consequence of repeated application of Theorem 2.

Corollary 1. *Let* $L = \{L_t\}_{t\in[0,T]}$ *and* $X^{(k)} = \{X_t^{(k)}\}_{t\in[0,T]}$*,* $k = 1,\dots,n$*, be processes in* $L^1(\mathbb{F})$ *such that the requirements of Theorem 2 hold for each pair* $(L, X^{(k)})$*,* $k = 1,\dots,n$*. Then, as* $m \to \infty$*,*

$$\sup_{t\in\tau_m} \left|V_t^{\tau_m}\Big(\sum_{k=1}^{n} X^{(k)} + L\Big) - \sum_{k=1}^{n} \mathbb{E}_t\big[X_T^{(k)} - X_t^{(k)}\big] - V_t(L)\right| \to 0 \ a.s.$$

Next we present an example of a wide class of stochastic processes X which satisfies conditions (7)-(10) in Lemma 1 and Theorem 2. These processes are strong solutions to stochastic differential equations driven by Brownian motion, see (18) below.

Let $\mu, \sigma : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ be jointly measurable and satisfy the uniform Lipschitz type growth conditions

$$\mu(t,x)^2 + \sigma(t,x)^2 < K_1(1+x^2) \tag{16}$$
$$|\mu(t,x) - \mu(t,y)| + |\sigma(t,x) - \sigma(t,y)| < K_1|x-y|, \tag{17}$$

for some constant $K_1 > 0$. Let B be an $\mathbb{R}$-valued $\mathbb{F}$-adapted Brownian motion and consider the stochastic differential equation

$$\mathrm{d}X_t = \mu(t,X_t)\mathrm{d}t + \sigma(t,X_t)\mathrm{d}B_t, \quad X_0 = x_0. \tag{18}$$

Conditions (16) and (17) ensure that (18) has a unique strong solution which is a strong Markov process (see Theorem E7 in [5], Appendix E). Moreover, (16) and (17) together imply that the solution X to (18) is in $L^1(\mathbb{F})$.

The following result gives sufficient conditions on the process $\{X_t\}_{t\in[0,T]}$, which is the strong solution to (18), under which $\{X_t\}_{t\in[0,T]}$ satisfies the conditions in Theorem 2.

Theorem 3. *Let* $\{X_t\}_{t\in[0,T]}$ *be the strong solution to* (18) *with* μ *and* σ *satisfying* (16) *and* (17) *and set* $u(t,x) := \mathbb{E}[X_T - X_t \mid X_t = x]$. *If there exists* $K_2 > 0$ *such that* u *satisfies either*

$$|u(t,x) - u(s,y)| < K_2(|t-s|(1+|x|) + |x-y|) \tag{19}$$

for all $(t,x),(s,y) \in [0,T]\times\mathbb{R}$, *or*

$$\Big|\frac{\partial}{\partial x}u(x,t)\Big| < K_2 \tag{20}$$

for all $(t,x) \in [0,T]\times\mathbb{R}$, *then* $\{X_t\}_{t\in[0,T]}$ *satisfies* (7)-(10) *for* $u = 1/2$ *and any* $\varepsilon \in (0,1)$.

Below we give an example of a fairly general class of Itô processes for which both (19) and (20) hold.

Example 1. *Consider a process* X *given by the SDE*

$$\mathrm{d}X_t = \big(a(t) + b(t)X_t\big)\mathrm{d}t + \sigma(t,X_t)\mathrm{d}B_t, \quad X_0 = x_0. \tag{21}$$

The functions a *and* b *are assumed to be continuous and* σ *is assumed to satisfy the usual Lipschitz and linear growth conditions, ensuring existence of a strong solution. Then,* $\widetilde{u}(t,x) := u(t,x) - x = \mathbb{E}[X_T \mid X_t = x]$ *is given by the Feynman-Kac equation*

$$\frac{\partial\widetilde{u}}{\partial t} + (a(t) + b(t)x)\frac{\partial\widetilde{u}}{\partial x} + \frac{\sigma^2(t,x)}{2}\frac{\partial^2\widetilde{u}}{\partial x^2} = 0,$$

which has the easily verifiable solution $\widetilde{u}(t,x) = A(t) + B(t)x$, *where*

$$B(t) = \exp\Big\{-\int_t^T b(s)\mathrm{d}s\Big\}, \quad A(t) = \int_t^T a(s)B(s)\mathrm{d}s.$$

As this yields $u(t,x) = A(t) + (B(t)-1)x$, *with* A, B *being Lipschitz continuous functions on* $[0,T]$, *it is easily seen that* u *satisfies* (20). *Furthermore,* u *also satisfies* (19)*:*

$$\begin{aligned}|u(t,x) - u(s,y)| &\leqslant |A(t) - A(s)| + |B(s)||x-y| + |B(t) - B(s)||x| \\ &\leqslant K_{A,B}(|s-t|(1+|x|) + |x-y|),\end{aligned}$$

where $K_{A,B}$ *is a Lipschitz constant for both* A *and* B.

Due to the independent increment property, additive processes (see Chapter 2 in [20]) provide examples of stochastic processes $\{L_t\}_{t\in[0,T]}$ for which the sequence discrete-time value processes converges to an explicit continuous-time limit $\{V_t(L)\}_{t\in[0,T]}$, where $V_t(L)$ is not equal to $\mathbb{E}_t[L_T - L_t]$.

Theorem 4. *Let $\{L_t\}_{t\in[0,T]}$ be an $\mathbb{R}$-valued additive process in $L^0(\mathbb{F})$ with system of generating triplets $\{(\sigma_t^2, \nu_t, \gamma_t)\}_{t\in[0,T]}$. For each $t \in [0,T]$, let $\dot\sigma_t^2$ and $\dot\gamma_t$ be constants and let $\dot\nu_t$ be a measure on $\mathbb{R}\backslash\{0\}$ whose restrictions to sets bounded away from 0 are finite.*

Consider the following statements:

(i) For each $t \in [0,T]$,

$$\frac{1}{\delta}\left(\sigma_{s+\delta}^2 - \sigma_s^2, \nu_{s+\delta} - \nu_s, \gamma_{s+\delta} - \gamma_s\right) \to \left(\dot\sigma_t^2, \dot\nu_t, \dot\gamma_t\right) \quad \text{as } \delta \downarrow 0, s \to t \tag{22}$$

where the convergence in the second component means that

$$\lim_{\delta\downarrow 0, s\to t} \frac{1}{\delta}\int_{\mathbb{R}\backslash\{0\}} f(x)(\nu_{s+\delta} - \nu_s)(\mathrm{d}x) = \int_{\mathbb{R}\backslash\{0\}} f(x)\dot\nu_t(\mathrm{d}x)$$

for all bounded and continuous functions vanishing in a neighborhood of 0.

(ii) For each $t \in [0,T)$,

$$\lim_{\varepsilon\downarrow 0}\limsup_{\delta\downarrow 0, s\to t}\int_{[-\varepsilon,\varepsilon]} x^2\frac{1}{\delta}(\nu_{s+\delta} - \nu_s)(\mathrm{d}x) = 0. \tag{23}$$

(iii) $[0,T] \times (0,\infty) \ni (t,x) \mapsto \dot\nu_t(x,\infty) \in (0,\infty)$ is continuous and $x \mapsto \dot\nu_t(x,\infty)$ is strictly decreasing on $(0,\infty)$.

(iv) For some $\underline{q} \in (0,\infty)$ and for each $t \in [0,T]$, there exists $q_t \in (\underline{q},\infty)$ solving $\dot\nu_t(q_t,\infty) = -\log\alpha$.

(v) $\sup_{\delta\in(0,T]}\sup_{t\in[0,T-\delta]}\delta^{-1}\mathbb{E}[(\Delta L_{t+\delta})^2] < \infty$.

Let $\{\tau_m\}_{m=1}^\infty$ be a sequence of partitions of $[0,T]$, $0 = \tau_{m,0} < \cdots < \tau_{m,m} = T$, such that $\lim_{m\to\infty}\mathrm{mesh}(\tau_m) = 0$. Let $\{V_t^{\tau_m}(L)\}_{t\in\tau_m}$ be given by (2) for $\tau = \tau_m$, with sequences $\{\alpha_{\delta_{m,k}}\}$ and $\{\eta_{\delta_{m,k}}\}$ satisfying (3). If (i)-(v) hold, then $L \in L^1(\mathbb{F})$ and

$$\sup_{t\in\tau_m}\left|V_t^{\tau_m}(L) - \mathbb{E}[L_T - L_t] - \int_t^T K_L(s)\mathrm{d}s\right| \to 0 \text{ a.s. as } m \to \infty, \tag{24}$$

where $K_L(t)$ is given by

$$K_L(t) = \log(1+\eta)q_t - \int_{q_t}^\infty \dot\nu_t(x,\infty)\mathrm{d}x. \tag{25}$$

Moreover, for $\beta \in (0,\infty)$,

$$\sup_{t\in\tau_m}\left|\widehat{V}_t^{\tau_m,\beta}(L) - \check{V}_t^{\tau_m,\beta}(L)\right| \to 0 \text{ a.s. as } m \to \infty.$$

For the special case of Lévy processes or processes obtained by deterministic time-changes of Lévy processes, Theorem 4 simplifies considerably. Notice that a Lévy process is an additive process with system of generating triplets $\{(\sigma_t^2, \nu_t, \gamma_t)\} = \{(t\sigma^2, t\nu, t\gamma)\}$.

Corollary 2. *Let $\{\widetilde{L}_t\}_{t\in[0,\infty)}$ be an $\mathbb{R}$-valued Lévy process with generating triplet $\{(\sigma^2, \nu, \gamma)\}$ and with respect to a filtration $\mathbb{G} = \{\mathcal{G}_t\}_{t\in[0,\infty)}$. Consider the following statements:*

(i) $\int_{\mathbb{R}} x^2\nu(\mathrm{d}x) < \infty$ *and* $\lim_{\varepsilon\downarrow 0}\int_{[-\varepsilon,\varepsilon]} x^2\nu(\mathrm{d}x) = 0$.

(ii) $x \mapsto \nu(x,\infty)$ *is continuous and strictly decreasing on* $(0,\infty)$.

Let $\lambda : [0,T] \to (0,\infty)$ be continuous, let $\mu : [0,T] \to (0,\infty)$ be given by $\mu(t) = \int_0^T \lambda(s)\mathrm{d}s$ and let $\{L_t\}_{t\in[0,T]}$ be given by $L_t = \widetilde{L}_{\mu(t)}$. Now consider $\{L_t\}_{t\in[0,T]}$ with respect to the filtration $\mathbb{F} := \{\mathcal{G}_{\mu(t)}\}_{t\in[0,T]}$. Let $\{\tau_m\}_{m=1}^{\infty}$ be a sequence of partitions of $[0,T]$, $0 = \tau_{m,0} < \cdots < \tau_{m,m} = T$, such that $\lim_{m\to\infty}\mathrm{mesh}(\tau_m) = 0$. Let $\{V_t^{\tau_m}(L)\}_{t\in\tau_m}$ be given by (2) for $\tau = \tau_m$, with sequences $\{\alpha_{\delta_{m,k}}\}$ and $\{\eta_{\delta_{m,k}}\}$ satisfying (3). If (i)-(ii) hold and, for some $\underline{q} \in (0,\infty)$ and for each $t \in [0,T)$, there exists $q_t > \underline{q}$ solving $\lambda(t)\nu(q_t,\infty) = -\log\alpha$, then (24) holds, where

$$K_L(t) = \log(1+\eta)q_t - \lambda(t)\int_{q_t}^{\infty}\nu(x,\infty)\mathrm{d}x. \tag{26}$$

Example 2. *A compound Poisson process driven by a Poisson process N with mean-value function $t \mapsto \mu(t)$ is an additive process with representation $L_t = \sum_{k=1}^{N_t} Z_k$, where N is independent of the iid sequence $\{Z_k\}$. The Lévy measure of L_t is $\mu(t)P_Z$, where P_Z is the common distribution of the variables Z_k. In the setting of Corollary 2, $\widetilde{L}$ is a compound Poisson process driven by a homogeneous Poisson process $\widetilde{N}$ with unit intensity and representation $\widetilde{L}_t = \sum_{k=1}^{\widetilde{N}_t} Z_k$. If further $1 + (\log\alpha)/\lambda(t) > 0$ for all $t \in [0,T)$, then (24) holds, where*

$$K_L(t) = \log(1+\eta)q_t - \lambda(t)\int_{q_t}^{\infty}\overline{F}_Z(x)\mathrm{d}x, \quad q_t = F_Z^{-1}(1 + (\log\alpha)/\lambda(t)),$$

where $F_Z(z) = P_Z(-\infty, z]$ and $\overline{F}_Z = 1 - F_Z$.

4 Proofs

Proof of Lemma 1. To ease notation, let $t := \tau_i$, $\delta := \delta_i = \tau_{i+1} - \tau_i$, $A := A_{\tau_{i+1}}$ and $U_t := \mathbb{E}_t[X_T - X_t]$. By Lemma 2,

$$\begin{aligned}
&\widehat{V}_t^{\tau,\varepsilon'}(X + L) - \mathbb{E}_t[X_T - X_t] \\
&= \widehat{W}_t^{\delta,\varepsilon'}(\Delta(X + L)_{t+\delta} + \widehat{V}_{t+\delta}^{\tau,\varepsilon'}(X + L) - U_t) \\
&= \frac{1}{1+\eta_\delta}\Big(\mathbb{E}_t[\Delta(X + L)_{t+\delta} + \widehat{V}_{t+\delta}^{\tau,\varepsilon'}(X + L) - U_t] \qquad (27)\\
&\quad - (1-\alpha_\delta)\,\mathrm{ES}_{t,1-\alpha_\delta}\big(-(\Delta(X + L)_{t+\delta} + \widehat{V}_{t+\delta}^{\tau,\varepsilon'}(X + L) - U_t)\big) \qquad (28)\\
&\quad + (1-\alpha_\delta+\eta_\delta)\,\mathrm{VaR}_{t,1-\alpha_\delta-\delta^{1+\varepsilon'}}\big(-(\Delta(X + L)_{t+\delta} + \widehat{V}_{t+\delta}^{\tau,\varepsilon'}(X + L) - U_t)\big)\Big) \qquad (29)
\end{aligned}$$

If we replace $\widehat{\cdot}$ by $\check{\cdot}$, the term (29) is replaced by

$$(1-\alpha_\delta+\eta_\delta)\,\mathrm{VaR}_{t,1-\alpha_\delta+\delta^{1+\varepsilon'}}\big(-(\Delta(X + L)_{t+\delta} + \widehat{V}_{t+\delta}^{\tau,\varepsilon'}(X + L) - U_t)\big). \qquad (30)$$

We now bound (27)-(30) individually. We will use the bounds (11) and (12) repeatedly throughout the arguments.

The term (27)**:** An upper bound for (27) is constructed as follows:

$$\begin{aligned}
&\mathbb{E}_t[\Delta(X_{t+\delta} + L_{t+\delta}) + \widehat{V}_{t+\delta}^{\tau,\varepsilon'}(X + L) - U_t] \\
&\quad \leqslant \mathbb{E}_t[\Delta L_{t+\delta} + \widehat{V}_{t+\delta}^{\tau,\varepsilon''}(L)] + \mathbb{E}_t[\Delta X_{t+\delta} + U_{t+\delta} - U_t + A(1 + X_{t+\delta}^2)] \\
&\quad = \mathbb{E}_t[\Delta L_{t+\delta} + \widehat{V}_{t+\delta}^{\tau,\varepsilon''}(L)] + A(1 + X_t^2 + \mathbb{E}_t[\Delta X_{t+\delta}^2]) \\
&\quad \leqslant \mathbb{E}_t[\Delta L_{t+\delta} + \widehat{V}_{t+\delta}^{\tau,\varepsilon''}(L)] + A(1 + X_t^2 + C_2\delta(1 + X_t^2)) \\
&\quad = \mathbb{E}_t[\Delta L_{t+\delta} + \widehat{V}_{t+\delta}^{\tau,\varepsilon''}(L)] + A(1 + C_2\delta)(1 + X_t^2),
\end{aligned}$$

where (11) was used for the first inequality, the tower property of conditional expectations was used for the equality, (10) was used for the second inequality and Lemma 7 was used for the third inequality. Similarly, a lower bound for (27) if we replace $\widehat{\cdot}$ with $\check{\cdot}$ is

$$\mathbb{E}_t[\Delta L_{t+\delta} + \check{V}_{t+\delta}^{\tau,\varepsilon''}(L)] - A(1 + C_2\delta)(1 + X_t^2).$$

The term (28)**:** We first construct an upper bound for (28). Using (11) and monotonicity and subadditivity of expected shortfall,

$$\begin{aligned}
&\mathrm{ES}_{t,1-\alpha_\delta}(-\Delta(X_{t+\delta}+L_{t+\delta})-\widehat{V}^{\tau,\varepsilon'}_{t+\delta}(X+L)+U_t)\\
&\leqslant \mathrm{ES}_{t,1-\alpha_\delta}(-\Delta L_{t+\delta}-\widehat{V}^{\tau,\varepsilon''}_{t+\delta}(L)-\Delta X_{t+\delta}-U_{t+\delta}+U_t-A(1+X^2_{t+\delta}))\\
&\leqslant \mathrm{ES}_{t,1-\alpha_\delta}(-\Delta L_{t+\delta}-\widehat{V}^{\tau,\varepsilon''}_{t+\delta}(L))\\
&\quad+\mathrm{ES}_{t,1-\alpha_\delta}(-\Delta X_{t+\delta}-U_{t+\delta}+U_t-A(1+X^2_{t+\delta}))\\
&\leqslant \mathrm{ES}_{t,1-\alpha_\delta}(-\Delta L_{t+\delta}-\widehat{V}^{\tau,\varepsilon''}_{t+\delta}(L))+\mathrm{ES}_{t,1-\alpha_\delta}(-\Delta X_{t+\delta})\\
&\quad+\mathrm{ES}_{t,1-\alpha_\delta}(-U_{t+\delta}+U_t)+\mathrm{ES}_{t,1-\alpha_\delta}(-A(1+X^2_{t+\delta}))\\
&\leqslant \mathrm{ES}_{t,1-\alpha_\delta}(-\Delta L_{t+\delta}-\widehat{V}^{\tau,\varepsilon''}_{t+\delta}(L))+\mathrm{ES}_{t,1-\alpha_\delta}(-|\Delta X_{t+\delta}|)\\
&\quad+\mathrm{ES}_{t,1-\alpha_\delta}(-|U_{t+\delta}-U_t|)+\mathrm{ES}_{t,1-\alpha_\delta}(-A(1+X^2_{t+\delta}))
\end{aligned}$$

Notice that

$$\mathrm{ES}_{t,1-\alpha_\delta}(-X^2_{t+\delta})=\mathrm{ES}_{t,1-\alpha_\delta}(-\Delta X^2_{t+\delta}-X^2_t)\leqslant X^2_t+\mathrm{ES}_{t,1-\alpha_\delta}(-|\Delta X^2_{t+\delta}|).$$

Applying Lemma 3 gives, for some function $g(\delta)\in o(1)$ as $\delta\to 0$,

$$\begin{aligned}
&\mathrm{ES}_{t,1-\alpha_\delta}(-\Delta(X_{t+\delta}+L_{t+\delta})-\widehat{V}^{\tau,\varepsilon'}_{t+\delta}(X+L)+U_t)\\
&\leqslant \mathrm{ES}_{t,1-\alpha_\delta}(-\Delta L_{t+\delta}-\widehat{V}^{\tau,\varepsilon''}_{t+\delta}(L))\\
&\quad+2g(\delta)(1+X^2_t)+A(1+g(\delta))(1+X^2_t).
\end{aligned}$$

We now construct a lower bound for (28), replacing $\widehat{\cdot}$ with $\check{\cdot}$. Using (12) and monotonicity and subadditivity of expected shortfall,

$$\begin{aligned}
&\mathrm{ES}_{t,1-\alpha_\delta}(-\Delta(X_{t+\delta}+L_{t+\delta})-\check{V}^{\tau,\varepsilon'}_{t+\delta}(X+L)+U_t)\\
&\geqslant \mathrm{ES}_{t,1-\alpha_\delta}(-\Delta L_{t+\delta}-\check{V}^{\tau,\varepsilon''}_{t+\delta}(L)-\Delta X_{t+\delta}-U_{t+\delta}+U_t+A(1+X^2_{t+\delta}))\\
&\geqslant \mathrm{ES}_{t,1-\alpha_\delta}(-\Delta L_{t+\delta}-\check{V}^{\tau,\varepsilon''}_{t+\delta}(L))\\
&\quad-\mathrm{ES}_{t,1-\alpha_\delta}(\Delta X_{t+\delta}+U_{t+\delta}-U_t-A(1+X^2_{t+\delta}))
\end{aligned}$$

An upper bound for $\mathrm{ES}_{t,1-\alpha_\delta}(\Delta X_{t+\delta}+U_{t+\delta}-U_t-A(1+X^2_{t+\delta}))$, derived as above, gives the lower bound for (28):

$$\begin{aligned}
&\mathrm{ES}_{t,1-\alpha_\delta}(-\Delta(X_{t+\delta}+L_{t+\delta})-\check{V}^{\tau,\varepsilon'}_{t+\delta}(X+L)+U_t)\\
&\geqslant \mathrm{ES}_{t,1-\alpha_\delta}(-\Delta L_{t+\delta}-\check{V}^{\tau,\varepsilon''}_{t+\delta}(L))\\
&\quad-2g(\delta)(2+X^2_t)-A(1+g(\delta))(1+X^2_t).
\end{aligned}$$

Notice that $\lim_{\delta\to 0}\delta^{-1}(1-\alpha_\delta)=-\log\alpha$. Therefore, there exist a function

$f(\delta) \in o(\delta)$ such that for all $\delta > 0$ sufficiently small

$$
\begin{aligned}
&(1-\alpha_\delta)\,\mathrm{ES}_{t,1-\alpha_\delta}(-\Delta(X_{t+\delta}+L_{t+\delta}) - \widehat{V}^{\tau,\varepsilon'}_{t+\delta}(X+L) + U_t)\\
&\quad \leqslant (1-\alpha_\delta)\,\mathrm{ES}_{t,1-\alpha_\delta}(-\Delta L_{t+\delta} - \widehat{V}^{\tau,\varepsilon''}_{t+\delta}(L))\\
&\qquad + ((1-\log\alpha)\delta A + f(\delta))(1+X_t^2),\\
&(1-\alpha_\delta)\,\mathrm{ES}_{t,1-\alpha_\delta}(-\Delta(X_{t+\delta}+L_{t+\delta}) - \check{V}^{\tau,\varepsilon'}_{t+\delta}(X+L) + U_t)\\
&\quad \geqslant (1-\alpha_\delta)\,\mathrm{ES}_{t,1-\alpha_\delta}(-\Delta L_{t+\delta} - \check{V}^{\tau,\varepsilon''}_{t+\delta}(L))\\
&\qquad - ((1-\log\alpha)\delta A + f(\delta))(1+X_t^2).
\end{aligned}
$$

The term (29)**:** Now we construct an upper and a lower bound for (29). Using (11) and monotonicity of value at risk,

$$
\begin{aligned}
&\mathrm{VaR}_{t,1-\alpha_\delta-\delta^{1+\epsilon'}}(-\Delta(X_{t+\delta}+L_{t+\delta}) - \widehat{V}^{\tau,\varepsilon'}_{t+\delta}(X+L) + U_t)\\
&\leqslant \mathrm{VaR}_{t,1-\alpha_\delta-\delta^{1+\epsilon'}}(-\Delta L_{t+\delta} - \widehat{V}^{\tau,\varepsilon''}_{t+\delta}(L) - |U_{t+\delta}-U_t| - A(1+X_{t+\delta}^2) - |\Delta X_{t+\delta}|)\\
&\leqslant \mathrm{VaR}_{t,1-\alpha_\delta-\delta^{1+\epsilon'}}(-\Delta L_{t+\delta} - \widehat{V}^{\tau,\varepsilon''}_{t+\delta}(L) - |\Delta X_{t+\delta}| - |U_{t+\delta}-U_t| - A|\Delta X_{t+\delta}^2|)\\
&\quad + A(1+X_t^2).
\end{aligned}
$$

Similarly

$$
\begin{aligned}
&\mathrm{VaR}_{t,1-\alpha_\delta+\delta^{1+\epsilon'}}(-\Delta(X_{t+\delta}+L_{t+\delta}) - V^{\tau}_{t+\delta}(X+L) + U_t)\\
&\geqslant \mathrm{VaR}_{t,1-\alpha_\delta+\delta^{1+\epsilon'}}(-\Delta L_{t+\delta} - \check{V}^{\tau,\varepsilon''}_{t+\delta}(L) + |\Delta X_{t+\delta}| + |U_{t+\delta}-U_t| + A|\Delta X_{t+\delta}^2|)\\
&\quad - A(1+X_t^2).
\end{aligned}
$$

Applying Lemma 4, for δ sufficiently small, yields the upper bound

$$
\begin{aligned}
&\mathrm{VaR}_{t,1-\alpha_\delta-\delta^{1+\epsilon'}}(-\Delta L_{t+\delta} - \widehat{V}^{\tau,\varepsilon''}_{t+\delta}(L) - |\Delta X_{t+\delta}| - |U_{t+\delta}-U_t| - A|\Delta X_{t+\delta}^2|)\\
&\quad \leqslant \mathrm{VaR}_{t,1-\alpha_\delta-\delta^{1+\varepsilon''}}(-\Delta L_{t+\delta} - \widehat{V}^{\tau,\varepsilon''}_{t+\delta}(L)) + 5\delta^{(u-\epsilon u)/2}(1+A)(1+X_t^2).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\mathrm{VaR}_{t,1-\alpha_\delta+\delta^{1+\epsilon'}}(-\Delta L_{t+\delta} - \check{V}^{\tau,\varepsilon''}_{t+\delta}(L) + |\Delta X_{t+\delta}| + |U_{t+\delta}-U_t| + A|\Delta X_{t+\delta}^2|)\\
&\quad \geqslant \mathrm{VaR}_{t,1-\alpha_\delta+\delta^{1+\epsilon''}}(-\Delta L_{t+\delta} - \check{V}^{\tau,\varepsilon''}_{t+\delta}(L)) - 5\delta^{(u-\epsilon u)/2}(1+A)(1+X_t^2).
\end{aligned}
$$

Notice that $\lim_{\delta\to 0}\delta^{-1}(1-\alpha_\delta+\eta_\delta) = \log(1+\eta) - \log(\alpha)$ and therefore, there exist a function $f(\delta) \in o(\delta)$ such that for all $\delta > 0$ sufficiently small

$$
\begin{aligned}
&(1-\alpha_\delta+\eta_\delta)\,\mathrm{VaR}_{t,1-\alpha_\delta-\delta^{1+\epsilon'}}(-\Delta(X_{t+\delta}+L_{t+\delta}) - V^{\tau}_{t+\delta}(X+L) - U_t)\\
&\quad \leqslant (1-\alpha_\delta+\eta_\delta)\,\mathrm{VaR}_{t,1-\alpha_\delta-\delta^{1+\epsilon''}}(-\Delta L_{t+\delta} - \widehat{V}^{\tau,\varepsilon''}_{t+\delta}(L))\\
&+ A(1+\log(1+\eta)-\log\alpha)\delta(1+X_t^2) + f(\delta)(1+X_t^2),\\
&(1-\alpha_\delta+\eta_\delta)\,\mathrm{VaR}_{t,1-\alpha_\delta+\delta^{1+\epsilon'}}(-\Delta(X_{t+\delta}+L_{t+\delta}) - V^{\tau}_{t+\delta}(X+L) - U_t)\\
&\quad \geqslant (1-\alpha_\delta+\eta_\delta)\,\mathrm{VaR}_{t,1-\alpha_\delta+\delta^{1+\epsilon''}}(-\Delta L_{t+\delta} - \check{V}^{\tau,\varepsilon''}_{t+\delta}(L))\\
&- A(1+\log(1+\eta)-\log\alpha)\delta(1+X_t^2) - f(\delta)(1+X_t^2).
\end{aligned}
$$

Summing up, there exist a function $f(\delta) \in o(\delta)$ such that for any sufficiently small $\delta > 0$, defining $B := C_2 + \log\eta + 2 - 2\log\alpha$,

$$\begin{aligned}
\widehat{V}_t^{\tau,\epsilon'}(X+L) &\leqslant \mathbb{E}_t[X_T - X_t] + \widehat{W}_t^{\delta,\epsilon''}(\Delta L_{t+\delta} + \widehat{V}_{t+\delta}^{\tau,\varepsilon''}(L)) \\
&\quad + A(1 + (C_2 + \log\eta + 2 - 2\log\alpha)\delta) + f(\delta), \\
&= \mathbb{E}_t[X_T - X_t] + \widehat{V}_t^{\tau,\epsilon''}(L) + A(1 + B\delta) + f(\delta), \\
V_t^{\tau}(X+L) &\geqslant \mathbb{E}_t[X_T - X_t] + \widecheck{W}_t^{\delta,\epsilon''}(\Delta L_{t+\delta} + \widecheck{V}_{t+\delta}^{\tau,\varepsilon''}(L)) \\
&\quad - A(1 + (C_2 + \log\eta + 2 - 2\log\alpha)\delta) + f(\delta) \\
&= \mathbb{E}_t[X_T - X_t] + \widecheck{V}_t^{\tau,\epsilon''}(L) - A(1 + B\delta) + f(\delta).
\end{aligned}$$

The proof is complete. □

Proof of Theorem 2. Consider any sequence of partitions $\{\tau_m\}_{m=1}^{\infty}$. Take m sufficiently large so that $\mathrm{mesh}(\tau_m)$ is small enough for the statements of Theorem 1 to hold for each $t \in \tau_m$, with respect to the triple $\beta_2 < \beta_1 < \epsilon$ and a function $f(\delta) \in o(\delta)$. We show via backward induction that

$$\widehat{V}_{\tau_{m,i}}^{\tau_m,\beta_1}(X+L) \leqslant \mathbb{E}_{\tau_{m,i}}[X_T - X_{\tau_i}] + A_{\tau_{m,i}}(1 + X_{\tau_{m,i}}^2) + \widehat{V}_{\tau_{m,i}}^{\tau_m,\beta_2}(L), \quad (31)$$

$$\widecheck{V}_{\tau_i}^{\tau_m,\beta_1}(X+L) \geqslant \mathbb{E}_{\tau_{m,i}}[X_T - X_{\tau_i}] - A_{\tau_{m,i}}(1 + X_{\tau_{m,i}}^2) + \widecheck{V}_{\tau_{m,i}}^{\tau_m,\beta_2}(L). \quad (32)$$

However, since the induction base $i = m$ is trivial and the induction step follows immediately from Lemma 1, (31) and (32) immediately follows. Now we note, by Lemma 5, that there exists $h(\delta) \in o(1)$ such that for each m large enough and $k = 0, ..., m$, $A_{\tau_{m,k}} \leqslant h(\mathrm{mesh}(\tau_m))$. Hence

$$\begin{aligned}
&\sup_{t\in\tau_m} \left| V_t^{\tau_m}(X+L) - \mathbb{E}_t[X_T - X_t] - V_t(L) \right| \\
&\leqslant \sup_{t\in\tau_m} \max\Big(\left| \widehat{V}_t^{\tau_m,\beta_1}(X+L) - \mathbb{E}_t[X_T - X_t] - V_t(L) \right|, \\
&\qquad \left| \widecheck{V}_t^{\tau_m,\beta_1}(X+L) - \mathbb{E}_t[X_T - X_t] - V_t(L) \right| \Big) \\
&\leqslant \sup_{t\in\tau_m} \max\Big(\left| \widehat{V}_t^{\tau_m,\beta_2}(L) - V_t(L) \right|, \left| \widecheck{V}_t^{\tau_m,\beta_2}(L) - V_t(L) \right| \Big) \\
&\qquad + \sup_{k\in\{0,\ldots,m\}} A_{\tau_{m,k}}(1 + X_{\tau_{m,k}}^2) \\
&\leqslant \sup_{t\in\tau_m} \max\Big(\left| \widehat{V}_t^{\tau_m,\beta_2}(L) - V_t(L) \right|, \left| \widecheck{V}_t^{\tau_m,\beta_2}(L) - V_t(L) \right| \Big) \\
&\qquad + h(\mathrm{mesh}(\tau_m))\Big(1 + \sup_{t\in[0,T]} X_t^2\Big).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\sup_{t\in\tau_m} \left| \widehat{V}_t^{\tau_m,\beta_1}(X+L) - \widecheck{V}_t^{\tau_m,\beta_1}(X+L) \right| \\
&\quad \leqslant \sup_{t\in\tau_m} \left| \widehat{V}_t^{\tau_m,\beta_2}(L) - \widecheck{V}_t^{\tau_m,\beta_2}(L) \right| + 2h(\mathrm{mesh}(\tau_m))\Big(1 + \sup_{t\in[0,T]} X_t^2\Big)
\end{aligned}$$

Now, if $\sup_{t\in[0,T]} X_t^2 < \infty$ almost surely, then

$$\sup_{t\in\tau_m} \max\Big(\big|\widehat{V}_t^{\tau_m,\beta_2}(L) - V_t(L)\big|, \big|\check{V}_t^{\tau_m,\beta_2}(L) - V_t(L)\big|\Big) \\ + h(\mathrm{mesh}(\tau_m))\Big(1 + \sup_{t\in[0,T]} X_t^2\Big) \to 0 \text{ a.s. as } m \to \infty$$

and

$$\sup_{t\in\tau_m} \big|\widehat{V}_t^{\tau_m,\beta_2}(L) - \check{V}_t^{\tau_m,\beta_2}(L)\big| + 2h(\mathrm{mesh}(\tau_m))\Big(1 + \sup_{t\in[0,T]} X_t^2\Big) \\ \to 0 \text{ a.s. as } m \to \infty.$$

This completes the proof. □

Proof of Theorem 3. Consider any $\varepsilon \in (0,1)$. By Lemma 6, for all $y \geqslant \delta^{(1-\varepsilon)/4}$ and $t \in [0, T-\delta]$, for $\delta > 0$ sufficiently small,

$$\mathbb{P}_t\Big(\sup_{s\in[t,t+\delta]} |X_s - X_t| > y(1+|X_t|)\Big) \leqslant C_1 \exp\Big\{-\frac{y^2}{C_2\delta}\Big\},$$

$$\mathbb{P}_t\Big(\sup_{s\in[t,t+\delta]} |X_s^2 - X_t^2| > y(1+|X_t|)^2\Big) \leqslant C_1 \exp\Big\{-\frac{y^2}{C_2\delta}\Big\}.$$

Notice that

$$C_1 \exp\Big\{-\frac{y^2}{C_2\delta}\Big\} \leqslant \frac{C_1}{1 + \frac{y^2}{C_2\delta} + \frac{y^4}{2C_2\delta^2}} \leqslant 2C_1C_2\delta^2 y^{-4}, \tag{33}$$

from which it is clear that we may bound $C_1 \exp\{-y^2/(C_2\delta)\}$ from above by $C\delta^2 y^{-2/u}$ for all $y \geqslant \delta^{u(1-\varepsilon)/2}$ for $u = 1/2$ and $C = 2C_1C_2$. We have therefore verified that conditions (7) and (8) in Lemma 1 hold.

Suppose first that the function u satisfies (19). Then

$$\mathbb{P}_t\Big(\sup_{s\in[t,t+\delta]} |u(s,X_s) - u(t,X_t)| > y(1+|X_t|)\Big) \\ \leqslant \mathbb{P}_t\Big(\sup_{s\in[t,t+\delta]} K_2(\delta(1+|X_t|) + |X_s - X_t|) > y(1+|X_t|)\Big) \\ = \mathbb{P}_t\Big(\sup_{s\in[t,t+\delta]} |X_s - X_t| > (y/K_2 - \delta)(1+|X_t|)\Big).$$

For any $\beta \in (0,\varepsilon)$, $\delta^{(1-\varepsilon)/4}/K_2 - \delta > \delta^{(1-\beta)/4}$ for δ sufficiently small. Hence, by Lemma 6 and (33), condition (9) holds for $y \geqslant \delta^{(1-\beta)/4}$. Since $\varepsilon \in (0,1)$ was arbitrarily, we have verified that (9) in Lemma 1 holds for $y \geqslant \delta^{(1-\varepsilon)/4}$.

Now suppose instead that the function u satisfies (20). Then the verification of (9) in Lemma 1 follows immediately from combining Lemma 8 and (33).

Finally, the verification of condition (10) in Lemma 1 follows immediately from Lemma 7. □

Proof of Theorem 4. First notice that, by (*v*), $\mathbb{E}[|L_t|] \leqslant \mathbb{E}[L_t^2]^{1/2} < \infty$ and therefore $L \in L^2(\mathbb{F}) \subset L^1(\mathbb{F})$. Write $\Delta L_{t+\delta} := L_{t+\delta} - L_t$ and notice that $\Delta L_{t+\delta}$ is infinitely divisible with Lévy measure $\nu_{t+\delta} - \nu_t$. We first show the statement

$$\lim_{\delta \downarrow 0, s \to t} \left| F^{-1}_{\Delta L_{s+\delta}}(\alpha_\delta) - q_t \right| = 0. \tag{34}$$

Write $\overline{F}_{\Delta L_{s+\delta}}(x) := 1 - F_{\Delta L_{s+\delta}}(x)$ and notice that

$$\begin{aligned} F^{-1}_{\Delta L_{s+\delta}}(\alpha_\delta) &= \min\{x : F_{\Delta L_{s+\delta}}(x) \geqslant \alpha_\delta\} \\ &= \min\Big\{x : \exp\Big\{\frac{1}{\delta}\log\big(1 - \overline{F}_{\Delta L_{s+\delta}}(x)\big)\Big\} \geqslant \alpha_\delta^{1/\delta}\Big\} \\ &= \min\Big\{x : \frac{1}{\delta}\log\big(1 - \overline{F}_{\Delta L_{s+\delta}}(x)\big) \geqslant \frac{1}{\delta}\log(1 - (1 - \alpha_\delta))\Big\}. \end{aligned}$$

Notice that $-y - y^2 \leqslant \log(1-y) \leqslant -y$ for $y \in [0,1)$ and recall that $\lim_{\delta \to 0}(1 - \alpha_\delta)/\delta = -\log \alpha$. By Lemma 9, Corollary 3 and the continuity of $x \mapsto \dot{\nu}_t(x, \infty)$ for $x > 0$,

$$\lim_{\delta \downarrow 0, s \to t} \frac{1}{\delta}\overline{F}_{\Delta L_{s+\delta}}(x) = \dot{\nu}_t(x, \infty) \quad \text{for every } x > 0. \tag{35}$$

Moreover, by combining the assumptions (*iii*) and (*iv*), there exists a unique $q_t > 0$ such that $\dot{\nu}_t(q_t, \infty) = -\log \alpha$. Statement (34) now follows.

We next show the statement

$$\lim_{\delta \downarrow 0, s \to t} \left| \frac{1}{\delta}\Big(W_s^\delta(\Delta L_{s+\delta}) - \frac{\mathbb{E}[\Delta L_{s+\delta}]}{1 + \eta_\delta}\Big) - K_L(t) \right| = 0, \tag{36}$$

where $K_L(t)$ is given in (25). Notice that due to the independent increments of additive processes, $W_s^\delta(\Delta L_{s+\delta})$ is independent of $\mathcal{F}_s$ and

$$\begin{aligned} \frac{1}{\delta}W_s^\delta(\Delta L_{s+\delta}) &= \frac{1}{\delta}F^{-1}_{\Delta L_{s+\delta}}(\alpha_\delta) - \frac{1}{(1+\eta_\delta)\delta}\mathbb{E}\big[\big(F^{-1}_{\Delta L_{s+\delta}}(\alpha_\delta) - \Delta L_{s+\delta}\big)_+\big] \\ &= \frac{(1 - F_{\Delta L_{s+\delta}}(F^{-1}_{\Delta L_{s+\delta}}(\alpha_\delta)) + \eta_\delta)F^{-1}_{\Delta L_{s+\delta}}(\alpha_\delta)}{(1+\eta_\delta)\delta} \\ &\quad + \frac{\mathbb{E}\big[\Delta L_{s+\delta} I\{\Delta L_{s+\delta} \leqslant F^{-1}_{\Delta L_{s+\delta}}(\alpha_\delta)\}\big]}{(1+\eta_\delta)\delta}. \end{aligned}$$

Notice that

$$\begin{aligned} &\frac{1}{\delta}\mathbb{E}\big[\Delta L_{s+\delta} I\{\Delta L_{s+\delta} \leqslant F^{-1}_{\Delta L_{s+\delta}}(\alpha_\delta)\}\big] \\ &\quad = \frac{1}{\delta}\mathbb{E}[\Delta L_{s+\delta}] - \frac{1}{\delta}\mathbb{E}\big[\Delta L_{s+\delta} I\{\Delta L_{s+\delta} > F^{-1}_{\Delta L_{s+\delta}}(\alpha_\delta)\}\big] \\ &\quad = \frac{1}{\delta}\mathbb{E}[\Delta L_{s+\delta}] - \frac{F^{-1}_{\Delta L_{s+\delta}}(\alpha_\delta)}{\delta}(1 - F_{\Delta L_{s+\delta}}(F^{-1}_{\Delta L_{s+\delta}}(\alpha_\delta)) \\ &\qquad - \frac{1}{\delta}\int_{F^{-1}_{\Delta L_{s+\delta}}(\alpha_\delta)}^{\infty} \mathbb{P}(\Delta L_{s+\delta} > x)\mathrm{d}x. \end{aligned}$$

Hence,

$$\frac{1}{\delta}\Big(W_s^\delta(\Delta L_{s+\delta}) - \frac{\mathbb{E}[\Delta L_{s+\delta}]}{1+\eta_\delta}\Big) = \frac{\eta_\delta F^{-1}_{\Delta L_{s+\delta}}(\alpha_\delta)}{(1+\eta_\delta)\delta} - \frac{1}{(1+\eta_\delta)\delta}\int_{F^{-1}_{\Delta L_{s+\delta}}(\alpha_\delta)}^\infty \mathbb{P}(\Delta L_{s+\delta} > x)\mathrm{d}x.$$

Combining (3) and (34) establishes the appropriate convergence to $\log(1+\eta)q_t$ of the first terms on the right-hand side as $\delta \to 0$. We now address the second term. First notice that, by (34), there exists $c \in (0,\infty)$ such that

$$\begin{aligned}
&\limsup_{\delta\downarrow 0, s\to t}\Big|\int_{F^{-1}_{\Delta L_{s+\delta}}(\alpha_\delta)}^\infty \frac{1}{(1+\eta_\delta)\delta}\mathbb{P}(\Delta L_{s+\delta} > x)\mathrm{d}x - \int_{q_t}^\infty \dot\nu_t(x,\infty)\mathrm{d}x\Big| \\
&\quad \leqslant \limsup_{\delta\downarrow 0, s\to t} c\Big|F^{-1}_{\Delta L_{s+\delta}}(\alpha_\delta) - q_t\Big| \\
&\qquad + \limsup_{\delta\downarrow 0, s\to t}\Big|\int_{q_t}^\infty \frac{1}{(1+\eta_\delta)\delta}\mathbb{P}(\Delta L_{s+\delta} > x)\mathrm{d}x - \int_{q_t}^\infty \dot\nu_t(x,\infty)\mathrm{d}x\Big| \\
&\quad = \limsup_{\delta\downarrow 0, s\to t}\Big|\int_{q_t}^\infty \frac{1}{(1+\eta_\delta)\delta}\mathbb{P}(\Delta L_{s+\delta} > x)\mathrm{d}x - \int_{q_t}^\infty \dot\nu_t(x,\infty)\mathrm{d}x\Big|. \qquad (37)
\end{aligned}$$

We will show that (37) = 0. By continuity of $x \mapsto \dot\nu_t(x,\infty)$ for $x > 0$ and the fact that all functions in (35) are monotone, the pointwise convergence in (35) is in fact uniform on any interval $[a,b]$, $0 < a < b < \infty$. Hence, for any $b \in (q_t,\infty)$,

$$\limsup_{\delta\downarrow 0, s\to t}\Big|\int_{q_t}^b \frac{1}{(1+\eta_\delta)\delta}\mathbb{P}(\Delta L_{s+\delta} > x)\mathrm{d}x - \int_{q_t}^b \dot\nu_t(x,\infty)\mathrm{d}x\Big| = 0$$

from which follows that

$$(37) \leqslant \limsup_{\delta\downarrow 0, s\to t}\int_b^\infty \frac{1}{(1+\eta_\delta)\delta}\mathbb{P}(\Delta L_{s+\delta} > x)\mathrm{d}x + \int_b^\infty \dot\nu_t(x,\infty)\mathrm{d}x.$$

Next we show that the above upper bound on (37) can be chosen arbitrary small by choosing b sufficiently large. By Markov's inequality follows that

$$\frac{1}{\delta}\mathbb{P}(\Delta L_{s+\delta} > x) \leqslant \frac{1}{\delta}\mathbb{P}((\Delta L_{s+\delta})^2 > x^2) \leqslant \frac{\mathbb{E}[(\Delta L_{s+\delta})^2]}{\delta x^2}$$

and further that

$$\int_b^\infty \frac{1}{\delta}\mathbb{P}(\Delta L_{s+\delta} > x)\mathrm{d}x \leqslant \frac{1}{\delta b}\mathbb{E}[(\Delta L_{s+\delta})^2].$$

In particular, the assumed property (v) gives

$$\lim_{b\to\infty}\limsup_{\delta\downarrow 0, s\to t}\int_b^\infty \frac{1}{(1+\eta_\delta)\delta}\mathbb{P}(\Delta L_{t+\delta} > x)\mathrm{d}x = 0.$$

By Fatou's lemma, assumption (v) and (35), for any $b \in (0, \infty)$,

$$\int_b^\infty \dot{\nu}_t(x,\infty)\mathrm{d}x \leqslant \liminf_{\delta\downarrow 0, s\to t} \int_b^\infty \frac{1}{\delta}\mathbb{P}(\Delta L_{s+\delta} > x)\mathrm{d}x$$
$$\leqslant \limsup_{\delta\downarrow 0, s\to t} \int_b^\infty \frac{1}{\delta}\mathbb{P}(\Delta L_{s+\delta} > x)\mathrm{d}x < \infty.$$

In particular,

$$\lim_{b\to\infty} \limsup_{\delta\downarrow 0} \int_b^\infty \dot{\nu}_t(x,\infty)\mathrm{d}x = 0.$$

Summing up, we have shown that (37) = 0 from which it follows that

$$\lim_{\delta\downarrow 0, s\to t} \int_{F^{-1}_{\Delta L_{s+\delta}}(\alpha_\delta)}^\infty \frac{1}{(1+\eta_\delta)\delta}\mathbb{P}(\Delta L_{s+\delta} > x)\mathrm{d}x = \int_{q_t}^\infty \dot{\nu}_t(x,\infty)\mathrm{d}x$$

and further that (36) holds. By combining the assumptions (iii) and (iv), there exists a unique $q_t > 0$ such that $\dot{\nu}_t(q_t,\infty) = -\log\alpha$. Moreover, by joint continuity of $(t,x) \mapsto \dot{\nu}_t(x,\infty)$, $t \mapsto q_t$ is continuous. Since also $t \mapsto q_t$ is uniformly bounded away from 0, $t \mapsto K_L(t)$ is continuous on $[0,T]$. Thus (36) and Lemma 10 together imply

$$\lim_{\delta\downarrow 0} \sup_{t\in[0,T-\delta]} \left|\frac{1}{\delta}\left(W_t^\delta(\Delta L_{t+\delta}) - \frac{\mathbb{E}[\Delta L_{t+\delta}]}{1+\eta_\delta}\right) - K_L(t)\right| = 0. \tag{38}$$

It remains to prove the convergence in (24). For any $k \in \{0, ..., m-1\}$,

$$\left|V^{\tau_m}_{\tau_{m,k}}(L) - \mathbb{E}[L_T - L_{\tau_{m,k}}] - \int_{\tau_{m,k}}^T K_L(s)\mathrm{d}s\right|$$
$$\leqslant \Big|\sum_{i=k}^{m-1} \frac{\eta_{\delta_{m,i}}}{1+\eta_{\delta_{m,i}}}\mathbb{E}[\Delta L_{\tau_{m,i}+\delta_{m,i}}]\Big| \tag{39}$$
$$+ \Big|W^{\delta_{m,k}}_{\tau_{m,k}} \circ \cdots \circ W^{\delta_{m,m-1}}_{\tau_{m,m-1}}\Big(\sum_{i=k}^{m-1} \Delta L_{\tau_{m,i}+\delta_{m,i}}\Big)$$
$$- \sum_{i=k}^{m-1} \frac{\mathbb{E}[\Delta L_{\tau_{m,i}+\delta_{m,i}}]}{1+\eta_{\delta_{m,i}}} - \sum_{i=k}^{m-1} K_L(\tau_{m,i})\delta_{m,i}\Big| \tag{40}$$
$$+ \Big|\sum_{i=k}^{m-1} K_L(\tau_{m,i})\delta_{m,i} - \int_{\tau_{m,k}}^T K_L(s)\mathrm{d}s\Big|. \tag{41}$$

The term (41) converges to 0 as $m \to \infty$ from the definition of the Riemann integral. Moreover, as $m \to \infty$,

$$(39) \leqslant \sup_{i\leqslant m-1} \left|\mathbb{E}[\Delta L_{\tau_{m,i}+\delta_{m,i}}]\right| \sum_{i=k}^{m-1} \frac{\eta_{\delta_{m,i}}}{1+\eta_{\delta_{m,i}}} \to 0,$$

since the sum is uniformly bounded in m and, by Hölder's inequality and assumption (v),

$$\begin{aligned}
&\limsup_{m\to\infty}\sup_{i\leqslant m-1}\big|\mathbb{E}[\Delta L_{\tau_{m,i}+\delta_{m,i}}]\big| \\
&\quad\leqslant \limsup_{m\to\infty}\sup_{i\leqslant m-1}\mathbb{E}\big[\big|\Delta L_{\tau_{m,i}+\delta_{m,i}}\big|\big] \\
&\quad\leqslant \limsup_{m\to\infty}\sup_{i\leqslant m-1}\mathbb{E}[(\Delta L_{\tau_{m,i}+\delta_{m,i}})^2]^{1/2} \\
&\quad= \limsup_{m\to\infty}\sup_{i\leqslant m-1}\delta_{m,i}^{1/2}\Big(\frac{\mathbb{E}[(\Delta L_{\tau_{m,i}+\delta_{m,i}})^2]}{\delta_{m,i}}\Big)^{1/2} \\
&\quad\leqslant \limsup_{m\to\infty}\sup_{i\leqslant m-1}\delta_{m,i}^{1/2}\Big(\limsup_{\delta\downarrow 0}\sup_{t\in[0,T-\delta]}\frac{1}{\delta}\mathbb{E}[(\Delta L_{t+\delta})^2]\Big)^{1/2} \\
&\quad= 0.
\end{aligned}$$

Using the translation invariance property in Theorem 1 (i) and (38),

$$\begin{aligned}
(40) &= \Big|\sum_{i=k}^{m-1}\delta_{m,i}\Big(\frac{1}{\delta_{m,i}}\Big(W_{\tau_{m,i}}^{\delta_{m,i}}(\Delta L_{\tau_{m,i}+\delta_{m,i}}) - \frac{\mathbb{E}[\Delta L_{\tau_{m,i}+\delta_{m,i}}]}{1+\eta_{\delta_{m,i}}}\Big) - K_L(\tau_{m,i})\Big)\Big| \\
&\leqslant T\max_{0\leqslant i\leqslant m-1}\Big|\frac{1}{\delta_{m,i}}\Big(W_{\tau_{m,i}}^{\delta_{m,i}}(\Delta L_{\tau_{m,i}+\delta_{m,i}}) - \frac{\mathbb{E}[\Delta L_{\tau_{m,i}+\delta_{m,i}}]}{1+\eta_{\delta_{m,i}}}\Big) - K_L(\tau_{m,i})\Big| \\
&\to 0
\end{aligned}$$

as $m\to\infty$. Hence, (38) implies (24).

Now it only remains to show is that, for any $\beta\in(0,\infty)$,

$$\sup_{t\in\tau_m}\big|\widehat{V}_t^{\tau_m,\beta}(L) - \check{V}_t^{\tau_m,\beta}(L)\big| \to 0 \quad \text{a.s. as } m\to\infty.$$

This is straightforward considering the fact that also the sequences $\{\alpha_{\delta_{m,k}} - \delta_{m,k}^{1+\beta}\}_{k=0}^{m-1}$ and $\{\alpha_{\delta_{m,k}} + \delta_{m,k}^{1+\beta}\}_{k=0}^{m-1}$ satisfy (3), yielding

$$\lim_{m\to\infty}\sup_{k\leqslant m-1}\big|F^{-1}_{\Delta L_{\tau_{m,k}+\delta_{m,k}}}(\alpha_{\delta_{m,k}} \pm \delta_{m,k}^{1+\beta}) - q_t\big| = 0.$$

Thus, the arguments in the above proof for V^{τ_m} hold for both $\widehat{V}^{\tau_m,\beta}$ and $\check{V}^{\tau_m,\beta}$. This concludes the proof. □

Proof of Corollary 2. Notice that $\{L_t\}_{t\in[0,T]}$ has system of generating triplets $\{(\sigma_t^2,\nu_t,\gamma_t)\} = \{(\mu(t)\sigma^2,\mu(t)\nu,\mu(t)\gamma)\}$. Now we verify the requirements $(i)-(v)$ in Theorem 4, noting that $(\dot{\sigma}_t^2,\dot{\nu}_t,\dot{\gamma}_t) = \lambda(t)(\sigma^2,\nu,\gamma)$.

(i): For each $\delta\in(0,T]$ and $s\in[0,T-\delta]$, by the integral mean value theorem there exist a $\theta_{s,\delta}\in[s,s+\delta]$ such that

$$\frac{1}{\delta}\big(\sigma_{s+\delta}^2-\sigma_s^2,\nu_{s+\delta}-\nu_s,\gamma_{s+\delta}-\gamma_s\big) = \lambda(\theta_{s,\delta})(\sigma^2,\nu,\gamma)$$

By the continuity of $t \mapsto \lambda(t)$, we immediately get

$$\begin{aligned}
&\lim_{\delta\downarrow 0, s\to t} \frac{1}{\delta}\big(\sigma^2_{s+\delta} - \sigma^2_s, \nu_{s+\delta} - \nu_s, \gamma_{s+\delta} - \gamma_s\big) \\
&\quad = \lim_{\delta\downarrow 0, s\to t} \lambda(\theta_{s,\delta})(\sigma^2, \nu, \gamma) \\
&\quad = \lambda(t)(\sigma^2, \nu, \gamma) \\
&\quad = \big(\dot\sigma^2_t, \dot\nu_t, \dot\gamma_t\big).
\end{aligned}$$

(ii): The statement follows from assumption (i) in Corollary 2 and the fact

$$\int_{[-\varepsilon,\varepsilon]} x^2 \frac{1}{\delta}(\nu_{t+\delta} - \nu_t)(\mathrm{d}x) = \frac{\mu(t+\delta) - \mu(t)}{\delta} \int_{[-\varepsilon,\varepsilon]} x^2 \nu(\mathrm{d}x).$$

(iii): The statement follows from $\dot\nu_t(x, \infty) = \lambda(t)\nu(x, \infty)$, which is jointly continuous by assumption (ii) in Corollary 2 and the assumed continuity of $t \mapsto \lambda(t)$.

(iv): Since $\dot\nu_t(q_t, \infty) = \lambda(t)\nu(q_t, \infty)$ and, by assumption, there exists $q_t > 0$ solving $\lambda(t)\nu(q_t, \infty) = -\log\alpha$, the statement follows.

(v): We first note that by assumption (ii) and Corollary 25.8 in [20], $\widetilde{L}_t \in L^2(\mathbb{G})$. For any $s > 0$ and $\delta \in [0, s]$, by the stationary and independent increments property,

$$s\frac{1}{\delta}\mathbb{E}[\widetilde{L}^2_\delta] - s\delta\mathbb{E}[\widetilde{L}_1]^2 = s\frac{1}{\delta}(\mathbb{E}[\widetilde{L}^2_\delta] - \mathbb{E}[\widetilde{L}_\delta]^2) = (s/\delta)\operatorname{Var}(\widetilde{L}_\delta) = s\operatorname{Var}(\widetilde{L}_1) < \infty$$

from which $\sup_{\delta\in(0,s]} \delta^{-1}\mathbb{E}[\widetilde{L}^2_\delta] < \infty$ for any $s > 0$ follows. Notice that, for any $\delta \in (0, T]$ and $t \in [0, T-\delta]$,

$$\Delta L_{t+\delta} = \widetilde{L}_{\mu(t+\delta)} - \widetilde{L}_{\mu(t)} \overset{d}{=} \widetilde{L}_{\mu(t+\delta)-\mu(t)}$$

and, from the mean-value theorem,

$$\underline{\lambda} := \min_{s\in[0,T]} \lambda(s) \leqslant \frac{\mu(t+\delta) - \mu(t)}{\delta} \leqslant \max_{s\in[0,T]} \lambda(s) =: \overline{\lambda},$$

where $0 < \underline{\lambda} \leqslant \overline{\lambda} < \infty$. Hence,

$$\begin{aligned}
\sup_{\delta\in(0,T]} \sup_{t\in[0,T-\delta]} \frac{1}{\delta}\mathbb{E}[(\Delta L_{t+\delta})^2] &\leqslant \sup_{\delta\in(0,T]} \overline{\lambda}\frac{1}{\overline{\lambda}\delta}\mathbb{E}[(\Delta\widetilde{L}_{\overline{\lambda}\delta})^2] \\
&= \overline{\lambda} \sup_{\delta\in(0,\overline{\lambda}T]} \frac{1}{\delta}\mathbb{E}[(\Delta\widetilde{L}_\delta)^2] \\
&< \infty
\end{aligned}$$

which verifies statement (v). □

5 Auxiliary results

Lemma 2. *For $Y \in L^1(\mathcal{F}_{t+\delta})$,*

$$W_t^\delta(Y) = \frac{1}{1+\eta_\delta}\Big(\mathbb{E}_t[Y] - (1-\alpha_\delta)\,\mathrm{ES}_{t,1-\alpha_\delta}(-Y) + (1-\alpha_\delta+\eta_\delta)\,\mathrm{VaR}_{t,1-\alpha_\delta}(-Y)\Big).$$

Proof. Notice that

$$\begin{aligned}
&-\mathbb{E}_t[(\mathrm{VaR}_{t,1-\alpha_\delta}(-Y)-Y)_+]\\
&\quad= -\mathbb{E}_t[(\mathrm{VaR}_{t,1-\alpha_\delta}(-Y)-Y)I_{\{Y\leqslant \mathrm{VaR}_{t,1-\alpha_\delta}(-Y)\}}]\\
&\quad= \mathbb{E}_t[(Y-\mathrm{VaR}_{t,1-\alpha_\delta}(-Y))(1-I_{\{Y>\mathrm{VaR}_{t,1-\alpha_\delta}(-Y)\}})]\\
&\quad= \mathbb{E}_t[Y]-\mathrm{VaR}_{t,1-\alpha_\delta}(-Y)-\mathbb{E}_t[(Y-\mathrm{VaR}_{t,1-\alpha_\delta}(-Y))_+].
\end{aligned}$$

Straightforward calculations, see Lemma 2.2 in [6], yields

$$\mathbb{E}_t[(Y-\mathrm{VaR}_{t,1-\alpha_\delta}(-Y))_+] = (1-\alpha_\delta)(\mathrm{ES}_{t,1-\alpha_\delta}(-Y)-\mathrm{VaR}_{t,1-\alpha_\delta}(-Y))$$

from which we conclude that

$$-\mathbb{E}_t[(\mathrm{VaR}_{t,1-\alpha_\delta}(-Y)-Y)_+] = \mathbb{E}_t[Y]-(1-\alpha_\delta)\,\mathrm{ES}_{t,1-\alpha_\delta}(-Y) - \alpha_\delta\,\mathrm{VaR}_{t,1-\alpha_\delta}(-Y).$$

Hence,

$$\begin{aligned}
W_t(Y) &= \mathrm{VaR}_{t,1-\alpha_\delta}(-Y) - \frac{1}{1+\eta_\delta}\mathbb{E}_t[(\mathrm{VaR}_{t,1-\alpha_\delta}(-Y)-Y)_+]\\
&= \frac{1}{1+\eta_\delta}\Big(\mathbb{E}_t[Y]-(1-\alpha_\delta)\,\mathrm{ES}_{t,1-\alpha_\delta}(-Y)\\
&\qquad + (1-\alpha_\delta+\eta_\delta)\,\mathrm{VaR}_{t,1-\alpha_\delta}(-Y)\Big).
\end{aligned}$$

□

Lemma 3. *Let $\{X_t\}_{t\in[0,T]}$ and $\{U_t\}_{t\in[0,T]}$ be adapted processes. Suppose that there exist constants $\delta_0 \in (0,1/2)$, $u \in (0,2)$, $\varepsilon \in (0,1)$ and $C_1 > 0$ such that for $\delta \in (0,\delta_0)$ and for any $y \geqslant \delta^{(u-\varepsilon u)/2}$ and any $t \in [0,T-\delta]$*

$$\begin{aligned}
\mathbb{P}_t\Big(|\Delta X_{t+\delta}| > y(1+|X_t|)\Big) &\leqslant C_1\delta^2 y^{-2/u},\\
\mathbb{P}_t\Big(|\Delta U_{t+\delta}| > y(1+|X_t|)\Big) &\leqslant C_1\delta^2 y^{-2/u},\\
\mathbb{P}_t\Big(|\Delta X^2_{t+\delta}| > y(1+|X_t|)^2\Big) &\leqslant C_1\delta^2 y^{-2/u}.
\end{aligned}$$

Then, for some $g(\delta) \in o(1)$,

$$\begin{aligned}
\mathrm{ES}_{t,1-\alpha_\delta}(-|\Delta X_{t+\delta}|) &\leqslant g(\delta)(1+|X_t|),\\
\mathrm{ES}_{t,1-\alpha_\delta}(-|\Delta U_{t+\delta}|) &\leqslant g(\delta)(1+|X_t|),\\
\mathrm{ES}_{t,1-\alpha_\delta}(-|\Delta X^2_{t+\delta}|) &\leqslant g(\delta)(1+X_t^2).
\end{aligned}$$

Proof. By assumption,

$$\mathbb{P}_t(|\Delta X_{t+\delta}| > y) \leqslant C_1\delta^2\Big(\frac{y}{1+|X_t|}\Big)^{-2/u}$$

for $y \geqslant (1+|X_t|)\delta^{(u-\varepsilon u)/2}$. Solving $C_1\delta^2 y^{-2/u}(1+|X_t|)^{2/u} = p$ for y gives

$$y(p) = (1+|X_t|)\Big(\frac{p}{C_1\delta^2}\Big)^{-u/2}.$$

Hence, for $r \in (0,1)$ and δ small enough,

$$\begin{aligned}
\mathrm{ES}_{t,1-\alpha_\delta}(-|\Delta X_{t+\delta}|) &\leqslant \frac{1}{\delta^{1+r}}\int_0^{\delta^{1+r}} y(p)dp \\
&= (1+|X_t|)C_1^{u/2}\delta^u \frac{1}{\delta^{1+r}}\int_0^{\delta^{1+r}} p^{-u/2}dp \\
&= \frac{(1+|X_t|)C_1^{u/2}}{1-u/2}\delta^{u(1-(1+r)/2)} \\
&=: g(\delta)(1+|X_t|).
\end{aligned}$$

The same argument shows the upper bound for $\mathrm{ES}_{t,1-\alpha_\delta}(-|\Delta U_{t+\delta}|)$. A similar argument applies for showing the upper bound for $\mathrm{ES}_{t,1-\alpha_\delta}(-|\Delta X_{t+\delta}^2|)$: By assumption,

$$\mathbb{P}(|\Delta X_{t+\delta}^2 > y \mid \mathcal{F}_t) \leqslant C_1\delta^2\Big(\frac{y}{(1+|X_t|)^2}\Big)^{-2/u}$$

for $y \geqslant (1+|X_t|)^2\delta^{(u-\varepsilon u)/2}$. The same argument as above gives

$$\mathrm{ES}_{t,1-\alpha_\delta}(-|\Delta X_{t+\delta}^2|) \leqslant g(\delta)(1+|X_t|)^2.$$

Noting that $(1+|X_t|)^2 \leqslant 2(1+X_t^2)$ yields the upper bound in the statement. ☐

Lemma 4. *Let $\{X_t\}_{t\in[0,T]}$, $\{Y_t\}_{t\in[0,T]}$ and $\{U_t\}_{t\in[0,T]}$ be adapted processes. Suppose that there exist constants $\delta_0 \in (0,1/2)$, $u \in (0,2)$, $\varepsilon \in (0,1)$ and $C_1 > 0$ such that for $\delta \in (0,\delta_0)$ and for any $y \geqslant \delta^{(u-\varepsilon u)/2}$ and any $t \in [0, T-\delta]$*

$$\begin{aligned}
&\mathbb{P}_t\Big(|\Delta X_{t+\delta}| > y(1+|X_t|)\Big) \leqslant C_1\delta^2 y^{-2/u}, \\
&\mathbb{P}_t\Big(|\Delta U_{t+\delta}| > y(1+|X_t|)\Big) \leqslant C_1\delta^2 y^{-2/u}, \\
&\mathbb{P}_t\Big(|\Delta X_{t+\delta}^2| > y(1+|X_t|)^2\Big) \leqslant C_1\delta^2 y^{-2/u}.
\end{aligned}$$

Then, for any $K > 0$, $\beta_1, \beta_2 \in (0, \varepsilon)$, with $\beta_2 < \beta_1$ and sufficiently small $\delta \in (0, \delta_0)$,

$$\begin{aligned}
&\mathrm{VaR}_{t,1-\alpha_\delta-\delta^{1+\beta_1}}(-Y_{t+\delta} - K(|\Delta X_{t+\delta}| + |\Delta U_{t+\delta}| + |\Delta X^2_{t+\delta}|)) \\
&\quad \leqslant \mathrm{VaR}_{t,1-\alpha_\delta-\delta^{1+\beta_2}}(-Y_{t+\delta}) + 5K\delta^{(u-\varepsilon u)/2}(1 + X_t^2), \\
&\mathrm{VaR}_{t,1-\alpha_\delta+\delta^{1+\beta_1}}(-Y_{t+\delta} + K(|\Delta X_{t+\delta}| + |\Delta U_{t+\delta}| + |\Delta X^2_{t+\delta}|)) \\
&\quad \geqslant \mathrm{VaR}_{t,1-\alpha_\delta+\delta^{1+\beta_2}}(-Y_{t+\delta}) - 5K\delta^{(u-\varepsilon u)/2}(1 + X_t^2).
\end{aligned}$$

Proof. Let

$$\begin{aligned}
&E := \{|\Delta X_{t+\delta}| + |\Delta U_{t+\delta}| + |\Delta X^2_{t+\delta}| \leqslant \delta^{(u-\varepsilon u)/2}(3 + 2|X_t| + X_t^2)\}, \\
&E_X := \{|\Delta X_{t+\delta}| \leqslant \delta^{(u-\varepsilon u)/2}(1 + |X_t|)\}, \\
&E_U := \{|\Delta U_{t+\delta}| \leqslant \delta^{(u-\varepsilon u)/2}(1 + |X_t|)\}, \\
&E_{X^2} := \{|\Delta X^2_{t+\delta}| \leqslant \delta^{(u-\varepsilon u)/2}(1 + |X_t|)^2\}.
\end{aligned}$$

From $\mathbb{P}_t(E) \geqslant \mathbb{P}_t(E_X \cap E_U \cap E_{X^2})$ follows that $\mathbb{P}_t(E^C) \leqslant \mathbb{P}_t(E_X^C) + \mathbb{P}_t(E_U^C) + \mathbb{P}_t(E_{X^2}^C)$. Hence, $\mathbb{P}_t(E^C) \leqslant 3C_1\delta^2(\delta^{(u-\varepsilon u)/2})^{-2/u} = 3C_1\delta^{1+\varepsilon}$. Notice that

$$\begin{aligned}
&\mathbb{P}_t(Y_{t+\delta} - K(|\Delta X_{t+\delta}| + |\Delta U_{t+\delta}| + |\Delta X^2_{t+\delta}|) \leqslant x) \\
&= \mathbb{P}_t(E \cap \{Y_{t+\delta} - K(|\Delta X_{t+\delta}| + |\Delta U_{t+\delta}| + |\Delta X^2_{t+\delta}|) \leqslant x\}) \\
&\quad + \mathbb{P}_t(E^C \cap \{Y_{t+\delta} - K(|\Delta X_{t+\delta}| + |\Delta U_{t+\delta}| + |\Delta X^2_{t+\delta}|) \leqslant x\}) \\
&\leqslant \mathbb{P}_t(Y_{t+\delta} \leqslant x + K\delta^{(u-\varepsilon u)/2}(3 + 4|X_t| + X_t^2)) + \mathbb{P}_t(E^C)
\end{aligned}$$

and similarly

$$\begin{aligned}
&\mathbb{P}_t(Y_{t+\delta} + K(|\Delta X_{t+\delta}| + |\Delta U_{t+\delta}| + |\Delta X^2_{t+\delta}|) \leqslant x) \\
&\geqslant \mathbb{P}_t(Y_{t+\delta} + |K|(|\Delta X_{t+\delta}| + |\Delta U_{t+\delta}| + |\Delta X^2_{t+\delta}|) \leqslant x) \\
&\geqslant \mathbb{P}_t(E \cap \{Y_{t+\delta} \leqslant x - K\delta^{(u-\varepsilon u)/2}(3 + 4|X_t| + X_t^2)\}) \\
&\geqslant \mathbb{P}_t(Y_{t+\delta} \leqslant x - K\delta^{(u-\varepsilon u)/2}(3 + 4|X_t| + X_t^2)) - \mathbb{P}_t(E^C).
\end{aligned}$$

Hence we conclude that, for δ small enough,

$$\begin{aligned}
&\mathrm{VaR}_{t,1-\alpha_\delta+\delta^{1+\beta_1}}(-Y_{t+\delta} + K(|\Delta X_{t+\delta}| + |\Delta U_{t+\delta}| + |\Delta X^2_{t+\delta}|)) \\
&\geqslant \mathrm{VaR}_{t,1-\alpha_\delta+\delta^{1+\beta_1}+\mathbb{P}_t(E^C)}(-Y_{t+\delta}) - K\delta^{(u-\varepsilon u)/2}(3 + 4|X_t| + X_t^2) \\
&\geqslant \mathrm{VaR}_{t,1-\alpha_\delta+\delta^{1+\beta_1}+3C_1\delta^{1+\varepsilon}}(-Y_{t+\delta}) - K\delta^{(u-\varepsilon u)/2}(3 + 4|X_t| + X_t^2)
\end{aligned}$$

and analogously that

$$\begin{aligned}
&\mathrm{VaR}_{t,1-\alpha_\delta-\delta^{1+\beta_1}}(-Y_{t+\delta} - K(|\Delta X_{t+\delta}| + |\Delta U_{t+\delta}| + |\Delta X^2_{t+\delta}|)) \\
&\leqslant \mathrm{VaR}_{t,1-\alpha_\delta-\delta^{1+\beta_1}-3C_1\delta^{1+\varepsilon}}(-Y_{t+\delta}) + K\delta^{(u-\varepsilon u)/2}(3 + 4|X_t| + X_t^2).
\end{aligned}$$

We note that $2|X_t| \leqslant 1 + X_t^2$, $3 + 4|X_t| + X_t^2 \leqslant 5 + 3X_t^2 \leqslant 5(1 + X_t^2)$. Moreover, $\delta^{1+\beta_1} + 3C_1\delta^{1+\varepsilon} < \delta^{1+\beta_2}$ for δ sufficiently small. The proof is complete. □

Lemma 5. *Let* $\{\tau_m\}_{m=1}^{\infty}$ *be a sequence of partitions of* $[0,T]$ *with* $0 = \tau_{m,0} < \cdots < \tau_{m,m} = T$*, let* $g(\delta) \in o(\delta)$ *and let* $B > 0$ *be a constant. Define*

$$A^{\tau_m}_{\tau_{m,k}} := A^{\tau_m}_{\tau_{m,k+1}}(1 + B(\tau_{m,k+1} - \tau_{m,k})) + g(\tau_{m,k+1} - \tau_{m,k}), \quad A^{\tau_m}_{\tau_{m,m}} := 0.$$

Then there exists $h(\delta) \in o(\delta)$ *such that* $A^{\tau_m}_{\tau_{m,k}} \leqslant h(\mathrm{mesh}(\tau_m))$ *for all* m, k.

Proof. Let $\delta_{m,k} := \tau_{m,k+1} - \tau_{m,k}$. Noticing that $1 + B\delta_{m,k} \leqslant e^{B\delta_{m,k}}$ gives

$$\begin{aligned} A^{\tau_m}_{\tau_{m,k}} &\leqslant \sum_{j=k}^{m-1} g(\delta_{m,j}) \exp\Big\{B \sum_{j=k}^{m-1} \delta_{m,j}\Big\} \\ &\leqslant e^{BT} \sum_{j=k}^{m-1} \delta_{m,k} \max_{k\leqslant j\leqslant m} \frac{g(\delta_{m,j})}{\delta_{m,j}} \\ &\leqslant Te^{BT} \sup_{\delta\leqslant \mathrm{mesh}(\tau_m)} \frac{g(\delta)}{\delta}. \end{aligned}$$

□

Lemma 6. *Let* $\{Y_t\}_{t\in[0,T]}$ *be the strong solution to* (18) *with* μ *and* σ *satisfying* (16) *and* (17)*. Then there are constants* $C_1, C_2 \in (0,\infty)$ *such that, for* $\delta \in (0,1)$ *sufficiently small and* $y > \delta^\beta$ *for any given* $\beta \in (0, 1/2)$,

$$\mathbb{P}_t\Big(\sup_{s\in[t,t+\delta]} |Y_s - Y_t| > y(1 + |Y_t|)\Big) \leqslant C_1 \exp\Big\{-\frac{y^2}{C_2\delta}\Big\}, \tag{42}$$

$$\mathbb{P}_t\Big(\sup_{s\in[t,t+\delta]} |Y_s^2 - Y_t^2| > 3y(1 + |Y_t|)^2\Big) \leqslant 2C_1 \exp\Big\{-\frac{y^2}{C_2\delta}\Big\}. \tag{43}$$

Proof. We first prove (42). Let $\tau := \inf\big\{s \in [t, t+\delta] : |Y_s - Y_t| > y(1+|Y_t|)\big\}$ and notice that

$$\begin{aligned} &\mathbb{P}_t\Big(\sup_{s\in[t,t+\delta]} |Y_s - Y_t| > y(1 + |Y_t|)\Big) \\ &\quad = \mathbb{P}_t(\tau \leqslant t + \delta) \\ &\quad \leqslant \mathbb{P}_t\Big(\tau \leqslant t + \delta, \sup_{t\leqslant s\leqslant \tau} \Big|\int_t^s \mu(u, Y_u)du\Big| > \frac{1}{2}y(1 + |Y_t|)\Big) \qquad (44) \\ &\qquad + \mathbb{P}_t\Big(\tau \leqslant t + \delta, \sup_{t\leqslant s\leqslant \tau} \Big|\int_t^s \sigma(u, Y_u)dB_u\Big| > \frac{1}{2}y(1 + |Y_t|)\Big). \qquad (45) \end{aligned}$$

Notice that $s \leqslant \tau$ implies $|Y_s| \leqslant |x| + y(1 + |x|)$ which in turn, by the growth condition (16) for μ and σ, implies that there is some finite constant M such that

$$\max\Big\{\sup_{s\in[t,\tau]} |\mu(s, Y_s)|, \sup_{s\in[t,\tau]} |\sigma(s, Y_s)|\Big\} \leqslant M(1 + |Y_t|). \tag{46}$$

Hence, for $\delta \in (0,1)$ sufficiently small, the probability in (44) is zero. The probability in (45) can be bounded from above as follows. Since $s \mapsto \int_0^s \sigma(u, Y_u) dB_u$ is a continuous local martingale, it may be expressed as a random time change $s \mapsto H(s)$,

$$H(s) := \int_0^s \sigma(u, Y_u)^2 du,$$

of a Brownian motion. By (46) and Theorem 18.4 in [10], there is standard Brownian motion $\widetilde{B}$ such that,

$$\begin{aligned}
(45) &\leqslant \mathbb{P}_t\Big(H(t+\delta) - H(t) \leqslant \delta M^2 (1+|Y_t|)^2, \sup_{s\in[t,t+\delta]} |\widetilde{B}_{H(s)}| > \frac{1}{2} y(1+|Y_t|)\Big) \\
&\leqslant \mathbb{P}_t\Big(\sup_{s\in[t,t+\delta]} |\widetilde{B}_s| > \frac{y}{2M}\Big).
\end{aligned}$$

Applying Lemma 5.2.1 in [5] to the last expression above gives

$$(45) \leqslant 4 \exp\Big\{ - \Big(\frac{y}{2M}\Big)^2 \frac{1}{2\delta} \Big\}.$$

We have proved (42). We now prove (43). Noting that $|Y_s^2 - x^2| = |Y_s - x||Y_s + x|$ we get:

$$\begin{aligned}
&\mathbb{P}_t\Big(\sup_{s\in[t,t+\delta]} |Y_s^2 - Y_t^2| > 3y(1+|Y_t|)^2\Big) \\
&\leqslant \mathbb{P}_t\Big(\Big\{\sup_{s\in[t,t+\delta]} |Y_t - Y_t| > y(1+|Y_t|)\Big\} \cup \Big\{\sup_{s\in[t,t+\delta]} |Y_t + Y_t| > 3(1+|Y_t|)\Big\}\Big) \\
&\leqslant \mathbb{P}_t\Big(\sup_{s\in[t,t+\delta]} |Y_t - Y_t| > y(1+|Y_t|)\Big) + \mathbb{P}_t\Big(\sup_{s\in[t,t+\delta]} |Y_t + Y_t| > 3(1+|Y_t|)\Big)
\end{aligned}$$

By Lemma 6, for $t > 0$ sufficiently small,

$$\mathbb{P}_t\Big(\sup_{s\in[t,t+\delta]} |Y_s - Y_t| > y(1+|Y_t|)\Big) \leqslant C_1 \exp\Big\{ - \frac{y^2}{C_2 \delta} \Big\}.$$

Moreover,

$$\begin{aligned}
&\mathbb{P}_t\Big(\sup_{s\in[t,t+\delta]} |Y_s + Y_t| > 3(1+|Y_t|)\Big) \\
&\leqslant \mathbb{P}_t\Big(\sup_{s\in[t,t+\delta]} |Y_s - Y_t| + 2|Y_t| > 3(1+|Y_t|)\Big) \\
&\leqslant \mathbb{P}_t\Big(\sup_{s\in[t,t+\delta]} |Y_s - Y_t| > 1 + |Y_t|\Big) \\
&\leqslant C_1 \exp\Big\{ - \frac{y^2}{C_2 \delta} \Big\}.
\end{aligned}$$

This concludes the proof. □

Lemma 7. *Let X be the solution to the stochastic differential equation* (18) *with coefficients μ and σ satisfying* (16) *and* (17)*. Then there exist a constant C such that*

$$|\mathbb{E}_t[\Delta X_{t+\delta}^2]| \leqslant C\delta(1+X_t^2). \tag{47}$$

Proof. Recall that $\Delta X_{t+\delta}^2 := X_{t+\delta}^2 - X_t^2$. Itô's Lemma yields

$$\Delta X_{t+\delta}^2 = \int_t^{t+\delta} \big(2X_s\mu(s,X_s) + \sigma(s,X_s)^2\big)\mathrm{d}s + 2\int_t^{t+\delta} X_s\sigma(s,X_s)\mathrm{d}B_s.$$

Hence,

$$\begin{aligned}
|\mathbb{E}_t[\Delta X_{t+\delta}^2]| &= \Big|\mathbb{E}_t\Big[\int_t^{t+\delta} \big(2X_s\mu(s,X_s) + \sigma(s,X_s)^2\big)\mathrm{d}s\Big]\Big| \\
&\leqslant \mathbb{E}_t\Big[\int_t^{t+\delta} \big|2X_s\mu(s,X_s) + \sigma(s,X_s)^2\big|\mathrm{d}s\Big] \\
&= \int_t^{t+\delta} \mathbb{E}_t\big[\big|2X_s\mu(s,X_s) + \sigma(s,X_s)^2\big|\big]\mathrm{d}s.
\end{aligned}$$

Since μ and σ satisfy (16),

$$\begin{aligned}
\mu(s,x) &\leqslant \big(K_1(1+x^2)\big)^{1/2} \leqslant K_1^{1/2}(1+|x|), \\
x\mu(s,x) &\leqslant K_1^{1/2}(|x|+x^2) \leqslant 2K_1^{1/2}(1+x^2), \\
x\mu(s,x) + \sigma(s,x)^2 &\leqslant (K_1 + 2K_1^{1/2})(1+x^2).
\end{aligned}$$

Hence, there is a constant C_1 such that

$$\int_t^{t+\delta} \mathbb{E}_t[|X_s\mu(s,X_s) + \sigma^2(s,X_s)|]\mathrm{d}s \leqslant \int_t^{t+\delta} \mathbb{E}_t[C_1(1+X_s^2)]\mathrm{d}s.$$

By Theorem 4.5.4 in [11], there is a constant C_2 such that $\mathbb{E}_t[X_s^2] \leqslant C_2(1+X_t^2)$ which immediately implies the existence of a constant C such that (47) holds. □

Lemma 8. *Let X be the solution to the stochastic differential equation* (18) *with coefficients μ and σ satisfying* (16) *and* (17)*. Define u as in Theorem 3 and assume u satisfies* (20)*. Then for any $\beta \in (0,1/2)$, for $\delta > 0$ sufficiently small and $y > \delta^\beta$, there exists constants $C_1, C_2 > 0$ such that*

$$\mathbb{P}_t\Big(\sup_{s\in[t,t+\delta]} |u(s+\delta, X_{s+\delta}) - u(t,X_t)| > y(1+|X_t|)\Big) \leqslant C_1 \exp\Big\{-\frac{y^2}{C_2\delta}\Big\} \tag{48}$$

Proof. We first notice that, by Itô's lemma and by Feynman-Kac, we have

$$\mathrm{d}u(t, X_t) = u_x(t, X_t)\sigma(t, X_t)\mathrm{d}B_t,$$

where subscript x denotes partial derivative with respect to the second argument of u. Let $\tau := \inf\{s \in [t, t+\delta] : |X_s - X_t| > y(1 + |X_t|)\}$ and notice that

$$\begin{aligned}
&\mathbb{P}_t\Big(\sup_{s\in[t,t+\delta]} |u(s, X_s) - u(t, X_t)| > y(1 + |X_t|)\Big) \\
&\quad \leqslant \mathbb{P}_t(\tau \leqslant t + \delta) \\
&\qquad + \mathbb{P}_t\Big(\tau > t + \delta, \sup_{s\in[t,t+\delta]} \Big|\int_0^s u_x(u, X_u)\sigma(u, X_u)dB_u\Big| > y(1 + |x|)\Big).
\end{aligned}$$

Since $s \mapsto \int_0^s u_x(u, X_u)\sigma(u, X_u)\mathrm{d}B_u$ is a continuous local martingale, it may be expressed as a random time change $s \mapsto H(s)$ of a Brownian motion:

$$H(s) := \int_t^{t+\delta} u_x(u, X_u)^2\sigma(u, X_u)^2\mathrm{d}u.$$

Notice that if $\tau > t + \delta$, then $H(t + \delta) \leqslant \delta K_1K_3(1 + |X_t|)^2$. By Theorem 18.4 in [10], there is standard Brownian motion $\widetilde{B}$ such that,

$$\begin{aligned}
&\mathbb{P}_t\Big(\tau > t + \delta, \sup_{s\in[t,t+\delta]} \Big|\int_0^s u_x(u, X_u)\sigma(u, X_u)dB_u\Big| > y(1 + |X_t|)\Big) \\
&\quad \leqslant \mathbb{P}\Big(\tau > t + \delta, \sup_{s\in[t,t+\delta]} |\widetilde{B}_{H(s)}| > y(1 + |X_t|)\Big) \\
&\quad \leqslant \mathbb{P}\Big(\sup_{s\in[t,t+\delta]} |\widetilde{B}_s| > \frac{y}{\delta^{1/2}K_1K_3}\Big).
\end{aligned}$$

Applying Lemma 5.2.1 in [5] to the last expression above gives

$$(45) \leqslant 4\exp\Big\{-\Big(\frac{y}{K_1K_3}\Big)^2\frac{1}{2\delta}\Big\}.$$

Noting that

$$\mathbb{P}_t(\tau \leqslant t + \delta) = \mathbb{P}_t\Big(\sup_{s\in[t,t+\delta]} |X_s - X_t| > y(1 + |X_t|)\Big)$$

can be bounded by Lemma 6 completes the argument showing (48). □

Lemma 9. *Let $\{L_t\}_{t\in[0,T]}$ be an $\mathbb{R}$-valued additive process with system of generating triplets $\{(\sigma_t^2, \nu_t, \gamma_t)\}_{t\in[0,T]}$. For each $t \in [0, T]$, let $\dot{\sigma}_t^2$ and $\dot{\gamma}_t$ be constants and let $\dot{\nu}_t$ be a measure on $\mathbb{R}\backslash\{0\}$ whose restrictions to sets bounded away from 0 are finite.*

Fix $t \in [0,T]$ and assume that

$$\frac{1}{\delta}\left(\sigma^2_{s+\delta} - \sigma^2_s, \nu_{s+\delta} - \nu_s, \gamma_{s+\delta} - \gamma_s\right) \to \left(\dot\sigma^2_t, \dot\nu_t, \dot\gamma_t\right) \quad \text{as } \delta \downarrow 0, s \to t, \tag{49}$$

where the convergence in the second component means that

$$\lim_{\delta\downarrow 0, s\to t} \frac{1}{\delta}\int_{\mathbb{R}\setminus\{0\}} f(x)(\nu_{s+\delta} - \nu_s)(\mathrm{d}x) = \int_{\mathbb{R}\setminus\{0\}} f(x)\dot\nu_t(\mathrm{d}x)$$

for all bounded and continuous functions vanishing in a neighborhood of 0. *Assume further that*

$$\lim_{\varepsilon\downarrow 0}\limsup_{\delta\downarrow 0, s\to t}\int_{[-\varepsilon,\varepsilon]} x^2\frac{1}{\delta}(\nu_{s+\delta} - \nu_s)(\mathrm{d}x) = 0. \tag{50}$$

Consider sequences $\delta_n \downarrow 0$, $t_n \to t$, with $t_n \in [0, T-\delta_n]$ and, for every n, let $\{L^{\delta_n,t_n}_s\}_{s\in[0,T]}$ be a Lévy process satisfying $\Delta L^{\delta_n,t_n}_{t+\delta_n} \overset{d}{=} \Delta L_{t_n+\delta_n}$ and let μ_{δ_n,t_n} be the probability distribution of L^{δ_n,t_n}_1. Then

$$\lim_{\delta_n\downarrow 0}\frac{1}{\delta_n}\int_{\mathbb{R}\setminus\{0\}} f(x)\mu^{\delta_n}_{\delta_n,t_n}(\mathrm{d}x) \to \int_{\mathbb{R}\setminus\{0\}} f(x)\dot\nu_t(\mathrm{d}x)$$

for bounded and continuous functions vanishing in a neighborhood of 0.

Proof. Notice that μ_{δ_n,t_n} is infinitely divisible with Lévy triplet

$$\frac{1}{\delta_n}\left(\sigma^2_{t_n+\delta_n} - \sigma^2_{t_n}, \nu_{t+\delta_n} - \nu_{t_n}, \gamma_{t_n+\delta_n} - \gamma_{t_n}\right)$$

By Theorem 8.7 in [20], (49) and (50) together imply that μ_{δ_n,t_n} converges weakly to an infinitely divisible distribution μ with Lévy triplet $(\dot\sigma^2_t, \dot\nu_t, \dot\gamma_t)$. In particular, the corresponding characteristic functions converges pointwise:

$$\lim_{\delta_n\downarrow 0}\hat\mu_{\delta_n,t_n}(z) = \hat\mu(z) \tag{51}$$

Define μ_n via its characteristic function $\hat\mu_n(z)$ as

$$\hat\mu_n(z) := \exp\left\{\delta_n^{-1}(\hat\mu_{\delta_n,t_n}(z)^{\delta_n} - 1)\right\} = \exp\left\{\delta_n^{-1}\int_{\mathbb{R}\setminus\{0\}}(e^{izx} - 1)\mu^{\delta_n}_{\delta_n,t_n}(\mathrm{d}x)\right\}.$$

From pp. 38-39 in [20], in particular (8.7) follows that μ_n is infinitely divisible with Lévy triplet $(0, \delta_n^{-1}\mu^{\delta_n}_{\delta_n,t_n}, 0)$. Moreover,

$$\begin{aligned}\hat\mu_n(z) &= \exp\left\{\delta_n^{-1}(\hat\mu_{\delta_n,,t_n}(z)^{\delta_n} - 1)\right\}\\ &= \exp\left\{\delta_n^{-1}(e^{\delta_n\log(\hat\mu_{\delta_n,t_n}(z))} - 1)\right\}\\ &= \exp\left\{\delta_n^{-1}(\delta_n\log(\hat\mu(z)) + O(\delta_n^2))\right\},\end{aligned}$$

where the last equality is due to (51) which, as in the proof of Theorem 8.7 in [20], implies that $\lim_{\delta_n \downarrow 0} \log \hat{\mu}_{\delta_n,t}(z) = \log \hat{\mu}(z)$ uniformly on any compact set. Hence, $\lim_{n\to\infty} \hat{\mu}_n(z) = \hat{\mu}(z)$ for every z, implying $\mu_n \to \mu$ weakly. Theorem 8.7 in [20] now gives

$$\lim_{\delta_n \downarrow 0} \frac{1}{\delta_n} \int_{\mathbb{R}\setminus\{0\}} f(x) \mu_{\delta_n,t_n}^{\delta_n}(\mathrm{d}x) = \int_{\mathbb{R}\setminus\{0\}} f(x) \dot{\nu}_t(\mathrm{d}x)$$

for all bounded and continuous functions vanishing in a neighborhood of 0. □

An important special case of Lemma 9 is the following:

Corollary 3. *If the conditions of Lemma 9 hold, and if $x > 0$ is a continuity point of $y \mapsto \dot{\nu}_t(y,\infty)$, then*

$$\lim_{\delta \downarrow 0, s \to t} \delta^{-1} \overline{F}_{\Delta L_{s+\delta}}(x) = \dot{\nu}_t(x,\infty).$$

Proof. Let $f(y) = 1_{(x,\infty)}(y)$ which is bounded, vanishes in a neighborhood of 0 but not continuous. For $m > 0$, let $\overline{f}$ and $\underline{f}$ be polygon functions given by

$$\overline{f}(y) = \begin{cases} 0, & y \leqslant x - 1/m, \\ m(y - x + 1/m), & y \in (x - 1/m, x), \\ 1, & y \geqslant x. \end{cases}$$

and

$$\underline{f}(y) = \begin{cases} 0, & y \leqslant x, \\ m(y - x + 1/m), & y \in (x, x + 1/m), \\ 1, & y \geqslant x + 1/m. \end{cases}$$

Then $\underline{f} \leqslant f \leqslant \overline{f}$,

$$\lim_{\delta_n \downarrow 0} \frac{1}{\delta_n} \int_{\mathbb{R}\setminus\{0\}} g(y) \mu_{\delta_n,t_n}^{\delta_n}(\mathrm{d}y) = \int_{\mathbb{R}\setminus\{0\}} g(y) \dot{\nu}_t(\mathrm{d}y), \quad g = \underline{f}, \overline{f},$$

and

$$\lim_{\delta_n \downarrow 0} \frac{1}{\delta_n} \int_{\mathbb{R}\setminus\{0\}} (\overline{f} - \underline{f}) \mu_{\delta_n,t_n}^{\delta_n}(\mathrm{d}y) \leqslant \dot{\nu}_t[x - 1/m, x + 1/m]$$

which tends to 0 as $m \to \infty$. □

Lemma 10. *Let $f : \{(t,\delta) \in [0,T) \times (0,T] : t + \delta \leqslant T\} \to \mathbb{R}$ and suppose there exists a continuous function $g : [0,T] \to \mathbb{R}$ such that, for all $t \in [0,T]$, $\lim_{\delta \downarrow 0, s \to t} f(s,\delta) = g(t)$. Then $\lim_{\delta \downarrow 0} \sup_{t \in [0,T)} |f(t,\delta) - g(t)| = 0$.*

Proof. By assumption, $\lim_{\delta\downarrow 0}|f(t,\delta)-g(t)|=0$ for every $t\in[0,T)$. Suppose that the convergence is not uniform in t. Then there exists $\varepsilon>0$ and a sequence $\{(t_n,\delta_n)\}_{n\geqslant 1}\subset\{(t,\delta)\in[0,T)\times(0,T]: t+\delta\leqslant T\}$ with $\delta_n\to 0$ such that $|f(t_n,\delta_n)-g(t_n)|>\varepsilon$ for all n. By the Bolzano Weierstrass theorem, there exists $t\in[0,T]$ and a subsequence $\{t_{n_k}\}_{k\geqslant 1}$ of $\{t_n\}_{n\geqslant 1}$ such that $\lim_k t_{n_k}=t$. Hence,

$$\varepsilon<|f(t_{n_k},\delta_{n_k})-f(t_{n_k})|\leqslant|f(t_{n_k},\delta_{n_k})-g(t)|+|g(t_{n_k})-g(t)|\to 0 \text{ as } k\to\infty.$$

From this contradiction we conclude that the convergence is indeed uniform, thereby proving the statement. □

Lemma 11. *Let* $\{\alpha_{\delta_{m,k}}\}$ *and* $\{\eta_{\delta_{m,k}}\}$ *satisfy* (3). *Then*

$$\lim_{m\to\infty}\prod_{k=0}^{m-1}\alpha_{\delta_{m,k}}=\alpha^T,\quad \lim_{m\to\infty}\prod_{k=0}^{m-1}(1+\eta_{\delta_{m,k}})=(1+\eta)^T.$$

Proof. We prove the first statement for $\alpha_{\delta_{m,k}}$. The proof of the second statement is completely analogous and omitted. Notice that

$$\alpha_{\delta_{m,k}}=\left(\left(1-\delta_{m,k}\Big(\frac{1-\alpha_{\delta_{m,k}}}{\delta_{m,k}}\Big)\right)^{1/\delta_{m,k}}\right)^{\delta_{m,k}},$$

We immediately use this to see that

$$\log\Big[\prod_{k=0}^{m-1}\alpha_{\delta_{m,k}}\Big]=\sum_{k=0}^{m-1}\delta_{m,k}\log\left[\left(1-\delta_{m,k}\Big(\frac{1-\alpha_{\delta_{m,k}}}{\delta_{m,k}}\Big)\right)^{1/\delta_{m,k}}\right]$$

By (3) and the well-known convergence result

$$\lim_{\delta\to 0}\Big(1+\delta^{-1}a+o(\delta)\Big)^{1/\delta}=e^a,$$

for any real a and any higher order term $o(\delta)$, we get

$$\lim_{m\to\infty}\sup_{k\leqslant m-1}\left|\left(1-\delta_{m,k}\Big(\frac{1-\alpha_{\delta_{m,k}}}{\delta_{m,k}}\Big)\right)^{1/\delta_{m,k}}-\alpha\right|=0.$$

Hence we conclude that

$$\lim_{m\to\infty}\log\Big[\prod_{k=0}^{m-1}\alpha_{\delta_{m,k}}\Big]=T\log(\alpha).$$

This proves the result for $\prod_{k=0}^{m-1}\alpha_{\delta_{m,k}}$. □

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