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On time-inconsistent stopping problems and mixed strategy stopping times

Sören Christensen* & Kristoffer Lindensjö†

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Abstract

A game-theoretic framework for time-inconsistent stopping problems where the time-inconsistency is due to the consideration of a non-linear function of an expected reward is developed. A class of mixed strategy stopping times that allows the agents in the game to choose the intensity function of a Cox process is introduced. A subgame perfect Nash equilibrium is defined. The equilibrium is characterized and other results with different necessary and sufficient conditions for equilibrium are proven. This includes a smooth fit result. A mean-variance problem and a variance problem are studied as examples. The state process is a general one-dimensional Itô diffusion.

Keywords: Conditional Poisson process, Cox process, Equilibrium stopping time, Mean-variance criterion, Mixed strategies, Optimal stopping, Subgame perfect Nash equilibrium, Time-inconsistency, Variance criterion.

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1 Introduction

Consider a diffusion X and the classical problem of choosing a stopping time τ that maximizes

$$\mathbb{E}_x(h(X_\tau)),$$

where h is a nice deterministic function. Recall that the solution to this problem is consistent in the sense that the optimal rule for stopping, i.e. 'stop the first time that X enters the stopping region', is independent of the initial state x. Now consider a non-linear nice deterministic function g and the problem of choosing a stopping time τ that maximizes

$$g(\mathbb{E}_x(h(X_\tau))).$$

The optimal stopping rule for this problem will, in contrast, typically depend on the initial state x, which means that it will not generally satisfy Bellman's principle of optimality. In the literature this is known as *time-inconsistency*. In the present paper we study a more general version of this problem, see (2.1) below.

Time-inconsistent problems are typically studied using one of the following approaches:

- The *game-theoretic approach*, which means formulating the problem as a game and look for equilibrium stopping times, cf. Remark 2.4 and Remark 2.5 below.
- The *pre-commitment approach*, which means formulating the problem for a fixed initial state and allowing the corresponding optimal stopping rule to depend on that initial state.
- The *dynamic optimality approach*, developed in [30]. See also [9] for a short description.

Time-inconsistent problems were originally studied in financial economics where the inconsistency is due to:

- Endogenous habit formation,
- Non-exponential discounting, or
- Mean-variance optimization/utility.

For a description of these terms in a stopping problem context we refer to [9]. Mean-variance optimization is also described in Section 4.2.

In [9] we developed a game-theoretic framework for time-inconsistent stopping problems covering endogenous habit formation and non-exponential discounting. In the present paper, a game-theoretic framework for time-inconsistent stopping problems that can handle e.g. mean-variance problems is developed. In the present paper we also define mixed strategy stopping times by allowing the agents in the game to jointly choose the intensity function of a Cox process that is used as a randomization device for the stopping decision, see Definition 2.2 and Remark 2.4 below. The use of the Cox process is crucial for the definition of mixed strategies in our framework as it allows us to identify equilibrium strategies. This type of mixed strategy stopping time appears to be novel, although other types of mixed strategies in stopping games have been considered and the Cox process has been used in other ways in different kinds of stopping games, see Section 1.1.

The rest of the paper is organized as follows: In Section 2 we formulate the type of time-inconsistent stopping problem that we consider and give the equilibrium definition. In Section 3 the equilibrium is characterized and other results with necessary and sufficient conditions for equilibrium are proven, these are the main results of the present paper, see Theorem 3.2, Theorem 3.5, Theorem 3.6 and Theorem 3.7. In Section 4 we formulate and study two well-known problems in our framework; a mean-variance problem and a variance problem.

1.1 Previous literature

Recently, there has been a substantial effort to develop the literature on the game-theoretic approach to time-inconsistent control problems, see e.g. [3, 4, 23] and the references therein. The development of the literature on the game-theoretic approach to time-inconsistent stopping problems is in an earlier stage. Recent papers include [1, 9, 11, 18, 19, 20, 21]. For short surveys of time-inconsistent stopping problems we refer to [1, 9, 30].

In [1] the game-theoretic approach is used to study mean-variance and mean-standard deviation stopping problems in discrete time. The authors consider mixed strategies defined as randomized stopping times and also equilibrium liquidation strategies. In [33] a continuous-time Dynkin game with mixed strategies defined as randomized stopping times is studied. It is instructive to note that the number of players in the games of these papers are countable and two, respectively, while the number of players in the game of the present paper is uncountable; in the framework of the present paper it is the Cox process construction of mixed stopping strategies that makes it possible to identify mixed equilibrium strategies.

Recent papers on time-inconsistent stopping problems and the dynamic optimality and pre-commitment approaches include [26, 30].

In [27], a mean-field optimal stopping game for e.g. bank-runs is studied. The default time is modeled as the first jump time of a given Cox process. In [10, 16], optimal stopping problems where stopping can only occur at exogenously determined Poisson jump times are studied.

See Section 4 for previous literature on mean-variance and variance problems.

2 Problem formulation

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}_x)$ be a filtered probability space carrying a one-dimensional Wiener process W. Let X be a one-dimensional diffusion living on an open interval $E = (\alpha, \beta)$, where $-\infty \leq \alpha \leq \beta \leq \infty$, which is the unique strong solution to the SDE

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \ \ X_0 = x.$$

The coefficients $\mu : E \to \mathbb{R}$ and $\sigma : E \to (0, \infty)$ are continuous and satisfy conditions guaranteeing the existence of a unique strong solution, see e.g. [22]. Moreover, for each continuous function $\lambda : E \to [0, \infty)$ the filtered probability space is assumed to carry an X-associated Cox process denoted by N^λ, meaning that N^λ is a Poisson process with intensity corresponding to $\lambda(X_t)$ conditional on the natural filtration generated by X, see e.g. [2, Sec. 6.6]. It is assumed that the filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions and that $x \mapsto \mathbb{P}_x(F)$ is measurable for each $F \in \mathcal{F}$. The associated expectations are denoted by $\mathbb{E}_x$. It is assumed that a measurable time shift operator θ with $X_\tau \circ \theta_{\tau_h} = X_{\tau\circ\theta_{\tau_h}+\tau_h}$ exists, where τ is a, possibly infinite, stopping time (with respect to $(\mathcal{F}_t)_{t\geq 0}$) and

$$\tau_h := \inf\{t \geq 0 : |X_t - X_0| \geq h\}.$$

Now consider the functions $f, h : E \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ satisfying Assumption 2.6 (below) and the problem of finding a stopping time τ that maximizes

$$J_\tau(x) := \mathbb{E}_x(f(X_\tau)) + g(\mathbb{E}_x(h(X_\tau))). \tag{2.1}$$

Remark 2.1. We use the convention that $h(X_\tau) := \limsup_{t\to\infty} h(X_t)$ on $\{\tau = \infty\}$ and similarly for f. We assume that the limits $g(\infty) := \lim_{x\to\infty} g(x)$ and $g(-\infty) := \lim_{x\to-\infty} g(x)$ exist, see Assumption 2.6.

The function g is allowed to be non-linear and hence, i) the standard theory for optimal stopping problems cannot be used to find an optimal stopping time for (2.1), and ii) even if we find an optimal stopping time for (2.1) it will generally be time-inconsistent. Specifically, a stopping rule that is optimal for (2.1) when the initial state is x is typically not optimal when the initial state is $y \neq x$. Based on the second fact we will reinterpret the problem as a game.

Let us specify which type of stopping times are admissible (Definition 2.2) and then give the equilibrium definition (Definition 2.3). For a fixed stopping time τ we define the functions φ_τ and ψ_τ by,

$$\varphi_\tau(x) = \mathbb{E}_x(f(X_\tau)) \text{ and } \psi_\tau(x) = \mathbb{E}_x(h(X_\tau)).$$

Definition 2.2. Consider a continuous function $\lambda : E \to [0, \infty)$ and the corresponding Cox process N^λ. Let $\tau^\lambda := \inf\{t \geq 0 : N_t^\lambda \neq N_{t-}^\lambda\}$. Let $C \subset E$ be an open set and let $\tau^C := \inf\{t \geq 0 : X_t \notin C\}$. Then $\tau^{\lambda,C} := \tau^\lambda \wedge \tau^C$ is said to be a mixed Markov strategy stopping time. A mixed Markov strategy stopping time $\tau^{\lambda,C}$ is said to be admissible if the function $J_{\tau^{\lambda,C}}$ in (2.1) is well-defined and the functions $\varphi_{\tau^{\lambda,C}}$ and $\psi_{\tau^{\lambda,C}}$ are continuous. The space of admissible mixed Markov strategy stopping times is denoted by $\mathcal{N}$.

Usually we write $\varphi_{\lambda,C}$ instead of $\varphi_{\tau^{\lambda,C}}$ and similarly for $\psi_{\tau^{\lambda,C}}$ and $J_{\tau^{\lambda,C}}$. We remark that the requirement that $\varphi_{\lambda,C}$ and $\psi_{\lambda,C}$ must be continuous in order for $\tau^{\lambda,C}$ to be admissible is a technical condition. For $\tau^{\lambda,C}, \tau^{\eta,D} \in \mathcal{N}$ we will use the notation

$$\tau^{\lambda,C} \diamond \tau^{\eta,D}(h) = I_{\{\tau^{\eta,D} \leq \tau_h\}} \tau^{\eta,D} + I_{\{\tau^{\eta,D} > \tau_h\}} (\tau^{\lambda,C} \circ \theta_{\tau_h} + \tau_h).$$

Definition 2.3. A stopping time $\hat{\tau} \in \mathcal{N}$ is said to be a (mixed Markov strategy) equilibrium stopping time if the equilibrium condition

$$\liminf_{h \searrow 0} \frac{J_{\hat{\tau}}(x) - J_{\hat{\tau} \diamond \tau^{\eta,D}(h)}(x)}{\mathbb{E}_x(\tau_h)} \geq 0 \tag{2.2}$$

is satisfied for each $\tau^{\eta,D} \in \mathcal{N}$ and each $x \in E$. If $\hat{\tau}$ is an equilibrium stopping time then $J_{\hat{\tau}}(x), x \in E$, is said to be the corresponding equilibrium value function.

This paper is devoted to the question of how to find equilibrium stopping times of the type in Definition 2.3.

Remark 2.4. The game corresponding to the equilibrium in Definition 2.3 is interpreted as follows (see also [9]): Consider a person who controls the process X and who wants to maximize $J_\tau(x)$ in (2.1). Interpret this person as comprising different versions of herself, one for each $x \in E$. These x-versions are interpreted as agents playing a sequential game against each other regarding when to stop the process X. The equilibrium condition (2.2) ensures that no agent wants to deviate from the equilibrium strategy $\hat{\tau}$ by using another strategy $\tau^{\eta,D}$ during the infinitesimally short time interval $[0, \tau_h]$ as long as every other agent plays $\hat{\tau}$; neither by stopping when $\hat{\tau}$ prescribes continuing, nor by continuing when $\hat{\tau}$ prescribes stopping, nor by using a different intensity than the one prescribed by $\hat{\tau}$. The equilibrium condition (2.2) is in line with the one in [9] and inspired by time-inconsistent stopping problems in financial economics, see e.g. [12, 17], and also by the equilibrium definition for time-inconsistent stochastic control problems, see [3, 4, 23] and the references therein.

Remark 2.5. In game theory, a pure Markov strategy determines the action of an agent based only on payoff relevant information. A mixed strategy determines the action of an agent using a randomization device that randomly selects a pure strategy. In the present model, a stopping strategy is thus pure if each x-version bases the decision to stop or not only on the current value of the state process. Hence, τ^C is a pure strategy stopping time. Strategies of the kind τ^λ in Definition 2.2 use Cox processes as randomization devices. This motivates calling $\tau^{\lambda,C}$ a mixed Markov strategy. We remark that Definition 2.3 is a (mixed strategy) subgame perfect Nash equilibrium, see e.g. [9, 25].

We denote the characteristic operator of X by A_X, i.e. for any function $f : E \to \mathbb{R}$,

$$A_X f(x) = \lim_{h \searrow 0} \frac{\mathbb{E}_x(f(X_{\tau_h})) - f(x)}{\mathbb{E}_x(\tau_h)},$$

whenever this expression exists. Recall that if $f \in \mathcal{C}^2(E)$ then

$$A_X f(x) = \mu(x) f'(x) + \frac{1}{2}\sigma^2(x) f''(x).$$

Throughout the paper we assume that the functions f, g and h in (2.1) satisfy the following conditions:

Assumption 2.6.

- *$f, h \in \mathcal{C}^2(E)$ and $g \in \mathcal{C}^3(\mathbb{R})$.*
- *$g(\infty)$ and $g(-\infty)$ exist in $[-\infty, \infty]$.*
- *f is either bounded from below or above on E. This also holds for h.*

3 Equilibrium conditions

This section contains a characterization of the equilibrium, see Theorem 3.2. It also contains other necessary and sufficient conditions for equilibrium, see Theorem 3.5, Theorem 3.6 and Theorem 3.7. These are the main results of the present paper. They rely on the results found in the appendix which mainly contain explicit expressions for the type of limit that is found in the left side of the equilibrium condition (2.2) for different values of the initial state, see Lemma 5.2, Lemma 5.3 and Lemma 5.4. The results in the appendix rely to a large extent on arguments similar to those in the proof of Lemma 3.1 and standard Taylor expansion. Theorem 3.5 and Theorem 3.6 rely on Proposition 3.3.

Lemma 3.1. *For any* $\tau^{\lambda,C},\tau^{\eta,D}\in\mathcal{N}$ *and* $x\in D$,

$$\lim_{h\searrow 0}\frac{\varphi_{\tau^{\lambda,C}\circ\theta_{\tau_h}+\tau_h}(x)-\varphi_{\tau^{\lambda,C}\diamond\tau^{\eta,D}(h)}(x)}{\mathbb{E}_x(\tau_h)}=\eta(x)(\varphi_{\lambda,C}(x)-f(x)).$$

Proof. Recall that D is open by definition of $\mathcal{N}$. This implies that for any $x\in D$ there exists a constant $\bar{h}>0$ such $\tau_h<\tau^D$ for each $0<h\leq\bar{h}$ (a.s.). Hence, for $0<h\leq\bar{h}$,

$$\begin{aligned}\tau^{\lambda,C}\diamond\tau^{\eta,D}(h)&=I_{\{\tau^{\eta,D}\leq\tau_h\}}\tau^{\eta,D}+I_{\{\tau^{\eta,D}>\tau_h\}}(\tau^{\lambda,C}\circ\theta_{\tau_h}+\tau_h)\\&=I_{\{\tau^{\eta}\leq\tau_h\}}\tau^{\eta,D}+I_{\{\tau^{\eta}>\tau_h\}}(\tau^{\lambda,C}\circ\theta_{\tau_h}+\tau_h)\\&=I_{\{\tau^{\eta}\leq\tau_h\}}\tau^{\eta}+I_{\{\tau^{\eta}>\tau_h\}}(\tau^{\lambda,C}\circ\theta_{\tau_h}+\tau_h).\end{aligned}$$

It follows that

$$\begin{aligned}f\left(X_{\tau^{\lambda,C}\diamond\tau^{\eta,D}(h)}\right)&=I_{\{\tau^{\eta,D}\leq\tau_h\}}f\left(X_{\tau^{\lambda,C}\diamond\tau^{\eta,D}(h)}\right)+I_{\{\tau^{\eta,D}>\tau_h\}}f\left(X_{\tau^{\lambda,C}\diamond\tau^{\eta,D}(h)}\right)\\&=I_{\{\tau^{\eta}\leq\tau_h\}}f\left(X_{\tau^{\lambda,C}\diamond\tau^{\eta,D}(h)}\right)+I_{\{\tau^{\eta}>\tau_h\}}f\left(X_{\tau^{\lambda,C}\diamond\tau^{\eta,D}(h)}\right)\\&=I_{\{\tau^{\eta}\leq\tau_h\}}f\left(X_{\tau^{\eta}}\right)+I_{\{\tau^{\eta}>\tau_h\}}f\left(X_{\tau^{\lambda,C}\circ\theta_{\tau_h}+\tau_h}\right).\end{aligned}\tag{3.1}$$

Using the above, the properties of the Poisson process and by conditioning on the filtration generated by X, we obtain, for $0<h\leq\bar{h}$, (here $\eta_t:=\eta(X_t)$)

$$\begin{aligned}&\varphi_{\tau^{\lambda,C}\circ\theta_{\tau_h}+\tau_h}(x)-\varphi_{\tau^{\lambda,C}\diamond\tau^{\eta,D}(h)}(x)\\&=\mathbb{E}_x\left(f\left(X_{\tau^{\lambda,C}\circ\theta_{\tau_h}+\tau_h}\right)-f\left(X_{\tau^{\lambda,C}\diamond\tau^{\eta,D}(h)}\right)\right)\\&=\mathbb{E}_x\left(I_{\{\tau^{\eta}\leq\tau_h\}}\left(f\left(X_{\tau^{\lambda,C}\circ\theta_{\tau_h}+\tau_h}\right)-f\left(X_{\tau^{\eta}}\right)\right)\right)\\&=\mathbb{E}_x\left(\int_0^\infty\eta_te^{-\int_0^t\eta_sds}I_{\{t\leq\tau_h\}}\left(f\left(X_{\tau^{\lambda,C}\circ\theta_{\tau_h}+\tau_h}\right)-f\left(X_t\right)\right)dt\right)\\&=\mathbb{E}_x\left(\int_0^{\tau_h}\eta_te^{-\int_0^t\eta_sds}\left(f\left(X_{\tau^{\lambda,C}\circ\theta_{\tau_h}+\tau_h}\right)-f\left(X_t\right)\right)dt\right)\end{aligned}$$

$$= \mathbb{E}_x \left(f \left(X_{\tau^{\lambda,C} \circ \theta_{\tau_h} + \tau_h} \right) \int_0^{\tau_h} \eta_t e^{-\int_0^t \eta_s ds} dt - \int_0^{\tau_h} \eta_t e^{-\int_0^t \eta_s ds} f(X_t)\, dt \right).$$

By conditioning on $\mathcal{F}_{\tau_h}$ and the strong Markov property we thus obtain

$$\begin{aligned}
&\varphi_{\tau^{\lambda,C} \circ \theta_{\tau_h} + \tau_h}(x) - \varphi_{\tau^{\lambda,C} \diamond \tau^{\eta,D}(h)}(x) \\
&= \mathbb{E}_x \left(\int_0^{\tau_h} \eta_t e^{-\int_0^t \eta_s ds} dt\, \mathbb{E}_x \left(f \left(X_{\tau^{\lambda,C} \circ \theta_{\tau_h} + \tau_h} \right) | \mathcal{F}_{\tau_h} \right) - \int_0^{\tau_h} \eta_t e^{-\int_0^t \eta_s ds} f(X_t)\, dt \right) \\
&= \mathbb{E}_x \left(\int_0^{\tau_h} \eta_t e^{-\int_0^t \eta_s ds} dt \varphi_{\lambda,C}(X_{\tau_h}) - \int_0^{\tau_h} \eta_t e^{-\int_0^t \eta_s ds} f(X_t)\, dt \right) \\
&= \mathbb{E}_x \left(\int_0^{\tau_h} \eta_t e^{-\int_0^t \eta_s ds} (\varphi_{\lambda,C}(X_{\tau_h}) - f(X_t)) dt \right). \qquad (3.2)
\end{aligned}$$

Now use the continuity of the functions $f, \eta, \varphi_{\lambda,C}$ and the paths of X, and that X is bounded on $[0, \tau_h]$, to obtain

$$\begin{aligned}
&\lim_{h \searrow 0} \frac{\varphi_{\tau^{\lambda,C} \circ \theta_{\tau_h} + \tau_h}(x) - \varphi_{\tau^{\lambda,C} \diamond \tau^{\eta,D}(h)}(x)}{\mathbb{E}_x(\tau_h)} \\
&= \lim_{h \searrow 0} \frac{\mathbb{E}_x \left(\int_0^{\tau_h} \eta(X_t) e^{-\int_0^t \eta(X_s) ds} \left(\varphi_{\lambda,C}(X_{\tau_h}) - f(X_t) \right) dt \right)}{\mathbb{E}_x(\tau_h)} \\
&= \eta(x) \left(\varphi_{\lambda,C}(x) - f(x) \right).
\end{aligned}$$

□

We are now ready to present the first main result, which characterizes the equilibrium.

Theorem 3.2. *A stopping time $\tau^{\lambda,C} \in \mathcal{N}$ is an equilibrium stopping time if and only if it is a solution to the following system,*

$$J_{\lambda,C}(x) - f(x) - g(h(x)) \geq 0, \quad \textit{for } x \in C, \qquad \text{(I)}$$

$$A_X f(x) + g'(h(x)) A_X h(x) \leq 0, \quad \textit{for } x \in \textit{int}(C^c), \qquad \text{(II)}$$

$$f(x) - \varphi_{\lambda,C}(x) + g'(\psi_{\lambda,C}(x)) \left(h(x) - \psi_{\lambda,C}(x) \right) = 0, \quad \textit{for } x \in C \textit{ with } \lambda(x) > 0, \qquad \text{(III)}$$

$$f(x) - \varphi_{\lambda,C}(x) + g'(\psi_{\lambda,C}(x)) \left(h(x) - \psi_{\lambda,C}(x) \right) \leq 0, \quad \textit{for } x \in C \textit{ with } \lambda(x) = 0, \qquad \text{(IV)}$$

$$\liminf_{h \searrow 0} \frac{-a(x,h)}{\mathbb{E}_x(\tau_h)} \geq 0, \quad \textit{for } x \in \partial C, \qquad \text{(V)}$$

where

$$a(x,h) := \mathbb{E}_x \left(\varphi_{\lambda,C}(X_{\tau_h}) \right) - \varphi_{\lambda,C}(x) + g \left(\mathbb{E}_x \left(\psi_{\lambda,C}(X_{\tau_h}) \right) \right) - g(\psi_{\lambda,C}(x)) \qquad (3.3)$$

See the appendix for a proof of Theorem 3.2. We will use the following general result.

Proposition 3.3. *Consider a fixed $x \in E$ and a function $k : E \to \mathbb{R}$. Suppose that there exists a constant $\bar{h} > 0$ such that k is $\mathcal{C}^2$ on $[x - \bar{h}, x]$ and $[x, x + \bar{h}]$ and continuous on $[x - \bar{h}, x + \bar{h}]$, then*

$$\lim_{h \searrow 0} \frac{\left(\mathbb{E}_x(k(X_{\tau_h})) - k(x)\right)^2}{\mathbb{E}_x(\tau_h)} = \left(\frac{k'(x+) - k'(x-)}{2}\right)^2 \sigma^2(x).$$

In particular, for the local time of X at x, denoted by $l_t^x(X)$, it holds that

$$\lim_{h \searrow 0} \frac{\mathbb{E}_x\left(l^x_{\tau_h}(X)\right)^2}{\mathbb{E}_x(\tau_h)} = \sigma^2(x).$$

Proof. Use the Itô-Tanaka formula, see e.g. [31] or [32, p. 75], to obtain, for $0 < h \leq \bar{h}$,

$$\begin{aligned} k(X_{\tau_h}) - k(x) &= \int_0^{\tau_h} A_X k(X_t) I_{\{X_t \neq x\}} dt + \int_0^{\tau_h} k'(X_t)\sigma(X_t) I_{\{X_t \neq x\}} dW_t \\ &\quad + \frac{1}{2}\int_0^{\tau_h} \left(k'(X_t+) - k'(X_t-)\right) I_{\{X_t = x\}} dl_t^x(X) \\ &= \int_0^{\tau_h} A_X k(X_t) I_{\{X_t \neq x\}} dt + \int_0^{\tau_h} k'(X_t)\sigma(X_t) I_{\{X_t \neq x\}} dW_t \\ &\quad + \frac{1}{2}\left(k'(x+) - k'(x-)\right) l^x_{\tau_h}(X). \end{aligned}$$

Thus,

$$\begin{aligned} &\lim_{h \searrow 0} \frac{\left(\mathbb{E}_x(k(X_{\tau_h})) - k(x)\right)^2}{\mathbb{E}_x(\tau_h)} \\ &= \lim_{h \searrow 0} \frac{\left(\mathbb{E}_x\left(\int_0^{\tau_h} A_X k(X_t) I_{\{X_t \neq x\}} dt\right) + \frac{1}{2}\left(k'(x+) - k'(x-)\right)\mathbb{E}_x\left(l^x_{\tau_h}(X)\right)\right)^2}{\mathbb{E}_x(\tau_h)}. \end{aligned}$$

Since $\lim_{h \searrow 0} \frac{\mathbb{E}_x\left(\int_0^{\tau_h} A_X k(X_t) I_{\{X_t \neq x\}} dt\right)}{\mathbb{E}_x(\tau_h)}$ converges and $\lim_{h \searrow 0} \mathbb{E}_x\left(l^x_{\tau_h}(X)\right) = 0$ it directly follows that

$$\lim_{h \searrow 0} \frac{\left(\mathbb{E}_x(k(X_{\tau_h})) - k(x)\right)^2}{\mathbb{E}_x(\tau_h)} = \left(\frac{(k'(x+) - k'(x-))}{2}\right)^2 \lim_{h \searrow 0} \frac{\mathbb{E}_x\left(l^x_{\tau_h}(X)\right)^2}{\mathbb{E}_x(\tau_h)} \tag{3.4}$$

Applying the result in (3.4) for $k(y) := |y - x|$ (recall that x is fixed) gives us

$$\lim_{h \searrow 0} \frac{\left(\mathbb{E}_x(|X_{\tau_h} - x|)\right)^2}{\mathbb{E}_x(\tau_h)} = \left(\frac{(1 - (-1))}{2}\right)^2 \lim_{h \searrow 0} \frac{\mathbb{E}_x\left(l^x_{\tau_h}(X)\right)^2}{\mathbb{E}_x(\tau_h)}$$

$$= \lim_{h \searrow 0} \frac{\mathbb{E}_x \left(l^x_{\tau_h}(X) \right)^2}{\mathbb{E}_x(\tau_h)}. \tag{3.5}$$

However, it is also easy to see that,

$$\begin{aligned} \lim_{h \searrow 0} \frac{(\mathbb{E}_x(|X_{\tau_h} - x|))^2}{\mathbb{E}_x(\tau_h)} &= \lim_{h \searrow 0} \frac{(p_h|x+h-x| + (1-p_h)|x-h-x|)^2}{\mathbb{E}_x(\tau_h)} \\ &= \lim_{h \searrow 0} \frac{h^2}{\mathbb{E}_x(\tau_h)}, \end{aligned} \tag{3.6}$$

where $p_h := \mathbb{P}_x(X_{\tau_h} = x + h)$. The result follows from (3.4), (3.5), and (3.6), if we can prove that

$$\lim_{h \searrow 0} \frac{h^2}{\mathbb{E}_x(\tau_h)} = \sigma^2(x). \tag{3.7}$$

Let us find a representation for $\mathbb{E}_x(\tau_h)$. Recall that the scale function can be represented as

$$s(x) = \int_a^x e^{-\int_a^y \frac{2\mu(z)}{\sigma^2(z)} dz} dy,$$

for an arbitrary $a \in E$, and that it solves the ODE $\mu(x)s'(x) + \frac{1}{2}\sigma^2(x)s''(x) = 0$, see [6, p.18]. Using this it is easy to verify that

$$u(x) := Bs(x) - \int_a^x s'(y) \int_a^y \frac{2}{s'(z)\sigma^2(z)} dzdy$$

solves

$$\mu(x)u'(x) + \frac{1}{2}\sigma^2(x)u''(x) = -1, \tag{3.8}$$

where B is an arbitrary constant. Let us choose B such that $u(b) = 0$ for some constant $b > a$. This gives us

$$B = \frac{\int_a^b s'(y) \int_a^y \frac{2}{s'(z)\sigma^2(z)} dzdy}{s(b)}.$$

Thus, for the chosen B, the function u solves (3.8) and satisfies the boundary conditions $u(b) = u(a) = 0$ (the last equality is trivial). According to the standard theory it therefore follows that $u(x) = \mathbb{E}_x(\tau^D)$ for $x \in D := (a, b)$, see e.g. [28, ch.9]. Now consider a and b to be variables and write the function u as

$$u(x, a, b) = \frac{s(x)}{s(b)} \int_a^b s'(y) \int_a^y \frac{2}{s'(z)\sigma^2(z)} dzdy - \int_a^x s'(y) \int_a^y \frac{2}{s'(z)\sigma^2(z)} dzdy.$$

where we note that also $s'(y) = e^{-\int_a^y \frac{2\mu(z)}{\sigma^2(z)}dz}$ and $s(x)$ depend on a. It follows that $u(x, x-h, x+h) = \mathbb{E}_x(\tau_h)$. Hence, with $\xi(\cdot) := \frac{2\mu(\cdot)}{\sigma^2(\cdot)}$, we obtain

$$\begin{aligned}
\mathbb{E}_x(\tau_h) &= \frac{s(x)}{s(x+h)} \int_{x-h}^{x+h} e^{-\int_{x-h}^{y}\xi(z)dz} \int_{x-h}^{y} \frac{2e^{\int_{x-h}^{z}\xi(u)du}}{\sigma^2(z)} dzdy \\
&\quad - \int_{x-h}^{x} e^{-\int_{x-h}^{y}\xi(z)dz} \int_{x-h}^{y} \frac{2e^{\int_{x-h}^{z}\xi(u)du}}{\sigma^2(z)} dzdy \\
&= \frac{\int_{x-h}^{x} e^{-\int_{x-h}^{y}\xi(z)dz}dy}{\int_{x-h}^{x+h} e^{-\int_{x-h}^{y}\xi(z)dz}dy} \int_{x-h}^{x+h} e^{-\int_{x-h}^{y}\xi(z)dz} \int_{x-h}^{y} \frac{2e^{\int_{x-h}^{z}\xi(u)du}}{\sigma^2(z)} dzdy \\
&\quad - \int_{x-h}^{x} e^{-\int_{x-h}^{y}\xi(z)dz} \int_{x-h}^{y} \frac{2e^{\int_{x-h}^{z}\xi(u)du}}{\sigma^2(z)} dzdy. \qquad (3.9)
\end{aligned}$$

Using the representation (3.9) it is possible to show, using lengthy but straightforward calculations, that (3.7) holds. Indeed, the basic observation is that the chain of equalities in (3.9) continues as follows, for small $h > 0$,

$$\begin{aligned}
&\approx \frac{1}{2}\int_{x-h}^{x+h}\int_{x-h}^{y} \frac{2}{\sigma^2(z)}dzdy - \int_{x-h}^{x}\int_{x-h}^{y} \frac{2}{\sigma^2(z)}dzdy \\
&= \frac{1}{2}\int_{x}^{x+h}\int_{x-h}^{y} \frac{2}{\sigma^2(z)}dzdy - \frac{1}{2}\int_{x-h}^{x}\int_{x-h}^{y} \frac{2}{\sigma^2(z)}dzdy \\
&= \int_{x}^{x+h}\int_{x-h}^{y} \frac{1}{\sigma^2(z)}dzdy - \int_{x-h}^{x}\int_{x-h}^{y} \frac{1}{\sigma^2(z)}dzdy \\
&= \int_{x}^{x+h}\int_{x-h}^{y} \frac{1}{\sigma^2(z)}dzdy - \int_{x}^{x+h}\int_{x-h}^{y-h} \frac{1}{\sigma^2(z)}dzdy \\
&= \int_{x}^{x+h}\int_{y-h}^{y} \frac{1}{\sigma^2(z)}dzdy,
\end{aligned}$$

which, when divided by h^2, clearly approaches $1/\sigma^2(x)$, as $h \to 0$. □

Remark 3.4. The last part of the proof of Proposition 3.3 consists of showing that (3.7) holds. This was also recently shown in [15].

Theorem 3.5 below presents a smooth fit condition that an equilibrium value function must satisfy at any $x \in \partial C$, under additional assumptions. We use this result when making an ansatz to finding an equilibrium stopping time in Section 4.2.

Theorem 3.5. *Suppose that $\tau^{\lambda,C}$ is an equilibrium stopping time. For a fixed $x \in \partial C$, if the functions $\varphi_{\lambda,C}$ and $\psi_{\lambda,C}$ are $\mathcal{C}^2$ on $[x-\bar{h}, x]$ and $[x, x+\bar{h}]$ for some* *constant $\bar{h} > 0$, then the equilibrium value function $J_{\lambda,C}$ satisfies smooth fit in the sense that*

$$J'_{\lambda,C}(x) = f'(x) + g'(h(x))h'(x).$$

Proof. Consider a fixed $x \in \partial C$. For any ϵ, satisfying $x + \epsilon \in E$ (both negative and positive such ϵ exist since E is open and ∂C is the boundary of C in E), it holds that

$$J_{\lambda,C}(x+\epsilon) \geq f(x+\epsilon) + g(h(x+\epsilon)).$$

To see this use that this inequality is an equality when $x + \epsilon \notin C$, and condition (I) for the case $x + \epsilon \in C$. Moreover, since $x \in \partial C$ it follows that $J_{\lambda,C}(x) = f(x) + g(h(x))$. Hence,

$$J_{\lambda,C}(x+\epsilon) - J_{\lambda,C}(x) \geq f(x+\epsilon) - f(x) + g(h(x+\epsilon)) - g(h(x)).$$

If $\epsilon < 0$ it follows that

$$\frac{J_{\lambda,C}(x+\epsilon) - J_{\lambda,C}(x)}{\epsilon} \leq \frac{f(x+\epsilon) - f(x)}{\epsilon} + \frac{g(h(x+\epsilon)) - g(h(x))}{\epsilon}.$$

Hence, the left derivative satisfies $J'^{(-)}_{\lambda,C}(x) \leq f'(x) + g'(h(x))h'(x)$. The right derivative can be similarly dealt with and we thus obtain

$$J'^{(-)}_{\lambda,C}(x) \leq f'(x) + g'(h(x))h'(x) \leq J'^{(+)}_{\lambda,C}(x). \tag{3.10}$$

Let us now prove that if we would not have smooth fit then condition (V) would be violated and hence smooth fit must hold, by Theorem 3.2. Note that if smooth fit would not hold then $J'^{(+)}_{\lambda,C}(x) - J'^{(-)}_{\lambda,C}(x) > 0$, cf. (3.10), which is equivalent to

$$\varphi'_{\lambda,C}(x+) + g'(\psi_{\lambda,C}(x))\psi'_{\lambda,C}(x+) > \varphi'_{\lambda,C}(x-) + g'(\psi_{\lambda,C}(x))\psi'_{\lambda,C}(x-).$$

To see this use that $J_{\lambda,C}(x) = \varphi_{\lambda,C}(x) + g(\psi_{\lambda,C}(x))$ and the chain rule, and then the differentiability assumptions (i.e. $\varphi_{\lambda,C}$ and $\psi_{\lambda,C}$ are $\mathcal{C}^2$ on $[x - \bar{h}, x]$ and $[x, x + \bar{h}]$) and continuity (of $\varphi_{\lambda,C}$ and $\psi_{\lambda,C}$, cf. admissibility, Definition 2.2). Rewrite the equation above as

$$\varphi'_{\lambda,C}(x+) - \varphi'_{\lambda,C}(x-) + g'(\psi_{\lambda,C}(x))(\psi'_{\lambda,C}(x+) - \psi'_{\lambda,C}(x-)) > 0. \tag{3.11}$$

The differentiability assumptions imply that we can use the Itô-Tanaka formula to obtain, for $0 < h < \bar{h}$,

$$\varphi_{\lambda,C}(X_{\tau_h}) - \varphi_{\lambda,C}(x)$$

$$= \int_0^{\tau_h} A_X \varphi_{\lambda,C}(X_t) I_{\{X_t \neq x\}} dt + \int_0^{\tau_h} \varphi'_{\lambda,C}(X_t)\sigma(X_t) I_{\{X_t \neq x\}} dW_t$$
$$+ \frac{1}{2}\left(\varphi'_{\lambda,C}(x+) - \varphi'_{\lambda,C}(x-)\right) l^x_{\tau_h}(X).$$

Thus,

$$\mathbb{E}_x\left(\varphi_{\lambda,C}(X_{\tau_h})\right) - \varphi_{\lambda,C}(x) = \ a_1(h) + a_2\,\mathbb{E}_x\left(l^x_{\tau_h}(X)\right),$$

for $a_1(h) := \mathbb{E}_x\left(\int_0^{\tau_h} A_X \varphi_{\lambda,C}(X_t) I_{\{X_t \neq x\}} dt\right)$ and $a_2 := \frac{1}{2}(\varphi'_{\lambda,C}(x+) - \varphi'_{\lambda,C}(x-))$. Similarly,

$$\mathbb{E}_x\left(\psi_{\lambda,C}(X_{\tau_h})\right) - \psi_{\lambda,C}(x) = \ b_1(h) + b_2\,\mathbb{E}_x\left(l^x_{\tau_h}(X)\right),$$

for $b_1(h) := \mathbb{E}_x\left(\int_0^{\tau_h} A_X \psi_{\lambda,C}(X_t) I_{\{X_t \neq x\}} dt\right)$ and $b_2 := \frac{1}{2}(\psi'_{\lambda,C}(x+) - \psi'_{\lambda,C}(x-))$. Hence, using standard Taylor expansion of the function g we write $a(x,h)$ in (3.3) as,

$$\begin{aligned} a(x,h) = \ & a_1(h) + a_2\,\mathbb{E}_x\left(l^x_{\tau_h}(X)\right) \\ & + g'(\psi_{\lambda,C}(x))\{b_1(h) + b_2\,\mathbb{E}_x\left(l^x_{\tau_h}(X)\right)\} \\ & + \frac{1}{2} g''(\psi_{\lambda,C}(x))\{b_1(h) + b_2\,\mathbb{E}_x\left(l^x_{\tau_h}(X)\right)\}^2 + \ldots \end{aligned}$$

This can be written as,

$$\frac{-a(x,h)}{\mathbb{E}_x(\tau_h)}$$
$$= -\frac{a_1(h)}{\mathbb{E}_x(\tau_h)} - g'(\psi_{\lambda,C}(x))\frac{b_1(h)}{\mathbb{E}_x(\tau_h)} \tag{3.12}$$
$$- (a_2 + g'(\psi_{\lambda,C}(x))b_2)\frac{\mathbb{E}_x\left(l^x_{\tau_h}(X)\right)}{\mathbb{E}_x(\tau_h)} \tag{3.13}$$
$$- \frac{1}{2} g''(\psi_{\lambda,C}(x))\frac{\left(\mathbb{E}_x\left(\psi_{\lambda,C}(X_{\tau_h})\right) - \psi_{\lambda,C}(x)\right)^2}{\mathbb{E}_x(\tau_h)} - \ldots \tag{3.14}$$

Let us see what happens to the liminf of $\frac{-a(x,h)}{\mathbb{E}_x(\tau_h)}$ when sending $h \searrow 0$: The liminf of the terms in (3.12) are finite due to the differentiability assumptions for $\varphi_{\lambda,C}$ and $\psi_{\lambda,C}$. The term in (3.13) can be written as

$$- (a_2 + g'(\psi_{\lambda,C}(x))b_2)\frac{\mathbb{E}_x\left(l^x_{\tau_h}(X)\right)}{\mathbb{E}_x(\tau_h)}$$
$$= -\frac{1}{2}\left(\varphi'_{\lambda,C}(x+) - \varphi'_{\lambda,C}(x-) + g'(\psi_{\lambda,C}(x))(\psi'_{\lambda,C}(x+) - \psi'_{\lambda,C}(x-))\right)$$

$$\times \frac{\mathbb{E}_x\left(l^x_{\tau_h}(X)\right)}{\mathbb{E}_x(\tau_h)}.$$

Thus, it follows from Proposition 3.3 and the contradiction assumption (3.11) that the liminf of the term in (3.13) is equal to $-\infty$ and that the liminf of the terms in (3.14) are finite (we remark that terms of order 3, and higher, vanish, since $\psi_{\lambda,C}$ is continous). This implies that condition (V) would indeed be violated if (3.11) were true and smooth fit must therefore hold. $\square$

Theorem 3.2 presents necessary and sufficient conditions for a stopping time $\tau^{\lambda,C}$ to be an equilibrium stopping time. If we for an equilibrium stopping time candidate $\tau^{\lambda,C}$ can find explicit expressions for the functions $\varphi_{\lambda,C}$ and $\psi_{\lambda,C}$ then it is easy to verify if conditions (I)–(IV) hold whereas condition (V) is not necessarily easy to verify. Theorem 3.6 below presents a more easily verified characterization of condition (V), given additional differentiability conditions. We will use Theorem 3.6 to verify an ansatz to finding an equilibrium in Section 4.2.

Theorem 3.6. *Consider a stopping time $\tau^{\lambda,C} \in \mathcal{N}$. If for any fixed $x \in \partial C$ there exists a constant $\bar{h} > 0$ such that the functions $\varphi_{\lambda,C}$ and $\psi_{\lambda,C}$ are $\mathcal{C}^2$ on $[x-\bar{h}, x]$ and $[x, x+\bar{h}]$ and such that the function $\varphi_{\lambda,C}(\cdot) + g'(\psi_{\lambda,C}(x))\psi_{\lambda,C}(\cdot)$ is $\mathcal{C}^1$ on $[x-\bar{h}, x+\bar{h}]$ then condition* (V) *is equivalent to,*

$$A_X\varphi_{\lambda,C}(x+) + g'(\psi_{\lambda,C}(x))A_X\psi_{\lambda,C}(x+) + A_X\varphi_{\lambda,C}(x-) + g'(\psi_{\lambda,C}(x))A_X\psi_{\lambda,C}(x-)$$

$$+ g''(\psi_{\lambda,C}(x))\left(\frac{\psi'_{\lambda,C}(x+) - \psi'_{\lambda,C}(x-)}{2}\right)^2 \sigma^2(x) \le 0. \tag{3.15}$$

Proof. Consider an arbitrary $x \in \partial C$. Use the Itô-Tanaka formula, to arrive at the same expression as in (3.12)–(3.14). Note that the $\mathcal{C}^1$ assumption in the statement of the theorem directly implies that $a_2 + g'(\psi_{\lambda,C}(x))b_2 = 0$. This implies, using (3.12)–(3.14), that the expression that we take the limit of in (V) can be written as

$$\begin{aligned}
&\frac{-a(x,h)}{\mathbb{E}_x(\tau_h)}\\
&= -\frac{\mathbb{E}_x\left(\int_0^{\tau_h}\left(A_X\varphi_{\lambda,C}(X_t)I_{\{X_t\neq x\}} + g'(\psi_{\lambda,C}(x))A_X\psi_{\lambda,C}(X_t)I_{\{X_t\neq x\}}\right)dt\right)}{\mathbb{E}_x(\tau_h)}\\
&\quad - \frac{1}{2}g''(\psi_{\lambda,C}(x))\frac{\left(\mathbb{E}_x\left(\psi_{\lambda,C}(X_{\tau_h})\right) - \psi_{\lambda,C}(x)\right)^2}{\mathbb{E}_x(\tau_h)} - \ldots
\end{aligned}$$

The differentiability assumptions and basic properties of diffusions imply that

$$\lim_{h\searrow 0}\left(\frac{\mathbb{E}_x\left(\int_0^{\tau_h}\left(A_X\varphi_{\lambda,C}(X_t)I_{\{X_t\neq x\}}+g'(\psi_{\lambda,C}(x))A_X\psi_{\lambda,C}(X_t)I_{\{X_t\neq x\}}\right)dt\right)}{\mathbb{E}_x(\tau_h)}\right)$$
$$=\frac{1}{2}\left(A_X\varphi_{\lambda,C}(x+)+g'(\psi_{\lambda,C}(x))A_X\psi_{\lambda,C}(x+)\right)$$
$$+\frac{1}{2}\left(A_X\varphi_{\lambda,C}(x-)+g'(\psi_{\lambda,C}(x))A_X\psi_{\lambda,C}(x-)\right).$$

Now use Proposition 3.3 to obtain the result. □

Theorem 3.7 below presents a necessary condition for equilibria for $x\in C$ in the case that the equilibrium intensity function is strictly positive, under additional assumptions. This result will be used when we make an ansatz to finding an equilibrium stopping time in Section 4.1.

Theorem 3.7. *Suppose that $\tau^{\lambda,C}$ is an equilibrium stopping time with $\lambda(x)>0$ for $x\in C$ and that $\psi_{\lambda,C}$ is $\mathcal{C}^2$ on C. Then $\psi_{\lambda,C}$ satisfies the (non-linear) ODE*

$$-\left(\mu(x)\psi'_{\lambda,C}(x)+\frac{1}{2}\sigma^2(x)\psi''_{\lambda,C}(x)\right)(h(x)-\psi_{\lambda,C}(x))g''(\psi_{\lambda,C}(x))$$
$$=\mu(x)\{f'(x)+h'(x)g'(\psi_{\lambda,C}(x))\}+\frac{1}{2}\sigma^2(x)\{f''(x)+d(x)\},\ \text{for } x\in C, \tag{3.16}$$

where

$$d(x):=g'''(\psi_{\lambda,C}(x))(\psi'_{\lambda,C}(x))^2\left(h(x)-\psi_{\lambda,C}(x)\right)$$
$$+2g''(\psi_{\lambda,C}(x))\psi'_{\lambda,C}(x)(h'(x)-\psi'_{\lambda,C}(x))+g'(\psi_{\lambda,C}(x))h''(x) \tag{3.17}$$

Moreover, the equilibrium intensity function λ satisfies

$$\lambda(x)(h(x)-\psi_{\lambda,C}(x))^2g''(\psi_{\lambda,C}(x))$$
$$=\mu(x)\{f'(x)+h'(x)g'(\psi_{\lambda,C}(x))\}+\frac{1}{2}\sigma^2(x)\{f''(x)+d(x)\},\ \text{for } x\in C. \tag{3.18}$$

Proof. Suppose that $\tau^{\lambda,C}$ is an equilibrium stopping time with $\lambda(x)>0$ for $x\in C$. Consider an arbitrary fixed $x\in C$. By definition $J_{\lambda,C}(x)=\varphi_{\lambda,C}(x)+g(\psi_{\lambda,C}(x))$ and hence

$$A_X(J_{\lambda,C}(x)-\varphi_{\lambda,C}(x)-g(\psi_{\lambda,C}(x)))=0. \tag{3.19}$$

Condition (III) holds by Theorem 3.2, i.e.

$$0=f(x)-\varphi_{\lambda,C}(x)+g'(\psi_{\lambda,C}(x))\left(h(x)-\psi_{\lambda,C}(x)\right)$$

$$= f(x) - (\varphi_{\lambda,C}(x) + g(\psi_{\lambda,C}(x))) + g(\psi_{\lambda,C}(x)) + g'(\psi_{\lambda,C}(x))\,(h(x) - \psi_{\lambda,C}(x))\,.$$

Since $J_{\lambda,C}(x) = \varphi_{\lambda,C}(x) + g(\psi_{\lambda,C}(x))$ we thus obtain

$$J_{\lambda,C}(x) = f(x) + g(\psi_{\lambda,C}(x)) + g'(\psi_{\lambda,C}(x))\,(h(x) - \psi_{\lambda,C}(x))\,.$$

This implies that

$$\begin{aligned} &J_{\lambda,C}(x) - \varphi_{\lambda,C}(x) - g(\psi_{\lambda,C}(x)) \\ &= f(x) - \varphi_{\lambda,C}(x) + g'(\psi_{\lambda,C}(x))\,(h(x) - \psi_{\lambda,C}(x))\,. \end{aligned} \tag{3.20}$$

We will notationally suppress λ, C and (x) in the rest of the proof. From (3.19) and (3.20) follows that $A_X f - A_X\varphi + A_X\,(g'(\psi)\,(h-\psi)) = 0$ which implies that

$$A_X\varphi = A_X f + A_X\,(g'(\psi)\,(h-\psi))\,. \tag{3.21}$$

Now use Lemma 5.1 and then condition (III) to see that

$$\begin{aligned} A_X\varphi &= \lambda(\varphi - f) \\ &= \lambda g'(\psi)\,(h-\psi)\,. \end{aligned} \tag{3.22}$$

Let us investigate the expressions in the right side of (3.21). The assumed differentiability implies that

$$\begin{aligned} A_X f &= \mu f' + \frac{1}{2}\sigma^2 f'', \quad \text{and} \\ A_X\,(g'(\psi)\,(h-\psi)) &= \mu(g'(\psi)\,(h-\psi))' + \frac{1}{2}\sigma^2(g'(\psi)\,(h-\psi))''. \end{aligned}$$

Use standard differentiation rules to find that the derivatives in the last expression can be written as

$$\begin{aligned} (g'(\psi)\,(h-\psi))' &= g''(\psi)\psi'\,(h-\psi) + g'(\psi)\,(h'-\psi') \\ &= \psi'\{g''(\psi)\,(h-\psi) - g'(\psi)\} + h'g'(\psi) \\ &= \psi' b + h'g'(\psi), \end{aligned}$$

where we use the temporary notation $b := g''(\psi)\,(h-\psi) - g'(\psi)$, and

$$\begin{aligned} &(g'(\psi)\,(h-\psi))'' \\ &= g'''(\psi)\psi'\psi'\,(h-\psi) + g''(\psi)\psi''\,(h-\psi) + g''(\psi)\psi'\,(h'-\psi') + g''(\psi)\psi'\,(h'-\psi') \\ &\quad + g'(\psi)\,(h''-\psi'') \end{aligned}$$

$$
\begin{aligned}
&= g'''(\psi)(\psi')^2 (h-\psi) + 2g''(\psi)\psi'(h'-\psi') + g'(\psi)h'' \\
&\quad + \psi''\{g''(\psi)(h-\psi) - g'(\psi)\} \\
&= d + \psi'' b,
\end{aligned}
$$

where d is defined in (3.17). It follows that the right side of (3.21) can be written as

$$
\begin{aligned}
&A_X f + A_X \left(g'(\psi)(h-\psi)\right) \\
&= \mu f' + \frac{1}{2}\sigma^2 f'' + \mu(\psi' b + h' g'(\psi)) + \frac{1}{2}\sigma^2 (d + \psi'' b) \\
&= \mu\{f' + h' g'(\psi)\} + \frac{1}{2}\sigma^2\{f'' + d\} + b\{\mu\psi' + \frac{1}{2}\sigma^2\psi''\} \\
&= \mu\{f' + h' g'(\psi)\} + \frac{1}{2}\sigma^2\{f'' + d\} + bA_X\psi \\
&= \mu\{f' + h' g'(\psi)\} + \frac{1}{2}\sigma^2\{f'' + d\} + b\lambda(\psi - h),
\end{aligned}
$$

where we in the last two rows relied on Lemma 5.1 (which analogously holds also for the function ψ) and the differential operator form of A_X (where we again relied on the assumed differentiability for ψ). Use the equality above, (3.21) and (3.22) to obtain

$$
\lambda g'(\psi)(h-\psi) = \mu\{f' + h' g'(\psi)\} + \frac{1}{2}\sigma^2\{f'' + d\} + b\lambda(\psi - h).
$$

This implies that

$$
\lambda(h-\psi)\{g'(\psi) + b\} = \mu\{f' + h' g'(\psi)\} + \frac{1}{2}\sigma^2\{f'' + d\}.
$$

Use that $b + g'(\psi) = g''(\psi)(h-\psi)$ to see that (3.18) follows. Now use Lemma 5.1 to obtain $A_X\psi = \lambda(\psi - h)$. Using the assumed differentiability for ψ we also obtain $A_X\psi = \mu\psi' + \frac{1}{2}\sigma^2\psi''$. Hence, $\lambda(h-\psi) = -\left(\mu\psi' + \frac{1}{2}\sigma^2\psi''\right)$ which, together with (3.18), implies that (3.16) holds. □

4 Examples

In this section we will use the general results of the previous section to solve two particular time-inconsistent stopping problems.

4.1 A variance stopping problem

The variance stopping problem corresponds to the time-inconsistent problem of trying to maximize

$$\mathrm{Var}_x(X_\tau).$$

An economic motivation for a variance stopping problem is found in [29] and the references therein. Variance stopping problems are also studied in [13, 14] using randomized stopping times. We also refer to [7, 8]. All these references consider the problem from the perspective of the pre-commitment approach.

The variance problem is given by $f(x) := x^2, g(x) := -x^2$ and $h(x) := x$. To see this note that

$$\begin{aligned}
J_\tau(x) &= \varphi_\tau(x) + g(\psi_\tau(x)) \\
&= \mathbb{E}_x(f(X_\tau)) + g(\mathbb{E}_x(h(X_\tau))) \\
&= \mathbb{E}_x(X_\tau^2) - \mathbb{E}_x^2(X_\tau) \\
&= \mathrm{Var}_x(X_\tau).
\end{aligned}$$

We consider a positive state process X. In this case Assumption 2.6 is satisfied. It follows that

$$g'(\psi_{\lambda,C}(x)) = -2\psi_{\lambda,C}(x) \text{ and } J_{\lambda,C}(x) = \varphi_{\lambda,C}(x) - \psi_{\lambda,C}^2(x). \tag{4.1}$$

Hence,

$$\begin{aligned}
& f(x) - \varphi_{\lambda,C}(x) + g'(\psi_{\lambda,C}(x))\,(h(x) - \psi_{\lambda,C}(x)) \\
&= x^2 - \varphi_{\lambda,C}(x) - 2\psi_{\lambda,C}(x)\,(x - \psi_{\lambda,C}(x)) \\
&= x^2 - \varphi_{\lambda,C}(x) - 2x\psi_{\lambda,C}(x) + 2\psi_{\lambda,C}^2(x) \\
&= -(\varphi_{\lambda,C}(x) - \psi_{\lambda,C}^2(x)) + x^2 - 2x\psi_{\lambda,C}(x) + \psi_{\lambda,C}^2(x) \\
&= -J_{\lambda,C}(x) + (\psi_{\lambda,C}(x) - x)^2.
\end{aligned} \tag{4.2}$$

Since the variance is trivially *minimized* by stopping immediately it follows that no equilibrium stopping time can ever recommend immediate stopping. Hence, we make an ansatz with $C = E$. Specifically, we make the ansatz that an equilibrium stopping time is given by $\tau^{\lambda,E}$ for some strictly positive intensity function λ which is to be determined. We will use the notation $\tau^{\lambda,E} = \tau^\lambda$, $\psi_{\lambda,E} = \psi_\lambda$ etc.

We immediately obtain the following result.

Theorem 4.1. *A stopping time $\tau^\lambda \in \mathcal{N}$, with $\lambda(x) > 0$ for each $x \in E$, is an equilibrium stopping time for the variance problem if and only if*

$$J_\lambda(x) = (\psi_\lambda(x) - x)^2, \quad \text{for } x \in E. \tag{4.3}$$

Moreover, if (4.3) *holds then J_λ given by* (4.3) *is the corresponding equilibrium value function.*

Proof. Use that $h(x) = x$, $-f(x)-g(x) = 0$ and (4.2) to see that if (4.3) holds then (I) and (III) hold, whereas (II), (IV) and (V) can be considered trivially fulfilled, since we use $C = E$ and $\lambda(x) > 0$. Now, if (III) holds then it follows from (4.2) and $C = E$ that (4.3) holds. Thus, the first assertion follows from Theorem 3.2. The second assertion follows immediately. □

Let us use the ODE condition (3.16) in Theorem 3.7 to identify a candidate for ψ_λ and then use the result (3.18) to identify the corresponding candidate equilibrium intensity function λ. In the present case the ODE (3.16) is

$$\begin{aligned}&-\left(\mu(x)\psi_\lambda'(x) + \frac{1}{2}\sigma^2(x)\psi_\lambda''(x)\right)(x-\psi_\lambda(x))(-2)\\&= \mu(x)\{2x - 2\psi_\lambda(x)\} + \frac{1}{2}\sigma^2(x)\{2 + d(x)\},\end{aligned}$$

with

$$\begin{aligned}d(x) &= 0 - 2\cdot 2\psi_\lambda'(x)(1-\psi_\lambda'(x))\\&= 4(\psi_\lambda'(x))^2 - 4\psi_\lambda'(x),\end{aligned}$$

where we used (4.1), $f'(x) = 2x$, $g'''(x) = 0$ etc. We note that if $x - \psi_\lambda(x) \neq 0$, then the ODE simplifies to

$$\begin{aligned}\mu(x)\psi_\lambda'(x) + \frac{1}{2}\sigma^2(x)\psi_\lambda''(x) &= \mu(x) + \frac{1}{2}\sigma^2(x)\frac{2+d(x)}{2x-2\psi_\lambda(x)}\\&= \mu(x) + \frac{1}{2}\sigma^2(x)\frac{1+2(\psi_\lambda'(x))^2 - 2\psi_\lambda'(x)}{x-\psi_\lambda(x)}\\&= \mu(x) + \frac{1}{2}\sigma^2(x)\frac{(\psi_\lambda'(x)-1)^2 + (\psi_\lambda'(x))^2}{x-\psi_\lambda(x)}.\end{aligned} \tag{4.4}$$

In case X is a geometric Brownian motion it turns out that the problem can be solved explicitly. Thus, from now we assume (in this example) that

$$dX_t = \mu X_t dt + \sigma X_t dW_t. \tag{4.5}$$

In this case (4.4) becomes

$$\mu x(\psi_\lambda'(x) - 1) + \frac{1}{2}\sigma^2 x^2\left(\psi_\lambda''(x) - \frac{(\psi_\lambda'(x)-1)^2 + (\psi_\lambda'(x))^2}{x-\psi_\lambda(x)}\right) = 0. \tag{4.6}$$

The ODE (4.6) has, under appropriate assumptions for the constants μ and σ, one solution (at least) on the form $\psi_\lambda(x) = cx$ for some constant $c \neq 0, 1$. To see this use that $\psi_\lambda''(x) = (cx)'' = 0$ and that $x > 0$, since $E = (0, \infty)$ for the GBM. Now use (3.18) and the candidate $\psi_\lambda(x) = cx$ to obtain the corresponding candidate intensity

$$\begin{aligned}\lambda(x) &= \frac{\mu(x)\{f'(x) + h'(x)g'(\psi_\lambda(x))\} + \frac{1}{2}\sigma^2(x)\{f''(x) + d(x).\}}{(h(x) - \psi_\lambda(x))^2 g''(\psi_\lambda(x))} \\ &= \frac{\mu x\{2x - 2cx\} + \frac{1}{2}\sigma^2 x^2\{2 + 4c^2 - 4c.\}}{(x - cx)^2(-2)} \\ &= \frac{\mu\{1 - c\} + \frac{1}{2}\sigma^2\{1 + 2c^2 - 2c.\}}{-(1 - c)^2}. \end{aligned} \tag{4.7}$$

This means the candidate solution $\psi_\lambda(x) = cx$ corresponds to using a constant intensity (depending on the constant c). This constant candidate intensity could, with some effort, be found by identifying the constant(s) c such that $\psi_\lambda(x) = cx$ solves (4.6), and inserting this c into (4.7) and thereby obtaining a corresponding constant equilibrium intensity candidate. We shall, however, instead use Theorem 4.1 to identify the constant equilibrium intensity (it turns out that only one constant equilibrium intensity exists) and thereby verify that the ansatz works. This is done in the proof of Theorem 4.2.

Theorem 4.2. *Let X be given by* (4.5) *where the constants μ and σ satisfy $\sigma^2 > 0$ and*

$$2\mu + \sigma^2 < 0. \tag{4.8}$$

Then τ^λ, with

$$\lambda = \sqrt{\frac{-\mu^2(2\mu + \sigma^2)}{\sigma^2}}, \tag{4.9}$$

is an equilibrium stopping time. The corresponding equilibrium value function is

$$J_\lambda(x) = \frac{1}{\left(\sqrt{\frac{-(2\mu+\sigma^2)}{\sigma^2}} + 1\right)^2} x^2.$$

Remark 4.3. From the formula for the variance of the log-normal X_t it follows that $\lim_{t\to\infty} \text{Var}_x(X_t) = 0$ for any $x \in E$ if (4.8) holds, whereas $\lim_{t\to\infty} \text{Var}_x(X_t) = \infty$ for any $x \in E$ if (4.8) does not hold. Hence, we only consider the case when (4.8) holds. We remark that condition (4.8) is also used in [29].

Proof. We remark that it follows from the calculations below that τ^λ is admissible. Using that $X_t = xe^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t}$ is log-normal and conditioning on the exponentially distributed stopping time τ^λ we directly obtain

$$\psi_\lambda(x) = \mathbb{E}_x(X_{\tau^\lambda}) = \frac{\lambda}{\lambda - \mu} x \quad \text{and} \quad \varphi_\lambda(x) = \mathbb{E}_x(X^2_{\tau^\lambda}) = \frac{\lambda}{\lambda - 2\mu - \sigma^2} x^2.$$

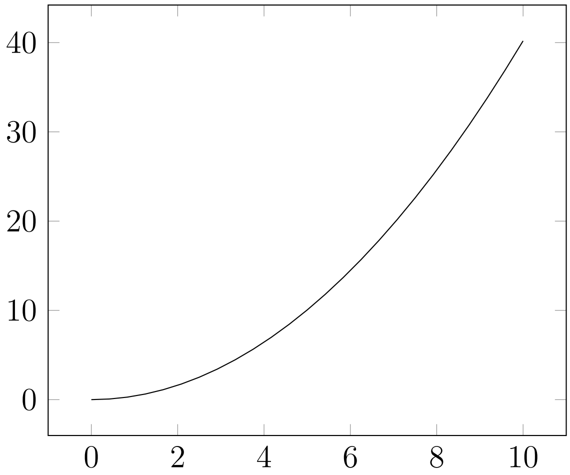

Figure 1: The equilibrium value function $x \mapsto J_\lambda(x)$, where λ is given by (4.9), for the parameters $\mu = -0.1$ and $\sigma^2 = 0.15$. In this case $\lambda \approx 0.0577$.

Here we relied on the denominators being positive, which follows directly from $\lambda > 0$ and $\mu < 0$, and $\lambda > 0$ and (4.8) respectively; where (4.8) implied that $\mu < 0$ and $\lambda > 0$. It follows that

$$\begin{aligned}
\psi_\lambda^2(x) &= \frac{\lambda^2}{(\lambda-\mu)^2}x^2, \\
(\psi_\lambda(x) - x)^2 &= \left(\frac{\lambda}{\lambda-\mu} - 1\right)^2 x^2 \\
&= \frac{\mu^2}{(\lambda-\mu)^2}x^2.
\end{aligned}$$

Using (4.1) and (4.9) we thus obtain, for any fixed $x \in E$,

$$\begin{aligned}
&\frac{J_\lambda(x) - (\psi_\lambda(x) - x)^2}{x^2} \\
&= \frac{\varphi_\lambda(x) - \psi_\lambda^2(x) - (\psi_\lambda(x) - x)^2}{x^2} \\
&= \frac{\lambda}{\lambda - 2\mu - \sigma^2} - \frac{\lambda^2 + \mu^2}{(\lambda-\mu)^2} \\
&= \frac{\lambda(\lambda-\mu)^2 - (\lambda - 2\mu - \sigma^2)(\lambda^2 + \mu^2)}{(\lambda - 2\mu - \sigma^2)(\lambda-\mu)^2} \\
&= \frac{(\lambda^3 - 2\lambda^2\mu + \lambda\mu^2) - (\lambda^3 - 2\mu\lambda^2 - \sigma^2\lambda^2 + \lambda\mu^2 - 2\mu^3 - \sigma^2\mu^2)}{(\lambda - 2\mu - \sigma^2)(\lambda-\mu)^2}
\end{aligned}$$

$$= \frac{2\mu^3 + \mu^2\sigma^2 + \sigma^2\lambda^2}{(\lambda - 2\mu - \sigma^2)(\lambda - \mu)^2}.$$

From Theorem 4.1 it therefore follows that τ^λ is an equilibrium stopping time when $2\mu^3 + \mu^2\sigma^2 + \sigma^2\lambda^2 = 0$, i.e. when λ satisfies $\lambda^2 = \frac{-\mu^2(2\mu+\sigma^2)}{\sigma^2}$. This proves the first assertion. Using the calculations above, $\mu < 0$, and Theorem 4.1, we obtain

$$\begin{aligned} J_\lambda(x) =& (\psi_\lambda(x) - x)^2 \\ =& \frac{\mu^2}{(\lambda-\mu)^2} x^2 \\ =& \frac{\mu^2}{\left(\sqrt{\frac{-\mu^2(2\mu+\sigma^2)}{\sigma^2}} - \mu\right)^2} x^2 \\ =& \frac{1}{\left(\sqrt{\frac{-(2\mu+\sigma^2)}{\sigma^2}} + 1\right)^2} x^2, \end{aligned}$$

which is the second assertion. □

Remark 4.4. In [14], the results from [29] on the pre-commitment version of the variance stopping problem are generalized to underlying geometric Lévy processes. In this paper, we have decided to developed the theory only for underlying diffusion processes to avoid certain technical difficulties. Therefore, applying our time-consistent approach to underlying jump processes would need some further work that we do not carry out here. We, nonetheless, want to mention that obtaining equilibrium conditions of the form (4.3) for the variance problem for underlying geometric Lévy processes of the form $X_t = X_0 e^{L_t}$, L a Lévy process, can also be obtained. It is then interesting to note that considering τ^λ for a constant $\lambda > 0$ yields – under suitable integrability conditions – that

$$\psi_\lambda(x) = \mathbb{E}_1(X_{\tau^\lambda})x = a_\lambda x, \;\; a_\lambda = \frac{\lambda}{\lambda - \Psi_L(1)} \quad \text{and}$$
$$\varphi_\lambda(x) = \mathbb{E}_1(X_{\tau^\lambda}^2)x^2 = b_\lambda x^2, \;\; b_\lambda = \frac{\lambda}{\lambda - \Psi_{2L}(1)},$$

where Ψ denotes the Laplace exponent. Hence, a similar calculation as in the previous proof yields both a formula for λ and the corresponding equilibrium value function also in this case.

4.2 A mean-variance stopping problem

Mean-variance optimization is one of the classical problems in financial economics. It was first studied in the context of optimal portfolio allocation in the seminal paper [24]. A vast number of papers on the topic have since then been published. For short surveys and economic motivation of mean-variance problems we refer to [5, 30] and the references therein. The mean-variance stopping problem corresponds to the time-inconsistent problem of trying to maximize

$$\mathbb{E}_x(X_\tau) - \gamma \mathrm{Var}_x(X_\tau), \text{ with } \gamma > 0.$$

Here γ is a given constant representing risk-aversion. In [30] a mean-variance stopping problem for a geometric Brownian motion is studied using the dynamic optimality approach and the pre-commitment approach. In [1] a mean-variance stopping problem for a general discrete time Markov chain is studied, see also Section 1.1. In [5] a mean-variance control problem is studied using the general game-theoretic framework for time-inconsistent stochastic control of [3].

The mean-variance stopping problem is given by $f(x) := -\gamma x^2$, $g(x) := x + \gamma x^2$ and $h(x) := x$. To see this note that

$$\begin{aligned} J_\tau(x) &= \varphi_\tau(x) + g(\psi_\tau(x)) \\ &= \mathbb{E}_x(f(X_\tau)) + g(\mathbb{E}_x(h(X_\tau))) \\ &= -\gamma\, \mathbb{E}_x(X_\tau^2) + \mathbb{E}_x(X_\tau) + \gamma\, \mathbb{E}_x^2(X_\tau) \\ &= \mathbb{E}_x(X_\tau) - \gamma \mathrm{Var}_x(X_\tau). \end{aligned}$$

We consider a positive state process X. In this case Assumption 2.6 is satisfied. Note that $g'(h(x)) = 1 + 2\gamma x$, $g'(\psi_{\lambda,C}(x)) = 1 + 2\gamma\psi_{\lambda,C}(x)$, and $J_{\lambda,C}(x) = \varphi_{\lambda,C}(x) + \psi_{\lambda,C}(x) + \gamma\psi_{\lambda,C}^2(x)$. Therefore,

$$\begin{aligned} & f(x) - \varphi_{\lambda,C}(x) + g'(\psi_{\lambda,C}(x))\,(h(x) - \psi_{\lambda,C}(x)) \\ &= -\gamma x^2 - \varphi_{\lambda,C}(x) + (1 + 2\gamma\psi_{\lambda,C}(x))\,(x - \psi_{\lambda,C}(x)) \\ &= -\gamma x^2 - \varphi_{\lambda,C}(x) + x - \psi_{\lambda,C}(x) + 2\gamma x\psi_{\lambda,C}(x) - 2\gamma\psi_{\lambda,C}^2(x) \\ &= x - (\varphi_{\lambda,C}(x) + \psi_{\lambda,C}(x) + \gamma\psi_{\lambda,C}^2(x)) - (\gamma x^2 - 2\gamma x\psi_{\lambda,C}(x) + \gamma\psi_{\lambda,C}^2(x)) \\ &= x - J_{\lambda,C}(x) - \gamma(\psi_{\lambda,C}(x) - x)^2. \end{aligned}$$

It follows that conditions (III) and (IV) can be written as

$$\begin{aligned} J_{\lambda,C}(x) &= x - \gamma(\psi_{\lambda,C}(x) - x)^2, \quad \text{for } x \in C \text{ with } \lambda(x) > 0, \\ J_{\lambda,C}(x) &\geq x - \gamma(\psi_{\lambda,C}(x) - x)^2, \quad \text{for } x \in C \text{ with } \lambda(x) = 0. \end{aligned}$$

Using that $f(x) + g(h(x)) = x$ we write condition (I) as,

$$J_{\lambda,C}(x) \geq x, \quad \text{for } x \in C. \tag{4.10}$$

Let us again consider the geometric Brownian motion. In the typical case it is reasonable to suppose that $J_{\lambda,C}(x) - x > 0$ for $x \in C$ and in this case we note, using Lemma 5.2, that if $\tau^{\eta,D} \in \mathcal{N}$ with $\eta = 0$, then, for $x \in C \cap D$,

$$\begin{aligned}
&\lim_{h \searrow 0} \frac{J_{\tau^{\lambda,C}}(x) - J_{\tau^{\lambda,C} \diamond \tau^{\eta,D}(h)}(x)}{\mathbb{E}_x(\tau_h)} \\
&= \lambda(x)\{f(x) - \varphi_{\lambda,C}(x) + g'(\psi_{\lambda,C}(x))\,(h(x) - \psi_{\lambda,C}(x))\} \\
&= \lambda(x)\{x - J_{\lambda,C}(x) - \gamma(\psi_{\lambda,C}(x) - x)^2\} \\
&< 0.
\end{aligned}$$

Consequently we make the ansatz $\lambda(x) = 0$ for $x \in C$. Specifically, we make the ansatz that τ^C for $C = (0, b)$ is an equilibrium stopping time for some b to be determined. We start by noting that if τ^C satisfies (4.10) then condition (I) and condition (IV) are satisfied, and condition (III) is irrelevant (since the ansatz is $\lambda = 0$ on C). Hence, if we can find a set $C = (0, b)$ such that (4.10), (II) and (V) are satisfied then τ^C is an equilibrium strategy.

Theorem 4.5. *Let X be given by*

$$dX_t = \mu X_t dt + \sigma X_t dW_t, \quad \text{where } \sigma^2 > 0.$$

If $\mu \in (0, \sigma^2/4]$, then $\hat{\tau} = \inf\{t \geq 0 : X_t \geq b\}$ with $b = \frac{\xi}{\gamma(1-\xi)}$, where $\xi := \frac{2\mu}{\sigma^2}$, is an equilibrium stopping time and the corresponding equilibrium value function is,

$$J_{\hat{\tau}}(x) = \begin{cases} x, & x \geq b, \\ x^{1-\xi}(b^{\xi} - \gamma b^{1+\xi}) + \gamma b^{2\xi} x^{2-2\xi}, & x < b. \end{cases}$$

Remark 4.6. If $\mu \leq 0$ then X is a supermartingale (with a last element) and it follows directly from $J_\tau(x) = \mathbb{E}_x(X_\tau) - \gamma \mathrm{Var}_x(X_\tau)$, Definition 2.3 and the optional sampling theorem that it is an equilibrium strategy to always stop immediately. If $\mu \geq \frac{\sigma^2}{2}$ then $\tau^b := \inf\{t \geq 0 : X_t \geq b\} < \infty$ a.s. for any initial state $x \leq b$ for each $b \in E$ and $J_{\tau^b}(x) = \mathbb{E}_x(X_{\tau^b}) - \gamma\left(\mathbb{E}_x(X^2_{\tau^b}) - \mathbb{E}^2_x(X_{\tau^b})\right) = b - \gamma(b^2 - b^2) = b$ can thus become arbitrarily large.

Remark 4.7. A mean-variance optimal stopping problem for a GBM is studied in [30]. There it is shown that the stopping time $\hat{\tau}$ in Theorem 4.5 is *dynamically optimal* when $\mu \in (0, \sigma^2/2)$, see [30, Theorem 3]. It is also argued that this stopping time is a subgame perfect Nash equilibrium when $\mu \in (0, \sigma^2/4]$, see [30, Sec. 4], which is in line with our findings in Theorem 4.5. For the case $\mu \in (\sigma^2/4, \sigma^2/2)$, it can be proved that the threshold time $\hat{\tau}$ is not an equilibrium stopping time in our setting.

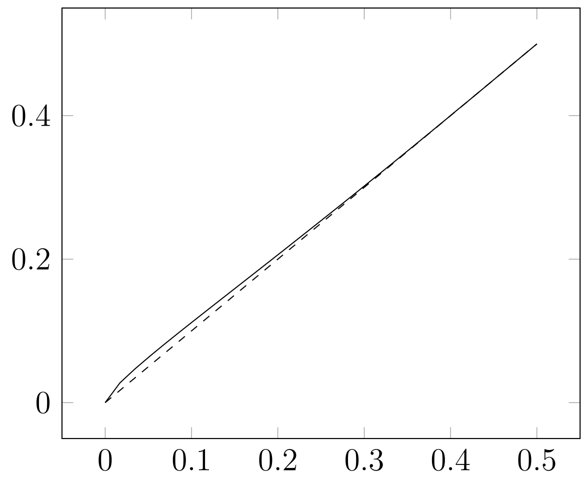

Figure 2: The equilibrium value function $x \mapsto J_{\hat{\tau}}(x)$ (solid) and $x \mapsto f(x) + g(x) = x$ (dashed) in the GBM case with parameters $\mu = 0.07, \sigma^2 = 0.45$ and $\gamma = 1.1$ (in this case $b \approx 0.4106$).

Proof. We remark that it follows from the calculations below that $\hat{\tau}$ is admissible. A stopping time is, according to Theorem 3.2, an equilibrium stopping time if and only if conditions (I)—(V) are satisfied. Note that we do not have to check (III) since $\hat{\tau}$ has no Cox process component, which corresponds to $\lambda(x) = 0$ for each x. Recall that if (4.10) is satisfied then (I) and (IV) are also satisfied. Note that (II) can in this case be written as

$$\begin{aligned}
&A_X f(x) + g'(h(x))A_X h(x)\\
&= \left(\mu x \frac{\partial}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2}\right)(-\gamma x^2) + (1 + 2\gamma x)\left(\mu x \frac{\partial}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2}\right) x\\
&= -\gamma\left(2\mu x^2 + \sigma^2 x^2\right) + (1 + 2\gamma x)\mu x\\
&= x(-\gamma\sigma^2 x + \mu)\\
&\leq 0, \quad \text{for } x \in \text{int}(C^c). \qquad (4.11)
\end{aligned}$$

It follows that if we can verify (4.10), (4.11) and (V) for $\hat{\tau}$ then we are done. Let us now consider the candidate equilibrium stopping time $\tau^b := \inf\{t \geq 0 : X_t \geq b\}$ and use the smooth fit condition to see that necessarily $b = \frac{\xi}{\gamma(1-\xi)}$. Recall, from standard theory, that for any b,

$$\mathbb{P}_x(\tau^b < \infty) = b^{\xi-1}x^{1-\xi}, \quad \text{for } x \leq b.$$

Since $X_t \to 0$ a.s. as $t \to \infty$, it hence holds, for any $x \leq b$, that

$$\psi_{\tau^b}(x) = \mathbb{E}_x(X_{\tau^b})$$

$$
\begin{aligned}
&= \mathbb{E}_x(X_{\tau^b} I_{\{\tau^b<\infty\}}) + \mathbb{E}_x(X_{\tau^b} I_{\{\tau^b=\infty\}}) \\
&= b\mathbb{P}_x(\tau^b < \infty) \\
&= b^\xi x^{1-\xi}.
\end{aligned}
$$

Similarly, $\mathbb{E}_x(X^2_{\tau^b}) = b^{1+\xi}x^{1-\xi}$. Hence, for $x \leq b$,

$$
\begin{aligned}
J_{\tau^b}(x) &= \mathbb{E}_x(X_{\tau^b}) - \gamma\left(\mathbb{E}_x(X^2_\tau) - \mathbb{E}^2_x(X_\tau)\right) \\
&= b^\xi x^{1-\xi} - \gamma(b^{1+\xi}x^{1-\xi} - (b^\xi x^{1-\xi})^2) \\
&= x^{1-\xi}(b^\xi - \gamma b^{1+\xi}) + \gamma b^{2\xi}x^{2-2\xi}.
\end{aligned}
$$

It is easy to verify that $J_{\tau^b}(b) = b$, for any b, and hence the function

$$
J_{\tau^b}(x) = \begin{cases} x, & x \geq b, \\ x^{1-\xi}(b^\xi - \gamma b^{1+\xi}) + \gamma b^{2\xi}x^{2-2\xi}, & x < b, \end{cases}
$$

is continuous. Note that

$$
J'_{\tau^b}(x) = \begin{cases} 1, & x > b, \\ (1-\xi)x^{-\xi}(b^\xi - \gamma b^{1+\xi}) + (2-2\xi)\gamma b^{2\xi}x^{1-2\xi}, & x < b, \end{cases}
$$

where the lower part is, for $x = b$, equal to:

$$
\begin{aligned}
(1-\xi)b^{-\xi}(b^\xi - \gamma b^{1+\xi}) + (2-2\xi)\gamma b^{2\xi}b^{1-2\xi} &= (1-\xi)(1-\gamma b) + 2(1-\xi)\gamma b \\
&= (1-\xi)(1+\gamma b).
\end{aligned}
$$

In order for the smooth fit condition (Theorem 3.5) to be satisfied we need that $J'_{\tau^b}(b)$ is equal to $f'(x)+g'(h(x))h'(x) = 1$. We thus need that $(1-\xi)(1+\gamma b) = 1$. Hence, the only possible b is given by

$$
b = \frac{1}{\gamma}\left(\frac{1}{(1-\xi)} - 1\right) = \frac{\xi}{\gamma(1-\xi)}.
$$

It is easily verified that (4.11) holds when $b = \frac{\xi}{\gamma(1-\xi)}$, using that $\mu \in (0, \sigma^2/4]$ and $\xi \in (0, 1/2]$. From the explicit form of $J_{\tau^b}(x)$ above it follows that (4.10) is satisfied exactly when

$$
x^{1-\xi}(b^\xi - \gamma b^{1+\xi}) + \gamma b^{2\xi}x^{2-2\xi} - x \geq 0, \quad \text{for } x < b.
$$

It is straightforward to show that this inequality is satisfied, using that $\xi \in (0, 1/2]$ and $b = \frac{\xi}{\gamma(1-\xi)}$, and thereby verifying (4.10). The only thing we have left is to verify (V), which we will do using Theorem 3.6. From the calculations above follows that

$$\varphi_{\tau^b}(x) = \mathbb{E}_x(-\gamma X_{\tau^b}^2) = \begin{cases} -\gamma x^2, & x \geq b, \\ -\gamma b^{1+\xi} x^{1-\xi}, & x < b, \end{cases}$$
$$\psi_{\tau^b}(x) = \mathbb{E}_x(X_{\tau^b}) = \begin{cases} x, & x \geq b, \\ b^{\xi} x^{1-\xi}, & x < b. \end{cases}$$

Let us drop the subscript τ^b in the rest of the example. It follows that

$$\varphi'(x) = \begin{cases} -2\gamma x, & x > b, \\ -(1-\xi)\gamma b^{1+\xi} x^{-\xi}, & x < b, \end{cases}$$
$$\varphi''(x) = \begin{cases} -2\gamma, & x > b, \\ \xi(1-\xi)\gamma b^{1+\xi} x^{-\xi-1}, & x < b. \end{cases}$$

Thus,

$$\begin{aligned} A_X\varphi(x) &= \mu x \varphi'(x) + \frac{1}{2}\sigma^2 x^2 \varphi''(x) \\ &= \begin{cases} -2\mu\gamma x^2 - \frac{1}{2}\sigma^2 x^2 2\gamma, & x > b, \\ -\mu x(1-\xi)\gamma b^{1+\xi} x^{-\xi} + \frac{1}{2}\sigma^2 x^2 \xi(1-\xi)\gamma b^{1+\xi} x^{-\xi-1}, & x < b, \end{cases} \\ &= \begin{cases} -2\gamma x^2(\mu + \frac{1}{2}\sigma^2), & x > b, \\ -(1-\xi)\gamma b^{1+\xi} x^{1-\xi}(\mu - \frac{1}{2}\sigma^2\xi), & x < b, \end{cases} \\ &= \begin{cases} -2\gamma x^2(\mu + \frac{1}{2}\sigma^2), & x > b, \\ 0, & x < b, \end{cases} \end{aligned}$$

where we in the last equality used that $\xi = 2\mu/\sigma^2$. Similarly,

$$\psi'(x) = \begin{cases} 1, & x > b, \\ (1-\xi)b^{\xi} x^{-\xi}, & x < b, \end{cases}$$
$$\psi''(x) = \begin{cases} 0, & x > b, \\ -\xi(1-\xi)b^{\xi} x^{-\xi-1}, & x < b, \end{cases}$$
$$\begin{aligned} A_X\psi(x) &= \mu x \psi'(x) + \frac{1}{2}\sigma^2 x^2 \psi''(x) \\ &= \begin{cases} \mu x, & x > b, \\ \mu x(1-\xi)b^{\xi} x^{-\xi} - \frac{1}{2}\sigma^2 x^2 \xi(1-\xi) b^{\xi} x^{-\xi-1}, & x < b, \end{cases} \end{aligned}$$

$$= \begin{cases} \mu x, & x > b, \\ (1-\xi) b^{\xi} x^{1-\xi}(\mu - \frac{1}{2}\sigma^2 \xi), & x < b, \end{cases}$$

$$= \begin{cases} \mu x, & x > b, \\ 0, & x < b. \end{cases}$$

Note that $g'(\psi(b)) = 1 + 2\gamma\psi(b) = 1 + 2\gamma b$. Thus,

$$A_X\varphi(x) + g'(\psi(b))A_X\psi(x) = \begin{cases} -2\gamma x^2(\mu + \frac{1}{2}\sigma^2) + (1+2\gamma b)\mu x, & x > b, \\ 0, & x < b. \end{cases}$$

$$= \begin{cases} x\{(1+2\gamma b)\mu - 2\gamma x(\mu + \frac{1}{2}\sigma^2)\}, & x > b, \\ 0, & x < b, \end{cases}$$

$$= \begin{cases} x\{\mu + 2\gamma\mu(b-x) - \gamma\sigma^2 x\}, & x > b, \\ 0, & x < b. \end{cases} \tag{4.12}$$

It is easily checked that φ and ψ are a $\mathcal{C}^2$ everywhere except at $x = b$ and that $\varphi(\cdot) + g'(\psi(b))\psi(\cdot)$ is $\mathcal{C}^1$ everywhere. Hence, we may use Theorem 3.6. Let us verify that (3.15) holds:

Trivially, $g''(b) = 2\gamma$. For the GBM it holds that $\sigma^2(x) = x^2\sigma^2$. Moreover, $\frac{\xi^2}{2}\sigma^2 = \xi\mu$, $\gamma b = \frac{\xi}{1-\xi}$ and $-1 + \sigma^2\xi/\mu = 1$. Using these findings, including (4.12), we obtain

$$\begin{aligned} &A_X\varphi(b+) + g'(\psi(b))A_X\psi(b+) + A_X\varphi(b-) + g'(\psi(b))A_X\psi(b-) \\ &\qquad + g''(\psi(b))\left(\frac{\psi'(b+) - \psi'(b-)}{2}\right)^2 \sigma^2(b) \\ &= b(\mu - \gamma\sigma^2 b) + 0 + 2\gamma\left(\frac{\xi}{2}\right)^2 b^2\sigma^2 \\ &= b\left(\mu - \gamma\sigma^2 b - \gamma b\frac{\xi^2}{2}\sigma^2\right) \\ &= b\left(\mu - \gamma\sigma^2 b + \gamma b\xi\mu\right) \\ &= b\left(\frac{\mu(1-\xi) - \sigma^2\xi + \xi^2\mu}{1-\xi}\right) \\ &= -b\mu\left(\frac{\xi - 1 + \sigma^2\xi/\mu - \xi^2}{1-\xi}\right) \\ &= -b\mu\frac{1+\xi-\xi^2}{1-\xi} \le 0, \end{aligned}$$

where the inequality follows from $\xi \in (0, 1/2]$. This means that (3.15) holds, which, by Theorem 3.6, implies that condition (V) holds and we are done.

□

5 Appendix

Lemma 5.1. *For any* $\tau^{\lambda,C} \in \mathcal{N}$ *and* $x \in C$,

$$A_X \varphi_{\lambda,C}(x) = \lim_{h \searrow 0} \frac{\varphi_{\tau^{\lambda,C} \circ \theta_{\tau_h} + \tau_h}(x) - \varphi_{\lambda,C}(x)}{\mathbb{E}_x(\tau_h)} = \lambda(x)(\varphi_{\lambda,C}(x) - f(x)). \tag{5.1}$$

Proof. Using arguments similar to those we used to arrive at (3.1) and the strong Markov property we obtain

$$\begin{aligned}\varphi_{\tau^{\lambda,C} \diamond \tau^{\lambda,C}(h)}(x) &= \mathbb{E}_x\left(f\left(X_{\tau^{\lambda,C} \diamond \tau^{\lambda,C}(h)}\right)\right) \\ &= \mathbb{E}_x\left(I_{\{\tau^\lambda \leq \tau_h\}} f\left(X_{\tau^{\lambda,C}}\right) + I_{\{\tau^\lambda > \tau_h\}} f\left(X_{\tau^{\lambda,C} \circ \theta_{\tau_h} + \tau_h}\right)\right) \\ &= \mathbb{E}_x\left(f(X_{\tau^{\lambda,C}})\right) \\ &= \varphi_{\lambda,C}(x),\end{aligned}$$

for $0 < h \leq \bar{h}$, for some $\bar{h} > 0$. This implies that the second equality in (5.1) follows from Lemma 3.1. Now use the strong Markov property to see that

$$\begin{aligned}\varphi_{\tau^{\lambda,C} \circ \theta_{\tau_h} + \tau_h}(x) &= \mathbb{E}_x\left(f\left(X_{\tau^{\lambda,C} \circ \theta_{\tau_h} + \tau_h}\right)\right) \\ &= \mathbb{E}_x\left(\mathbb{E}_x\left(f\left(X_{\tau^{\lambda,C} \circ \theta_{\tau_h} + \tau_h}\right) | \mathcal{F}_{\tau_h}\right)\right) \\ &= \mathbb{E}_x\left(\varphi_{\lambda,C}(X_{\tau_h})\right).\end{aligned}$$

Hence, the first equality in (5.1) follows from the definition of the characteristic operator A_X. □

Lemma 5.2. *For any* $\tau^{\lambda,C}, \tau^{\eta,D} \in \mathcal{N}$ *and* $x \in C \cap D$,

$$\begin{aligned}&\lim_{h \searrow 0} \frac{J_{\lambda,C}(x) - J_{\tau^{\lambda,C} \diamond \tau^{\eta,D}(h)}(x)}{\mathbb{E}_x(\tau_h)} \\ &\quad = (\lambda(x) - \eta(x))\{f(x) - \varphi_{\lambda,C}(x) + g'(\psi_{\lambda,C}(x))\,(h(x) - \psi_{\lambda,C}(x))\}.\end{aligned}$$

Proof. Use the same argument as in the proof of Lemma 5.1 to obtain

$$J_{\tau^{\lambda,C}}(x) - J_{\tau^{\lambda,C} \diamond \tau^{\eta,D}(h)}(x)$$

$$= J_{\tau^{\lambda,C}\diamond\tau^{\lambda,C}(h)}(x) - J_{\tau^{\lambda,C}\circ\theta_{\tau_h}+\tau_h}(x) - (J_{\tau^{\lambda,C}\diamond\tau^{\eta,D}(h)}(x) - J_{\tau^{\lambda,C}\circ\theta_{\tau_h}+\tau_h}(x))\tag{5.2}$$

The second part of (5.2) can, by definition, be written as

$$\begin{aligned} &J_{\tau^{\lambda,C}\diamond\tau^{\eta,D}(h)}(x) - J_{\tau^{\lambda,C}\circ\theta_{\tau_h}+\tau_h}(x) \\ &= \varphi_{\tau^{\lambda,C}\diamond\tau^{\eta,D}(h)}(x) - \varphi_{\tau^{\lambda,C}\circ\theta_{\tau_h}+\tau_h}(x) + g(\psi_{\tau^{\lambda,C}\diamond\tau^{\eta,D}(h)}(x)) - g(\psi_{\tau^{\lambda,C}\circ\theta_{\tau_h}+\tau_h}(x)). \end{aligned}$$

From Lemma 3.1 it follows that

$$\lim_{h\searrow 0}\frac{\varphi_{\tau^{\lambda,C}\diamond\tau^{\eta,D}(h)}(x) - \varphi_{\tau^{\lambda,C}\circ\theta_{\tau_h}+\tau_h}(x)}{\mathbb{E}_x(\tau_h)} = \eta(x)(f(x) - \varphi_{\lambda,C}(x)).$$

Use the same arguments as for (3.2) to obtain (here $\eta_t := \eta(X_t)$)

$$\begin{aligned} &\psi_{\tau^{\lambda,C}\diamond\tau^{\eta,D}(h)}(x) \\ &= \psi_{\tau^{\lambda,C}\circ\theta_{\tau_h}+\tau_h}(x) + \mathbb{E}_x\left(\int_0^{\tau_h}\eta_t e^{-\int_0^t \eta_s ds}\left(h(X_t) - \psi_{\lambda,C}\left(X_{\tau_h}\right)\right)dt\right). \end{aligned}$$

Using standard Taylor expansion we thus obtain

$$\begin{aligned} &g(\psi_{\tau^{\lambda,C}\diamond\tau^{\eta,D}(h)}(x)) - g(\psi_{\tau^{\lambda,C}\circ\theta_{\tau_h}+\tau_h}(x)) \\ &= g\left(\psi_{\tau^{\lambda,C}\circ\theta_{\tau_h}+\tau_h}(x) + \mathbb{E}_x\left(\int_0^{\tau_h}\eta_t e^{-\int_0^t \eta_s ds}\left(h(X_t) - \psi_{\lambda,C}\left(X_{\tau_h}\right)\right)dt\right)\right) \\ &\quad - g(\psi_{\tau^{\lambda,C}\circ\theta_{\tau_h}+\tau_h}(x)) \\ &= g'\left(\psi_{\lambda,C\circ\theta_{\tau_h}+\tau_h}(x)\right)\mathbb{E}_x\left(\int_0^{\tau_h}\eta_t e^{-\int_0^t \eta_s ds}\left(h(X_t) - \psi_{\lambda,C}\left(X_{\tau_h}\right)\right)dt\right) + o(\mathbb{E}_x(\tau_h)). \end{aligned}$$

Use the equality above and $\psi_{\tau^{\lambda,C}\circ\theta_{\tau_h}+\tau_h}(x) = \mathbb{E}_x\left(\psi_{\lambda,C}(X_{\tau_h})\right)$ to obtain

$$\begin{aligned} &\lim_{h\searrow 0}\frac{g(\psi_{\tau^{\lambda,C}\diamond\tau^{\eta,D}(h)}(x)) - g(\psi_{\tau^{\lambda,C}\circ\theta_{\tau_h}+\tau_h}(x))}{\mathbb{E}_x(\tau_h)} \\ &= \lim_{h\searrow 0}\frac{g'\left(\psi_{\tau^{\lambda,C}\circ\theta_{\tau_h}+\tau_h}(x)\right)\mathbb{E}_x\left(\int_0^{\tau_h}\eta_t e^{-\int_0^t \eta_s ds}\left(h(X_t) - \psi_{\lambda,C}\left(X_{\tau_h}\right)\right)dt\right)}{\mathbb{E}_x(\tau_h)} \\ &= g'(\psi_{\lambda,C}(x))\eta(x)(h(x) - \psi_{\tau^{\lambda,C}}(x)). \end{aligned}$$

Putting the above together gives us that the limit for the second part of (5.2) satisfies

$$\begin{aligned} &\lim_{h\searrow 0}\frac{J_{\tau^{\lambda,C}\diamond\tau^{\eta,D}(h)}(x) - J_{\tau^{\lambda,C}\circ\theta_{\tau_h}+\tau_h}(x)}{\mathbb{E}_x(\tau_h)} \\ &\qquad = \eta(x)\{f(x) - \varphi_{\lambda,C}(x) + g'(\psi_{\lambda,C}(x))\left(h(x) - \psi_{\lambda,C}(x)\right)\}. \end{aligned}\tag{5.3}$$

In the same way we obtain that the limit for the first part of (5.2) satisfies

$$\lim_{h \searrow 0} \frac{J_{\tau^{\lambda,C} \diamond \tau^{\lambda,C}(h)}(x) - J_{\tau^{\lambda,C} \circ \theta_{\tau_h} + \tau_h}(x)}{\mathbb{E}_x(\tau_h)}$$
$$= \lambda(x)\{f(x) - \varphi_{\lambda,C}(x) + g'(\psi_{\lambda,C}(x))\,(h(x) - \psi_{\lambda,C}(x))\}. \quad (5.4)$$

The result follows from (5.2), (5.3) and (5.4). □

Lemma 5.3. *For any $\tau^{\lambda,C}, \tau^{\eta,D} \in \mathcal{N}$ and $x \in int(C^c) \cap D$,*

$$\lim_{h \searrow 0} \frac{J_{\lambda,C}(x) - J_{\tau^{\lambda,C} \diamond \tau^{\eta,D}(h)}(x)}{\mathbb{E}_x(\tau_h)} = -A_X f(x) - g'(h(x))A_X h(x).$$

Proof. Since D and $\mathrm{int}(C^c)$ are open it follows that there exists a constant $\bar{h} > 0$ such that, for $0 < h \leq \bar{h}$,

$$\begin{aligned}
\tau^{\lambda,C} \diamond \tau^{\eta,D}(h) &= I_{\{\tau^{\eta,D} \leq \tau_h\}}\tau^{\eta,D} + I_{\{\tau^{\eta,D} > \tau_h\}}(\tau^{\lambda,C} \circ \theta_{\tau_h} + \tau_h) \\
&= I_{\{\tau^{\eta} \leq \tau_h\}}\tau^{\eta} + I_{\{\tau^{\eta} > \tau_h\}}(\tau^{\lambda,C} \circ \theta_{\tau_h} + \tau_h) \\
&= I_{\{\tau^{\eta} \leq \tau_h\}}\tau^{\eta} + I_{\{\tau^{\eta} > \tau_h\}}\tau_h \\
&= \tau^{\eta} \wedge \tau_h.
\end{aligned}$$

Since $x \in \mathrm{int}(C^c) \cap D$ it follows that

$$J_{\tau^{\lambda,C}}(x) - J_{\tau^{\lambda,C} \diamond \tau^{\eta,D}(h)}(x)$$
$$= f(x) - \varphi_{\tau^{\lambda,C} \diamond \tau^{\eta,D}(h)}(x) + g(h(x)) - g(\psi_{\tau^{\lambda,C} \diamond \tau^{\eta,D}(h)}(x)). \quad (5.5)$$

Use Itô's formula to rewrite the first part of (5.5) as

$$\begin{aligned}
f(x) - \varphi_{\tau^{\lambda,C} \diamond \tau^{\eta,D}(h)}(x) &= f(x) - \mathbb{E}_x\left(f\left(X_{\tau^{\lambda,C} \diamond \tau^{\eta,D}(h)}\right)\right) \\
&= f(x) - \mathbb{E}_x\left(f\left(X_{\tau^{\eta} \wedge \tau_h}\right)\right) \\
&= -\mathbb{E}_x\left(\int_0^{\tau^{\eta} \wedge \tau_h} A_X f(X_t)dt\right).
\end{aligned}$$

It follows that

$$\lim_{h \searrow 0} \frac{f(x) - \varphi_{\tau^{\lambda,C} \diamond \tau^{\eta,D}(h)}(x)}{\mathbb{E}_x(\tau_h)} = -A_X f(x).$$

Use similar arguments and standard Taylor expansion to rewrite the second part of (5.5)

$$g(h(x)) - g(\psi_{\tau^{\lambda,C} \diamond \tau^{\eta,D}(h)}(x))$$
$$= g(h(x)) - g\left(\mathbb{E}_x\left(h(X_{\tau^{\eta} \wedge \tau_h})\right)\right)$$

$$= g(h(x)) - g\left(h(x) + \mathbb{E}_x\left(\int_0^{\tau^\eta \wedge \tau_h} A_X h(X_t)dt\right)\right)$$
$$= g(h(x)) - \left(g(h(x)) + g'(h(x))\,\mathbb{E}_x\left(\int_0^{\tau^\eta \wedge \tau_h} A_X h(X_t)dt\right) + o(\mathbb{E}_x(\tau_h))\right)$$
$$= -g'(h(x))\,\mathbb{E}_x\left(\int_0^{\tau^\eta \wedge \tau_h} A_X h(X_t)dt\right) + o(\mathbb{E}_x(\tau_h)).$$

Thus,

$$\lim_{h\searrow 0} \frac{g(h(x)) - g(\psi_{\tau^{\lambda,C}\diamond\tau^{\eta,D}(h)}(x))}{\mathbb{E}_x(\tau_h)} = -g'(h(x))A_X h(x).$$

The result follows. □

Lemma 5.4. *For any $\tau^{\lambda,C}, \tau^{\eta,D} \in \mathcal{N}$ and $x \in \partial C \cap D$,*

$$\liminf_{h\searrow 0} \frac{J_{\lambda,C}(x) - J_{\tau^{\lambda,C}\diamond\tau^{\eta,D}(h)}(x)}{\mathbb{E}_x(\tau_h)}$$
$$= \liminf_{h\searrow 0} \frac{\varphi_{\lambda,C}(x) - \mathbb{E}_x\left(\varphi_{\lambda,C}(X_{\tau_h})\right) + g(\psi_{\lambda,C}(x)) - g\left(\mathbb{E}_x\left(\psi_{\lambda,C}(X_{\tau_h})\right)\right)}{\mathbb{E}_x(\tau_h)}.$$

Proof. Here we use the temporary notation (A), (B) etc defined below. Write

$$J_{\lambda,C}(x) - J_{\tau^{\lambda,C}\diamond\tau^{\eta,D}(h)}(x)$$
$$= J_{\lambda,C}(x) - J_{\tau^{\lambda,C}\circ\theta_{\tau_h}+\tau_h}(x) - \left(J_{\tau^{\lambda,C}\diamond\tau^{\eta,D}(h)}(x) - J_{\tau^{\lambda,C}\circ\theta_{\tau_h}+\tau_h}(x)\right)$$
$$= (A) - (B).$$

Write,

$$(B) := J_{\tau^{\lambda,C}\diamond\tau^{\eta,D}(h)}(x) - J_{\tau^{\lambda,C}\circ\theta_{\tau_h}+\tau_h}(x)$$
$$= \varphi_{\tau^{\lambda,C}\diamond\tau^{\eta,D}(h)}(x) - \varphi_{\tau^{\lambda,C}\circ\theta_{\tau_h}+\tau_h}(x) + g(\psi_{\tau^{\lambda,C}\diamond\tau^{\eta,D}(h)}(x)) - g(\psi_{\tau^{\lambda,C}\circ\theta_{\tau_h}+\tau_h}(x))$$
$$= (B1) + (B2).$$

Use that $x \in D$ and the same arguments as for (3.2) to see that there exists a constant $\bar{h} > 0$ such that, for each $0 < h \leq \bar{h}$,

$$(B1) := \varphi_{\tau^{\lambda,C}\diamond\tau^{\eta,D}(h)}(x) - \varphi_{\tau^{\lambda,C}\circ\theta_{\tau_h}+\tau_h}(x)$$
$$= \mathbb{E}_x\left(\int_0^{\tau_h} \eta(X_t)e^{-\int_0^t \eta(X_s)ds}\left(f(X_t) - \varphi_{\lambda,C}\left(X_{\tau_h}\right)\right)dt\right).$$

Similarly, using Taylor expansion, we obtain

$$(B2) := g(\psi_{\tau^{\lambda,C}\diamond\tau^{\eta,D}(h)}(x)) - g(\psi_{\tau^{\lambda,C}\circ\theta_{\tau_h}+\tau_h}(x))$$

$$
\begin{aligned}
&= g\left(\psi_{\tau^{\lambda,C}\circ\theta_{\tau_h}+\tau_h}(x) + \mathbb{E}_x\left(\int_0^{\tau_h} \eta(X_t)e^{-\int_0^t \eta(X_s)ds}\left(h(X_t) - \psi_{\lambda,C}\left(X_{\tau_h}\right)\right)dt\right)\right) \\
&\quad - g\left(\psi_{\tau^{\lambda,C}\circ\theta_{\tau_h}+\tau_h}(x)\right) \\
&= g'(\psi_{\tau^{\lambda,C}\circ\theta_{\tau_h}+\tau_h}(x))\,\mathbb{E}_x\left(\int_0^{\tau_h} \eta(X_t)e^{-\int_0^t \eta(X_s)ds}\left(h(X_t) - \psi_{\lambda,C}\left(X_{\tau_h}\right)\right)dt\right) \\
&\quad + o(\mathbb{E}_x(\tau_h)) \\
&= g'(\mathbb{E}_x\left(\psi_{\lambda,C}(X_{\tau_h})\right))\,\mathbb{E}_x\left(\int_0^{\tau_h} \eta(X_t)e^{-\int_0^t \eta(X_s)ds}\left(h(X_t) - \psi_{\lambda,C}\left(X_{\tau_h}\right)\right)dt\right) \\
&\quad + o(\mathbb{E}_x(\tau_h)).
\end{aligned}
$$

Since $\varphi_{\lambda,C}(x)-f(x)=0$ and $\psi_{\lambda,C}(x)-h(x)=0$ for $x\in\partial C$, and these functions are continuous (cf. admissibility), it follows that

$$
\liminf_{h\searrow 0}\frac{-(B)}{\mathbb{E}_x(\tau_h)} = \liminf_{h\searrow 0}\frac{-(B1)-(B2)}{\mathbb{E}_x(\tau_h)} = 0.
$$

Write

$$
\begin{aligned}
(A) :&= J_{\lambda,C}(x) - J_{\tau^{\lambda,C}\circ\theta_{\tau_h}+\tau_h}(x) \\
&= \varphi_{\lambda,C}(x) + g(\psi_{\lambda,C}(x)) - \left(\varphi_{\tau^{\lambda,C}\circ\theta_{\tau_h}+\tau_h}(x) + g(\psi_{\tau^{\lambda,C}\circ\theta_{\tau_h}+\tau_h}(x))\right) \\
&= \varphi_{\lambda,C}(x) + g(\psi_{\lambda,C}(x)) - \left(\mathbb{E}_x\left(\varphi_{\lambda,C}(X_{\tau_h})\right) + g(\mathbb{E}_x\left(\psi_{\lambda,C}(X_{\tau_h})\right))\right)
\end{aligned}
$$

The result follows. □

Proof. (of Theorem 3.2). In this proof we use the notation $\hat{\tau}=\tau^{\lambda,C}$. Let us first suppose that $\hat{\tau}$ is an equilibrium stopping time, i.e. that it satisfies (2.2) for each $x\in E$ and each $\tau^{\eta,D}\in\mathcal{N}$, and show that this implies that conditions (I)–(V) are satisfied. Let us consider different cases for x.

- $x\in C$: Set $D=C$ and use Lemma 5.2 to see that (2.2) can in this case be written as

  $$
  (\lambda(x)-\eta(x))\{f(x)-\varphi_{\lambda,C}(x)+g'(\psi_{\lambda,C}(x))\left(h(x)-\psi_{\lambda,C}(x)\right)\}\geq 0.
  $$

  It follows that conditions (III) and (IV) are satisfied. To see this recall that the non-negative function η can be chosen so that $\eta(x)$ is arbitrarily large or $\eta(x)=0$.

  Now set $D=\emptyset$, which implies that the numerator of the left side of (2.2) is $J_{\lambda,C}(x)-f(x)-g(h(x))$, which does not depend on the constant h. This implies that (I) holds.

- $x \in \text{int}(C^c)$: Set $D = \text{int}(C^c)$ and use Lemma 5.3 to see that (2.2) can in this case be written as $-A_X f(x) - g'(h(x))A_X h(x) \geq 0$. It follows that condition (II) is satisfied.
- $x \in \partial C$: Set $D = E$ and use Lemma 5.4 to see that the left side of (2.2) is equal to the left side of the inequality in (V), which directly implies that condition (V) holds.

Le us now suppose that $\hat{\tau}$ solves the system (I)–(V) and show that this implies that $\hat{\tau}$ is an equilibrium stopping time, i.e. that it satisfies (2.2) for each $x \in E$ and each $\tau^{\eta,D} \in \mathcal{N}$. Let us consider an arbitrary $\tau^{\eta,D} \in \mathcal{N}$ and different cases for x.

- $x \in D$:
  - If $x \in C$ and $\lambda(x) > 0$, then the left side of (2.2) is, by Lemma 5.2, equal to $(\lambda(x)-\eta(x))\{f(x)-\varphi_{\lambda,C}(x)+g'(\psi_{\lambda,C}(x))\,(h(x) - \psi_{\lambda,C}(x))\}$ and hence (III) implies that (2.2) must hold.
  - If $x \in C$ and $\lambda(x) = 0$, then the left side of (2.2) is, by Lemma 5.2, equal to $-\eta(x)\{f(x) - \varphi_{\lambda,C}(x) + g'(\psi_{\lambda,C}(x))\,(h(x) - \psi_{\lambda,C}(x))\}$ and hence (IV) implies that (2.2) must hold.
  - If $x \in \text{int}(C^c)$, then Lemma 5.3 implies that the left side of (2.2) is equal to $-A_X f(x) - g'(h(x))A_X h(x)$ and hence (II) implies that (2.2) must hold.
  - If $x \in \partial C$, then Lemma 5.4 and (V) implies that (2.2) must hold.
- $x \in D^c$: The numerator of the left side of (2.2) is in this case $J_{\lambda,C}(x) - f(x) - g(h(x))$ and hence (I) implies that (2.2) holds for $x \in C$. In case $x \notin C$ then the numerator is zero.

□

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