Mathematical Statistics

Stockholm University

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Research Report 2018:21

ISSN 1650-0377

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Mathematical Statistics
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Research Report **2018:21**,
http://www.math.su.se

Quantile-based optimal portfolio selection

December 2018

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Abstract

In this paper we introduce the concept of quantile-based optimal portfolio selection and a specific portfolio connected to it, the Conditional Value of Return (CVoR) portfolio. The portfolio selection consists solely of quantile-based risk and return measures. The portfolio has several advantages. It circumvents the estimation problem of mean while still taking the positive part of the return distribution into consideration. It constrains the negative values of the return distribution by a quantile based risk measure. Thus, it takes both tails of the return distribution into account.

Financial institutions that work in the context of Basel 4 use Conditional Value-at-Risk as a risk measure. Under these conditions we provide sufficient and necessary conditions for optimality of the CVoR portfolio under a general distributional assumption. The financial institutions that work in the context of the Solvency 2 insurance regulation must use Value-at-Risk as a risk measure. We provide a verification type theorem for a global optimum under the use of Value-at-Risk as a risk measure. Moreover, we show that the CVoR portfolio is mean-variance efficient when the returns are assumed to follow an elliptically contoured distribution. Under this assumption we derive closed form expressions for the weights and characteristics of the CVoR portfolio.

The introduced methods are illustrated based on weekly stock data, and the results obtained by elliptically contoured asset return distribution are compared with nonparametric CVoR portfolios. For the data at hand, the CVoR portfolio performs best when assuming elliptically contoured distributions in comparison to the nonparametric portfolio.

Keywords: Quantile-based return measure, VaR, CVaR, optimal portfolios, elliptical distributions

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1 Introduction

Since Markowitz (1952) posed the allocation problem of portfolio theory several extensions have been introduced. In Markowitz (1952) a portfolio which provided the smallest risk given an expected return was proposed. Here the variance of the portfolio was used as a risk measure. The use of variance as a risk measure has been critized by practioners and researchers in finance. One of the critiques is that when an asset returns are large the variance scales accordingly. An asset with higher return need not be riskier. It also depend on the whole loss distribution which might not be desirable. This has led to one of many extensions to Markovitz portfolio theory, the change of risk measure. One generalisation is that the variance has been exchanged for a quantile-based risk measure (see e.g. Linsmeier and Pearson (2000), Rockafellar and Uryasev (2002)). The two most commonly used are Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR). This is a consequence of the Solvency (EP (2009)) and Basel (BIS (2017)) requirements. In Solvency 2 restrictions on insurance companies are imposed by using the VaR as a risk measure while the recent Basel requirements enforce financial institutions to transition from VaR to CVaR for measuring risk.

Albeit a quantile-based measure for risk has generally been accepted by academics and practitioners, the expected return is most commonly taken as a default for the measure of return. Quantile-based risk measures do not depend on the positive values of the portfolio return distribution; one can ask why the measure of return should rely on negative values of the portfolio return distribution? Also, using the portfolio mean as a metric for the investor preferences of the portfolio return comes with limitations. An example of such a limitation is if the portfolio return distribution is complicated, such as skew or a mixture of distributions, the mean might be misleading. In such situations the portfolio's expected return is not informative. The mean of a distribution has a nice interpretation in probability theory only for symmetric distributions where it indicates the location of the distribution. Moreover, since the quantities of interest are usually not known when a portfolio is constructed, we must use estimates of the unknown quantities to realise our positions. As an estimate, the sample mean is known to be poor in terms of stability and convergence when compared to the quantities used for constructing risk measures, such as the covariance matrix (see e.g. Merton (1980), Best and Grauer (1991) and Chan et al. (1999)). Albeit the investor can use any estimate, the issue an investor poses is then to motivate why he or she chooses to optimize towards the mean. In this paper we propose an extension to the existing quantile-based portfolio selection problems to tackle these described shortcomings.

Portfolios using quantile-based risk measures together with the mean have been investigated in Alexander and Baptista (2002), Alexander and Baptista (2004) and Yao et al. (2013) to name a few. Alexander and Baptista (2002) investigated a mean-VaR portfolio selection problem under gaussian returns. The authors used the mean as a return measure while minimizing the risk, measured by VaR. Goh et al. (2012) also looked upon then same portfolio selection problem but extended it by modifying a worst-case scenario VaR to specifically cover asymmetry in the returns. Alexander and Baptista (2004) then extended their work from 2002 by considering the mean-CVaR portfolio under gaussian returns. Huang et al. (2010) considered a robust version of the portfolio selection problem, placing bounds on the unknown parameters of interest while considering the mean-CVaR portfolio selection problem. Since Rockafellar and Uryasev (2002) showed that the CVaR has the property of being coherent (convex) under a general distribution a number of purely data-driven portfolios have emerged. They proposed the use of the empirical distribution function as an approximation of the true underlying density. Yao et al. (2013) considered a nonparametric mean-CVaR portfolio by using kernel density estimators to approximate the true density.

To overcome these discussed shortcomings, we introduce the Conditional Value of Return (CVoR) portfolio which is solely constructed from quantile-based measures. The mean is replaced by a quantile-based measure which depends on the positive part of the portfolio return distribution. The risk is constrained by using a quantile-based risk measure. By doing so we are able to take both tails of the portfolio return distribution into account. To our knowledge, no such portolio selection problem exists in the literature. Our aim is to show the great applicability and flexibility of such a portfolio for an investor who is interested in maximizing their return (how he or she chooses to define it) while constraining their risk. This is in accordance with the modern portfolio theory of Markowitz (1952).

The CVoR portfolio will also be connected to the work of Merton (1972) who showed that the Markowitz portfolio lies on the efficient frontier, a parabola in the mean-variance space. Merton also showed that the parabola is completely determined by a set of three parameters. One determines the shape and the other determines its location. The properties of the parameters that constitute the efficient frontier has been widely investigated under different assumptions. Bodnar and Gupta (2009) derived the parametric form of the efficient frontier under elliptically distributed returns which we will connect to as a special case of the CVoR portfolio.

The remainder of the paper is outlined as follows. In Section 2 we present the CVoR portfolio in its most general form. Here, we discuss the implications of using such a portfolio. In Section 3 we present the CVoR portfolio assuming that the returns are elliptically contoured distributed. Under this assumption we will connect it to the efficient frontier in the mean-variance space and give a closed form solution to the portfolio weights and its characteristics. We end the paper with an empirical analysis in Section 4 and conclude with a discussion in Section 5.

2 The Conditional Value of Return Portfolio

In this section we will introduce the conditional value of return portfolio selection problem. We first define coherency for a functional according to Artzner et al. (1999). We thereafter connect this to the conditional value of return portfolio. We have the following

Definition 1. *Let $(\Omega, \mathcal{F})$ denote a measure space, $\mathcal{X}$ a linear space of $\mathcal{F}$ measurable functions such that any $X \in \mathcal{X}$, $X : \Omega \to \mathbb{R}$. A coherent functional $\mathcal{A}$ from $\mathcal{X} \to \mathbb{R}$ must fulfill*

1. *Concavity:* $\forall X, Y \in \mathcal{X}, \lambda \in [0,1]$ *then* $\mathcal{A}(\lambda X + (1-\lambda)Y) \geq \lambda\mathcal{A}(X) + (1-\lambda)\mathcal{A}(Y)$
2. *Monotonicity: if* $X \leq Y$ *then* $\mathcal{A}(X) \leq \mathcal{A}(Y)$
3. *Translation equivariance:* $\mathcal{A}(Y + a) = \mathcal{A}(Y) + a$, *where* $a \in \mathbb{R}$
4. *Positive homogeneity:* $\lambda \geq 0$ *we have that* $\mathcal{A}(\lambda Y) = \lambda\mathcal{A}(Y)$.

A coherent *risk* functional is then on the form $\rho = -\mathcal{A}$. This small rewriting of the definition is of great interest since it explicitly defines the concept of coherency for return measures. It can be seen that the functional $\mathcal{A}$ works exactly like one would think a return measure should. Taking $\mathcal{A}$ to be the ordinary expectation we can see that it fulfills all properties of being a coherent return measure.

Let $\mathbf{X}$ and $\mathbf{w}$ be p-dimensional vectors consisting of asset returns and portfolio weights, respectively. We define the portfolio as $X_{\mathbf{w}} = f(\mathbf{w}, \mathbf{X})$ for a given set of weights $\mathbf{w}$. Most often f will be an affine function, i.e $f(\mathbf{w}, \mathbf{X}) = \mathbf{w}^\top\mathbf{X}$. Now, let $q_\beta(X) := \inf_x\{F_X(x) \geq \beta\}$ where $\beta \in (1/2, 1)$, be the β percentile of the distribution X. We define the conditional value of return (CVoR) as $\mathrm{E}[X_{\mathbf{w}}|X_{\mathbf{w}} \geq q_{\alpha_1}(X_{\mathbf{w}})]$ at significance level α_1 with weights $\mathbf{w}$. It is easy to see from Definition 1 that the CVoR is a coherent return measure and that it truly behaves like a return measure. If we have two positions X and Y for which $F(X \geq x) \leq F(Y \geq x)$, then the same inequality holds for the CVoR. Also, assuming that α_1 is fixed and $q_{\alpha_1}(X_{\mathbf{w}}) = 0$

we can interpret the CVoR return measure as the expected return, given that we actually make a profit. Rockafellar and Uryasev (2002) showed that the the CVaR functional is a coherent function for a general loss distribution and investigated its properties as a function of the portfolio (decision vector) $\mathbf{w}$. Through their results, more specifically Theorem 10 of Rockafellar and Uryasev (2002), the CVoR is concave in $\mathbf{w}$ as long as f is concave in $\mathbf{w}$.

Let $F_{X_{\mathbf{w}}}(\cdot)$ denote the cumulative distribution function (CDF) of the portfolio return. Let $\rho(X_{\mathbf{w}};\alpha_2)$ denote a quantile-based risk measure at the significance level α_2 constructed for the loss distribution of $X_{\mathbf{w}}$. Consider the following optimization problem

$$\begin{aligned} \max_{\mathbf{w}} \ & \mathrm{E}[X_{\mathbf{w}}|X_{\mathbf{w}} \geq q_{\alpha_1}(\mathbf{w})] \\ s.t. \ & \mathbf{w}^\top \mathbf{1} = 1 \\ & \rho(X_{\mathbf{w}};\alpha_2) \leq v_0, \end{aligned} \tag{1}$$

where $\alpha_1, \alpha_2 \in (1/2, 1)$. The optimization problem (1) and its optimal solution will henceforth be called the Conditional Value of Return (CVoR) portfolio. The CVoR portfolio can be seen as a mean-variance portfolio but with quantile-based measure for the return and risk. Also note that we define the CVoR portfolio under a *general* return distribution and risk measure. By optimizing towards the return measure and constraining the risk in terms of the risk measure $\rho(X_{\mathbf{w}};\alpha_2)$, both tails of the return distribution are accounted for.

Next, we investigate the properties of the CVoR portfolio. Let value-at-risk (VaR) of a loss distribution Y be defined as $\mathrm{VaR}_\beta(Y) := q_\beta(Y)$ where $\beta \in (1/2, 1)$. The conditional value-at-risk (CVaR) is defined as $\mathrm{E}[Y|Y \geq \mathrm{VaR}_{\alpha_1}(Y)]$. By the results of Rockafellar and Uryasev (2002) we retrieve necessary and sufficient conditions for the existence of the CVoR portfolio when $\rho(X_{\mathbf{w}};\alpha_2)$ is chosen to be the CVaR. These are summarized in Theorem 1, whose proof follows immediately from the Karush-Kuhn-Tucker conditions presented in Theorems 4.3.7 and 4.3.8 of Bazaraa et al. (2013).

Theorem 1. *Let* $\rho(X_{\mathbf{w}};\alpha_2) = \mathrm{E}[-X_{\mathbf{w}}|-X_{\mathbf{w}} \geq \mathrm{VaR}_{\alpha_2}(-X_{\mathbf{w}})]$ *and* $X_{\mathbf{w}}$ *is concave in* $\mathbf{w}$. *A portfolio* $\mathbf{w}^*$ *is a global solution to* (1) *if and only if the Karush-Kuhn-Tucker conditions hold.*

By construction the CVoR portfolio inherits sufficient and necessary conditions under a *general* return distribution. Not only does it imply an extreme flexibility in terms of modelling in the context of the CVoR portfolio but also gives great comfort in terms of its economical applicability. If an investor needs to work in the context of the new Basel requirements, then he or she will choose the CVaR as a risk measure. The investor can then be sure that the solution and optimum of (1) is the unique maximum. He or she cannot do any better.

The results of Theorem 1 is not limited to the use of CVaR as a risk measure. However, the practical relevance of the CVoR portfolio is then lost (to some extent) because of the Basel requirements. As long as the investor can limit themselves to a certain class of risk measures, the results of Theorem 1 still applies. We have the following

Remark 1. *Assume that $\rho(X_{\mathbf{w}}; \alpha_2)$ is a coherent risk measure. Then the results of Theorem 1 still apply.*

An example of risk measures that are coherent are the class of spectral risk measures. For a more thorough introduction to spectral risk measures, see e.g. Acerbi (2002) and Adam et al. (2008).

In an insurance context, European insurers have to follow the Solvency 2 regulation. The risk measure is now chosen to be the Value-at-Risk (VaR). However, all quantile-based risk measures are not obviously coherent (convex) (see e.g. Rockafellar and Uryasev (2000)) and one such examples is the VaR. This poses several difficulties for the construction of the CVoR portfolio under a general return distribution when $\rho(X_{\mathbf{w}}; \alpha_2) = \mathrm{VaR}_{\alpha_2}(-X_{\mathbf{w}})$ in (1), since the distribution function $F_{X_{\mathbf{w}}}(x)$ may contain atoms. However, by imposing regularity conditions we may provide somewhat weaker conditions in comparisons to Theorem 1. Under these assumptions, which are to be disclosed, we are able to present a verification type theorem for the CVoR portfolio using the VaR as a risk meaure.

Theorem 2. *Let $\mathcal{X}$ denote the class of random variables which are absolutely continuous, have support on $\mathbb{R}^p$ and whose cumulative distribution function is quasiconcave*[2]*. Let $\rho(X_{\mathbf{w}}; \alpha_2) = \mathrm{VaR}_{\alpha_2}(-X_{\mathbf{w}})$, $\mathbf{X}_{\mathbf{w}}$ be an affine function in $\mathbf{w}$ and assume that the return distribution $\mathbf{X} \in \mathcal{X}$. A portfolio $\mathbf{w}^*$ which fulfills the Karush-Kuhn-Tucker conditions of* (1) *is a global optimum.*

Proof. We need only to devote ourselves to the risk constraint. By absolute continuity, we may rewrite the constraint according to $1 - \alpha_2 \leq F_{X_{\mathbf{w}}}(v_0)$. By Theorem 4.39 of Shapiro et al. (2009) we have that the constraint is a quasiconcave function of $\mathbf{w}$. The rest of the proof follows from Theorem 4.3.8 of Bazaraa et al. (2013). □

[2]Let $g : \mathcal{S} \to \mathbb{R}$ have support on $\mathcal{S}$, a convex set in $\mathbb{R}^p$. The function g is quasiconcave at $\mathbf{x}^* \in \mathcal{S}$ if
$$g(\lambda \mathbf{x}^* + (1 - \lambda)\mathbf{x}) \geq \max(g(\mathbf{x}^*), g(\mathbf{x}))$$
for each $\lambda \in (0, 1)$ and each $\mathbf{x} \in \mathcal{S}$.

The sufficient conditions give us some comfort in the applicability of the CVoR portfolio under Solvency 2. The class of absolutely continuous distributions is grand. Distributions of complicated forms, such as skew, fat tailed or mixture distributions are covered. Note that the assumption, that the cumulative distribution function of asset returns is quasiconcave, in turn implies that the cumulative distribution function of the portfolio return is also quasiconcave.

2.1 Numerical approaches for constructing the CvoR Portfolio and their complexity

In this section we shortly discuss the applicability of numerical methods for finding solutions to the CVoR portfolio. We continue to divide the discussion of algorithms in the context of the Basel and Solvency requirements.

Under the Basel requirements we use CVaR as a risk measure. By convexity, the optimum is easily attained by standard optimization algorithms. It poses no problem to optimize the CVoR portfolio under a *general* return distribution. In the context of the Basel requirements, we may choose any distribution function and optimize thereafter. As Rockafellar and Uryasev (2002) noted, if one believes that the true portfolio distribution function is determined by its empirical counterpart, the Empirical Cumulative Distribution Function (ECDF), or that it approximates the true portfolio distribution close enough, then the portfolio allocation problem (1) becomes a linear programming problem. To show that it holds for the CVoR portfolio as well assume that we select CVaR as a risk measure and that we approximate F by its ECDF $\hat{F}$ and it is based on a sample of size N. Also, let $[t]^+ = \max(t, 0)$. Let $\mathbf{X}$ denote the full sample where each column, denoted $\mathbf{X}_k$, $k = 1, ..., N$, corresponds to an observation and each row represents a specific asset. Also, let $\mathbf{0}_{a\times b}$ denote the zero matrix consisting of a rows and b columns, $\mathbf{1}_{a\times b}$ be a matrix containing ones of size $a \times b$ and $\mathbf{I}_{a\times a}$ be the diagonal matrix of size a. By Theorem 10 Rockafellar and Uryasev (2002), we have that

$$E[X_{\mathbf{w}}|X_{\mathbf{w}} > q_{\alpha_1}(\mathbf{w})] = \min_q \left\{ q + \frac{1}{(1-\alpha_1)N} \sum_{k=1}^{N} [f(\mathbf{w}, \mathbf{X})_k - q]^+ \right\}. \tag{2}$$

By using (2) we are able to rewrite (1) and arrive at the following computational lemma.

Lemma 1. *Let* $\mathbf{X}_{\mathbf{w}} = \mathbf{w}^\top \mathbf{X}$, $\mathbf{w} = \mathbf{u} - \mathbf{v}$, $q = q_1 - q_2$, $\xi = \xi_1 - \xi_2$ *and let* $\eta_k, \theta_k \geq 0$, $k = 1, ..., N$ *such that* $-(\mathbf{u}-\mathbf{v})^\top \mathbf{X}_k - (\gamma_1 - \gamma_2) - \eta_k \leq 0$ *and* $(\mathbf{u}-\mathbf{v})^\top \mathbf{X}_k - (\xi_1 - \xi_2) - \theta_k \leq 0$, *and let* $\mathbf{z} = (u_1, u_2, ...u_p, v_1, v_2, ..., v_p, q_1, q_2, \xi_1, \xi_2, \eta_1, \eta_2, ..., \eta_N, \theta_1, \theta_2, ..., \theta_N)$.

By approximating the distribution function of the portfolio return by its sample counterpart, the optimization problem (1) *can be rewritten as*

$$\begin{aligned} &\min_{\mathbf{z}} \ \mathbf{c}^\top \mathbf{z} \\ &s.t. \ \mathbf{A}\mathbf{z} - \mathbf{b} \leq 0 \end{aligned} \tag{3}$$

where all elements of $\mathbf{z}$ *are greater than zero,* $\mathbf{c} = \begin{pmatrix} \mathbf{0}_{1\times 2p} & 1 & -1 & 0 & 0 & \frac{1}{(1-\alpha_1)N}\mathbf{1}_{1\times p} & \mathbf{0}_{1\times N} \end{pmatrix}^\top$, $\mathbf{b} = \begin{pmatrix} 1 & -1 & v_0 & \mathbf{0}_{2N\times 1} \end{pmatrix}^\top$ *and*

$$\mathbf{A} = \begin{pmatrix} \mathbf{1}_{1\times p} & -\mathbf{1}_{1\times p} & 0 & 0 & 0 & 0 & \mathbf{0}_{1\times N} & \mathbf{0}_{1\times N} \\ -\mathbf{1}_{1\times p} & \mathbf{1}_{1\times p} & 0 & 0 & 0 & 0 & \mathbf{0}_{1\times N} & \mathbf{0}_{1\times N} \\ \mathbf{0}_{1\times p} & \mathbf{0}_{1\times p} & 0 & 0 & 1 & -1 & \frac{1}{(1-\alpha_2)N}\mathbf{1}_{1\times N} & \mathbf{0}_{1\times N} \\ -\mathbf{X} & \mathbf{X} & -\mathbf{1}_{N\times 1} & \mathbf{1}_{N\times 1} & 0 & 0 & -\mathbf{I}_{N\times N} & \mathbf{0}_{N\times N} \\ \mathbf{X} & -\mathbf{X} & 0 & 0 & -\mathbf{1}_{N\times 1} & \mathbf{1}_{N\times 1} & \mathbf{0}_{N\times N} & -\mathbf{I}_{N\times N} \end{pmatrix}$$

The proof is presented in the appendix. For actual computation and optimization of problem (3) one should consider multiplying the constraint with $(1-\alpha_2)N$ to increase numerical stability. We can further note that the number of constraints grows quadratically with N and linearly with p so an application in higher dimensions could be done without a considerable effort.

By Theorem 2 we still have some guarantees on the optimum of the CVoR portfolio under Solvency 2, using the VaR as a risk measure. The regularity conditions constrains the random variables to absolutely continuous return distributions, implying that their densities exists. An optimal portfolio $\mathbf{w}$ may then be found using algorithms such as gradient ascent or Newton's method (Bazaraa et al. (2013, ch. 8.2)) from the Langragian. However, this relies on the fact that we are able to evaluate a large number of integrals since the portfolio return distribution is determined by a (potentially large) convolution of the asset return distribution. The evaluation of the objective function may be costly and time-consuming. It may also be hard to attain the gradients since these will contain convolutions. The computational complexity to evaluate the objective function for a given portfolio $\mathbf{w}$ may in itself be large. However, under a known set of portfolio weights $\mathbf{w}$ a scenario model is easy to implement. Under a known set of weights, we can evaluate the VaR constraint using the empirical cumulative distribution function. As suggested in Meucci (2009, Sec. 8.2) an investor may determine a set of portfolios $\mathcal{W}$ which is feasible for their purpose. The investor may then try to perform an exhaustive search on this space and calculate the empirical quantities of VaR and CVoR. However, he or she then faces the issue of dimensionality. If one looks at a large asset class and take the cardinality of the space $\mathcal{W}$ into account, a grid search on that space may not be practically feasible because of computational time it takes.

One specific class of distributions has been widely considered in financial applications, the elliptically contoured distribution. Some examples of applications and reviews of the topic are Owen and Rabinovitch (1983), Hamada and Valdez (2008) and Gupta et al. (2013). In the next section, we derive an analytical solution to (1) under this large class of probability distributions.

3 The CVoR Portfolio for elliptically contoured distribution

3.1 Elliptically contoured distributions

If a random vector $\mathbf{Y}$ has the following characteristics function

$$E[\exp(i\boldsymbol{t}^\top \mathbf{x})] = \exp\{i\boldsymbol{\mu}^\top \mathbf{t}\}\phi(\mathbf{t}^\top \mathbf{D}\mathbf{t}),\ \text{for } \mathbf{t} \in \mathbb{R}^p,$$

it is said to have a p-dimensional elliptically contoured distribution with location parameter $\boldsymbol{\mu}$, dispersion matrix $\mathbf{D}$ and $\phi(\cdot)$ is a function determined by the family of distribution. In the following we denote this class of multivariate distributions by $ECD_p(\boldsymbol{\mu}, \mathbf{D}, \phi(\cdot))$. If the second moment of $\mathbf{Y}$ exists, then $\boldsymbol{\mu} = E[\mathbf{Y}]$ and $\boldsymbol{\Sigma} = Var[\mathbf{Y}] = \gamma^2\mathbf{D}$ with $\gamma = \sqrt{-\phi'(0)/2}$. Moreover, assuming that $\mathbf{Y}$ has a density $f_{\mathbf{Y}}(\mathbf{y})$, we get

$$f_{\mathbf{Y}}(\mathbf{y}) = |\mathbf{D}|^{-1/2} g((\mathbf{y}-\boldsymbol{\mu})^\top \mathbf{D}^{-1}(\mathbf{y}-\boldsymbol{\mu})). \tag{4}$$

where $g(\cdot)$ is the density generator. For the interested reader, the technical conditions when $\mathbf{Y}$ actually has a density can be found in Fang and Zhang (1990). We will simply assume that the density exists in the following sections.

Elliptically contoured distributions constitute a large class of multivariate (and also matrix-variate) distributions. Some examples of these are the multivariate normal distribution, the t-distribution and the Laplace distribution (see e.g. Fang and Zhang (1990)). Elliptically contoured distributions have many desirable properties. One of interest is the following: if $Y = (\mathbf{l}^\top \mathbf{Y} - \mathbf{l}^\top \boldsymbol{\mu})/\sqrt{\mathbf{l}^\top \mathbf{D}\mathbf{l}}$ then the distribution of Y is independent of the value of $\mathbf{l}$ by Fang and Zhang (1990, Theorem 2.6.3). It only depends on the specific family of elliptical distributions $\mathbf{Y}$ belongs to. A classical example of this property is the multivariate normal distribution.

We are now ready to introduce the closed-form solution to the CVoR portfolio choice problem under elliptically contoured distributed asset returns.

3.2 Closed form solution

In this section we will consider the class of elliptically countoured distributions which are absolutely continuous for which the second moment exist. We will also consider the special case where $X_{\mathbf{w}} = \mathbf{w}^\top \mathbf{X}$.

The expected return of the portfolio with weights $\mathbf{w}$ is given by $\mathrm{E}[X_{\mathbf{w}}] = \mathbf{w}^\top \boldsymbol{\mu}$ and its variance by $\mathrm{Var}(X_{\mathbf{w}}) = \mathbf{w}^\top \boldsymbol{\Sigma} \mathbf{w}$. Let d_{α_1} be the α_1-percentile of the standardized portfolio return $X \overset{\mathrm{d}}{=} (\mathbf{w}^\top \mathbf{X} - \mathbf{w}^\top \boldsymbol{\mu})/\sqrt{\mathbf{w}^\top \mathbf{D} \mathbf{w}}$ and $f_X(\cdot)$ and $F_X(\cdot)$ denote the density and cumulative distribution function of X, respectively. When using CVaR as a risk measure the optimization problem in (1) can then be rewritten as

$$\begin{aligned}
&\max_{\mathbf{w}} \mathbf{w}^\top \boldsymbol{\mu} + k_{\alpha_1} \sqrt{\mathbf{w}^\top \boldsymbol{\Sigma} \mathbf{w}} \\
s.t.\ &\mathbf{w}^\top \mathbf{1} = 1 \\
&- \mathbf{w}^\top \boldsymbol{\mu} - k_{1-\alpha_2} \sqrt{\mathbf{w}^\top \boldsymbol{\Sigma} \mathbf{w}} \leq v_0
\end{aligned} \tag{5}$$

where

$$k_\alpha = \frac{\int_{d_\alpha}^\infty x f_X(x) dx}{(1-\alpha)\gamma}, \quad d_\alpha = F_X^{-1}(\alpha).$$

The risk measure CVaR can easily be changed to VaR in this setting. This is simply done by replacing the constant $k_{1-\alpha_2}$ with $d_{1-\alpha_2}$ in the risk-constraint. We will proceed with deriving all results using CVaR as a risk measure, but note that the results hold true with VaR as a risk measure.

Since the risk-constraint is a convex function of $\mathbf{w}$ there exists a global optimum. A question is whether or not the risk constraint results in equality. We have the following result

Lemma 2. *Let $\mathbf{w}_{CVoR}$ denote the global optimum of* (5), *we then have that*

$$-\mathbf{w}_{CVoR}^\top \boldsymbol{\mu} - k_{1-\alpha_2} \sqrt{\mathbf{w}_{CVoR}^\top \boldsymbol{\Sigma} \mathbf{w}_{CVoR}} = v_0,$$

the risk constraint of (5) *results in equality, i.e. the constraint is active.*

The proof of Lemma 2 is presented in the appendix. By Lemma 2 we may impose an equality on the risk constraint in the CVoR portfolio. We will do so throughout the remainder of this section. A consequence of the equality constraint is that the CVoR portfolio can be attained by considering an easier optimization problem. Let

W denote the constraint set of (5), which allows us to write

$$
\begin{aligned}
\mathbf{w}_{CVoR} &= \arg\max_{\mathbf{w}\in W}\left\{\mathbf{w}^\top\boldsymbol{\mu} + k_{\alpha_1}\sqrt{\mathbf{w}^\top\boldsymbol{\Sigma}\mathbf{w}}\right\} \\
&= \arg\max_{\mathbf{w}\in W}\left\{\mathbf{w}^\top\boldsymbol{\mu} - \frac{k_{\alpha_1}}{k_{1-\alpha_2}}\left(-k_{1-\alpha_2}\sqrt{\mathbf{w}^\top\boldsymbol{\Sigma}\mathbf{w}} - \mathbf{w}^T\boldsymbol{\mu}\right) - \frac{k_{\alpha_1}}{k_{1-\alpha_2}}\mathbf{w}^T\boldsymbol{\mu}\right\} \\
&= \arg\max_{\mathbf{w}\in W}\left\{\left(1 - \frac{k_{\alpha_1}}{k_{1-\alpha_2}}\right)\mathbf{w}^\top\boldsymbol{\mu} - \frac{k_{\alpha_1}}{k_{1-\alpha_2}}v_0\right\} \\
&= \arg\max_{\mathbf{w}\in W}\left\{\mathbf{w}^\top\boldsymbol{\mu}\right\},
\end{aligned}
$$

where we use that $-\frac{k_{\alpha_1}}{k_{1-\alpha_2}} > 0$ since $\alpha_1, \alpha_2 \in (1/2, 1)$. Hence, the CVoR portfolio, retrieved from (5) does not depend on k_{α_1} which can be explained by the symmetry of the distribution of $\mathbf{X}$. Therefore, if a solution exists to (5) then the same solution can be obtained by solving

$$
\begin{aligned}
&\max_{\mathbf{w}} \mathbf{w}^\top\boldsymbol{\mu} \\
s.t.\ &\mathbf{w}^\top\mathbf{1} = 1 \\
&-\mathbf{w}^\top\boldsymbol{\mu} - k_{1-\alpha_2}\sqrt{\mathbf{w}^\top\boldsymbol{\Sigma}\mathbf{w}} = v_0.
\end{aligned}
\tag{6}
$$

The above problem is closely related to the portfolio dicussed in Alexander and Baptista (2002) and Alexander and Baptista (2004). Here, the authors introduced the mean-VaR and mean-CVaR efficient frontier in the context of an equivalent optimization problem to (6) under the assumptions of normality. The authors considered minimizing the portfolio CVaR (VaR) with a constraint on the expected return. They discussed the economical implications of using the portfolio VaR as the objective function compared to using the variance, as a risk measure. Under the assumption of Gaussian returns they showed that the portfolio is mean-variance efficient. To show that the same holds for the CVoR portfolio, let

$$
\mathbf{w}_{\mathrm{GMV}} = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}^\top\boldsymbol{\Sigma}^{-1}\mathbf{1}}, \quad R_{GMV} = \frac{\boldsymbol{\mu}^\top\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}^\top\boldsymbol{\Sigma}^{-1}\mathbf{1}}, \quad V_{GMV} = \frac{1}{\mathbf{1}^\top\boldsymbol{\Sigma}^{-1}\mathbf{1}}.
$$

The efficient frontier, in its parametric form, is then defined as

$$
(R_{GMV} - R)^2 = s\left(V - V_{GMV}\right) \tag{7}
$$

where $s = \boldsymbol{\mu}^\top\mathbf{Q}\boldsymbol{\mu}$ and $\mathbf{Q} = \boldsymbol{\Sigma}^{-1} - (\boldsymbol{\Sigma}^{-1}\mathbf{1}\mathbf{1}^\top\boldsymbol{\Sigma}^{-1})/\mathbf{1}^\top\boldsymbol{\Sigma}^{-1}\mathbf{1}$. Note that for the mean-variance efficient frontier to exist we must assume that $\boldsymbol{\mu}$ is not proportional to $\mathbf{1}$. For all practical purposes this poses no issue. In the following theorem we show the CVoR portfolio is mean-variance efficient under elliptically distributed returns.

Theorem 3. *Assume that* $\mathbf{X} \sim ECD_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \phi(\cdot))$*, where* $rank(\boldsymbol{\Sigma}) = p$ *and let* $\mathbf{w}_{CVoR}$ *denote the CVoR portfolio. Assume that* $\boldsymbol{\mu}$ *is not proportional to* $\mathbf{1}$*,* $\mathbf{w}_{CVoR}^\top \boldsymbol{\mu} > R_{GMV}$ *and* $k_{1-\alpha_2}^2 > s$ *then the CVoR portfolio is mean-variance efficient.*

The proof of the theorem is presented in the appendix. It is obtained by rewriting the solution of the Langragian on the form of the mean-variance portfolio for elliptically contoured models. There are a number of constraints in the construction. The assumption that $\boldsymbol{\mu}$ can not be proportional to the vector of ones is merely technical, such that the efficient frontier exists at all. The constraint $\mathbf{w}_{CVoR}^\top \boldsymbol{\mu} > R_{GMV}$ and $k_{1-\alpha_2}^2 > s$ guarantees that the portfolio lies on the efficient frontier, not only on the mean-variance parabola.

We are now ready to present the closed form solution to the CVoR portfolio.

Theorem 4. *Assume that* $\mathbf{X} \sim ECD_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \phi(\cdot))$*, where* $rank(\boldsymbol{\Sigma}) = p$*. Also, assume that* $\mathbf{w}_{CVoR}^\top \boldsymbol{\mu} > R_{GMV}$*,* $\alpha_2 \in (1/2, 1)$*,* $k_{1-\alpha_2}^2 > s$ *and* $v_0 \geq CVaR_{\alpha_2}(X_{\mathbf{w}_{gmv}})$*, then the CVoR portfolio exists and it has the following weights and characteristics*

$$\mathbf{w}_{CVoR} = \mathbf{w}_{GMW} + \frac{\eta}{s}\mathbf{Q}\boldsymbol{\mu}, \tag{8}$$

$$R_{CVoR} = E[X_{\mathbf{w}}] = R_{GMV} + \eta, \tag{9}$$

$$V_{CVoR} = Var[X_{\mathbf{w}}] = \mathbf{w}_{CVoR}^\top \boldsymbol{\Sigma} \mathbf{w}_{CVoR} = \left(V_{GMV} + \frac{\eta^2}{s}\right) \tag{10}$$

$$E[X_{\mathbf{w}} | X_{\mathbf{w}} > q_{\alpha_1}] = R_{GMV} + \eta + k_{\alpha_1}\sqrt{\left(V_{GMV} + \frac{\eta^2}{s}\right)} \tag{11}$$

where R_{CVoR}*,* V_{CVoR} *is the portfolio return and variance respectively, and*

$$\eta = \frac{(R_{GMV} + v_0)s + \left(k_{1-\alpha_2}^2 s \left((R_{GMV} + v_0)^2 + (s - k_{1-\alpha_2}^2)V_{GMV}\right)\right)^{1/2}}{k_{1-\alpha_2}^2 - s} \tag{12}$$

The proof is presented in the appendix. By Theorem 3 we know that the CVoR portfolio is on the mean-variance efficient frontier. Therefore, the proof consists of solving the CVaR constraint in (6).

Through Theorems 3 and 4 two especially interesting facts arise. Under the use of CVaR as a risk measure, the CVoR portfolio exists only if $k_{1-\alpha_2}^2 > s$, and under VaR replacing $k_{1-\alpha_2}$ with $d_{1-\alpha_2}$. The same inequality is presented in Bodnar et al. (2012) to ensure the existence of the minimum CVaR portfolio. Alexander and Baptista (2002) showed that the inequality $d_{1-\alpha_2}^2 > s$ is the criteria for existence of the minimum VaR portfolio. Also, the CVoR portfolio puts a constraint on the constant v_0 in order for it to exist. This has many interesting economical interpretations. We are *never* limited to small confidence levels α_2, but we are limited in the choice of v_0 given that confidence level. This is intuitively pleasing. If an investor want to pick a large confidence level then he or she must be comitted to place more capital at risk, i.e. a larger v_0. The following proposition explains the behaviour of η through the choice of v_0 and α_2.

Proposition 1. *Let $R_{CVoR} > R_{GMV}$, $\alpha_2 \in (1/2, 1)$, $k_{1-\alpha_2} > s$ and $v_0 \geq CVaR_{\alpha_2}(X_{\mathbf{w}_{GMV}})$. Then η is increasing in v_0. If additionally $k^2_{1-\alpha_2} > \max\{s, 2\}$, then η is decreasing in α_2.*

The proof is presented in the appendix. Note that the constraint $k^2_{1-\alpha_2} > \max\{s, 2\}$ is equally restrictive as $k^2_{1-\alpha_2} > s$ for all practical purposes. If $\alpha_2 \geq 0.95$ then $k^2_{1-\alpha_2} \geq d^2_{1-\alpha_2}$ and if the normal distribution is assumed (which is among the least fat tailed in the EC class) then $d^2_{1-\alpha_2} \geq (1.96)^2 > 2$. This proposition implies that the characteristics of the CVoR portfolio is a strictly increasing function of v_0 through their dependence of η. An investor will accept more return and risk by increasing the value of v_0. In the context of the CVoR portfolio, a risk-averse investor might choose v_0 to be equal to the CVaR (or VaR) for the GMV portfolio for a given α_2. He or she might even be interested in placing less money at risk, thus decreasing their v_0.The constraint on the constant v_0 in Theorem 4 can be replaced by a more tight one. We can choose smaller values of v_0 and still have that the CVoR portfolio exists. This is displayed in the remark below.

Remark 2. *Assume that $\mathbf{X} \sim ECD_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \phi(\cdot))$, where $rank(\boldsymbol{\Sigma}) = p$, and assume that $\mathbf{w}^\top_{CVoR}\boldsymbol{\mu} > R_{GMV}$ together with $\alpha_2 \in (1/2, 1)$ such that $k^2_{1-\alpha_2} > s$. If*

$$v_0 \geq -R_{GMV} + \sqrt{(k^2_{1-\alpha_2} - s)}\sqrt{V_{GMV}} \tag{13}$$

then the CVoR portfolio exists.

The inequality follows immediately from the proof of Theorem 4. The CVoR portfolio exists under a tighter constraint on v_0, but the economical implications are somewhat lost. There is a possibility to choose less capital at risk when constructing the CVoR portfolio. Note that the constraint $k^2_{1-\alpha_2} > s$ appears once again. We can now show that if equality holds in (13) then under this assumption, the constant η takes on the explicit form of $\sqrt{V_{GMV}/(k^2_{1-\alpha_2} - s)}s$ and the weights and the characteristics of the CVoR portfolio change thereafter. Under this assumption, increasing the confidence level α_2 towards one implies that the CVoR portfolio tends towards the GMV portfolio for both CVaR and VaR as a risk measure.

4 Financial application

In this section we perform an empirical illustration where we investigate the characteristics of the CVoR portfolio under different circumstances. In this section we assume that $X_{\mathbf{w}} = \mathbf{w}^\top \mathbf{X}$. We will present their numerical differences and also investigate their inherent location in the mean-variance space. The data used in this illustration consists of 29 stocks from the Dow Jones Index. There are a total of 317 weekly log returns of the closing prices. The data covers the period *mars* of 2008 to *mars* of 2016 . We will use a hold out period of 100 weeks to observe the out-of-sample performance of the portfolios.

In Figure 1 we display a boxplot of the weekly log returns for each stock. For illustrational purposes we have chosen to display both data-sets on the same scale. In the out-of-sample period there exists a large loss in AAPL equal to -1.956 not shown in the figure. There is no reason for it to be an outliers and is therefore not removed. It is noted that the univariate return distributions of the asset returns seem to be roughly symmetric. This observation motivates the application of elliptically contoured distributions as a model of asset return distributions, although it does not provide any strong statistical argument to conclude that the elliptical distributions are the best model for the considered data. Later on, we compare the performance of the optimal portfolio constructed under the assumption of elliptically contoured distribution to those obtained by the nonparametric portfolio described in Section 2.1. The CVoR portfolio will be constructed for two elliptically contoured distributions; the Laplace distribution and the t-distribution with a number of different degrees of freedom. On occasions we will restrict ourselves to the t-distribution with a certain value of degrees of freedom to keep the combinations of visualisations short.

The estimated parameters of the efficient frontier from the first sub-sample are equal to $\widehat{R}_{GMV} = 0.0013, \widehat{V}_{GMV} = 3 \times 10^{-4}$ and $\widehat{s} = 0.0396$. In the following we set $\alpha_2 = 0.999$. By plugging in the estimated quantities we have that the CVaR for the GMV portfolio is equal to -0.0012 for a t-distribution with 5 degrees of freedom and -0.0011 for a Laplace. Thus choosing any v_0 above that will suffice when assuming an elliptically contoured distribution for the portfolio returns. Given the confidence level 0.999, the inequality $k^2_{1-\alpha_2} > \widehat{s}$ holds as well.

In Figure 2 the CVoR portfolio weights are displayed for different portfolio configurations. Notice the scale of the y-axis. Figure 2 we have restricted ourselves to the use of the t-distribution but the result are similar for the Laplace distribution. From Figure 2a we can see that increasing the confidence level α_2 we take less aggressive positions. This is in line with the portfolio weights given in (8) where increasing α_2 will essentially imply convergence towards the GMV portfolio. On the

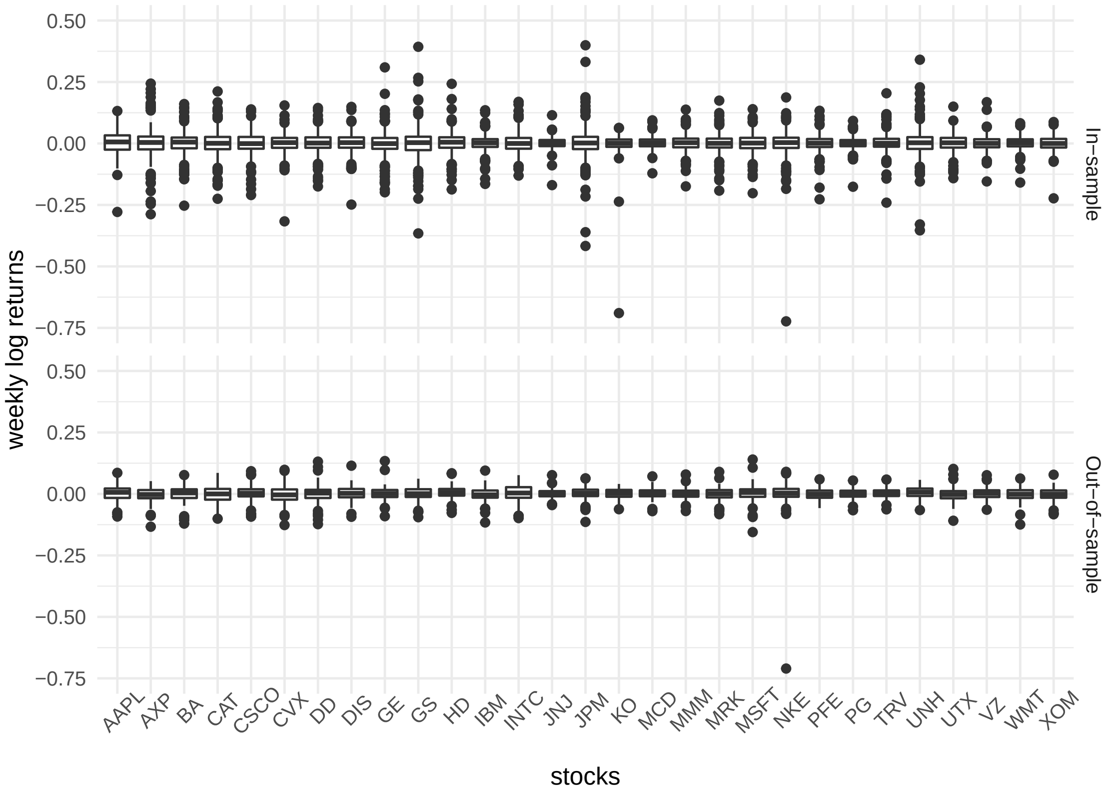
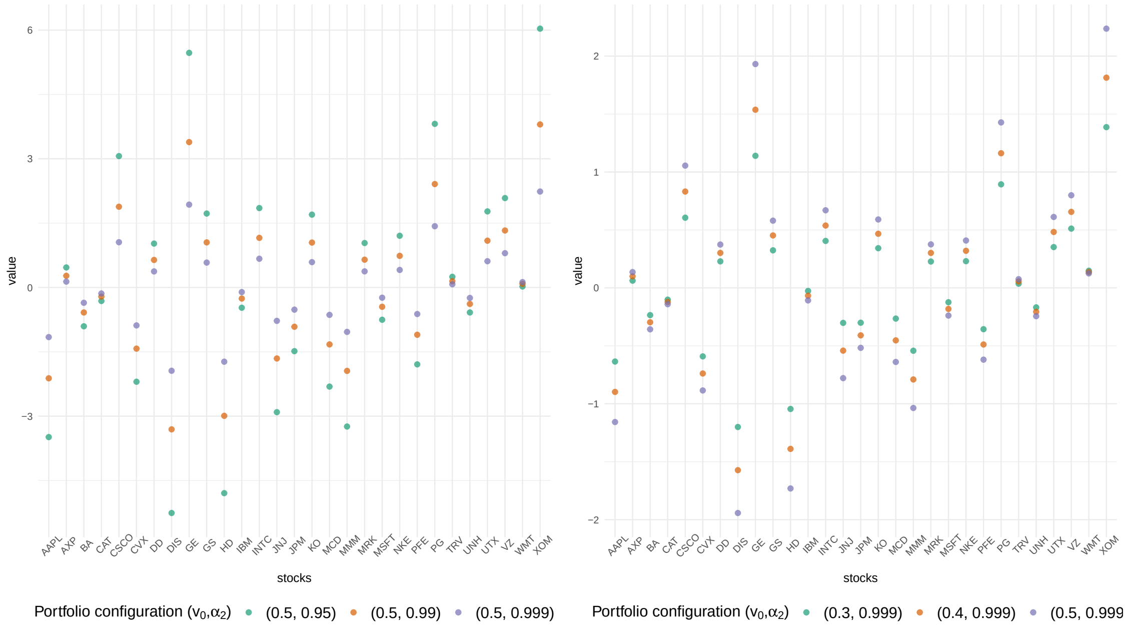


Figure 1: Boxplot of the weekly log returns of the in-sample dataset as well as the out-of-sample dataset. In sample data range from 2008-03-24 to 2014-04-14 while the out-of-sample data ranges from 2014-04-21 to 2016-03-14.

other hand, from Figure 2b we can see that increasing v_0 we take more aggressive positions in each stock. Further, from Figure 2a it is clear that the optimal portfolio consists of heavily leveraged and short sold positions.

Under the assumption of elliptical returns, we are able to connect the portfolios to the efficient frontier. In Figure 3 we display a number of different configurations for the Laplace distribution and a t-distribution with 5 degrees of freedom denoted by $t(5)$. We can see that they lie on the efficient frontier. When assuming the t(5) distribution we traverse quickly along the efficient frontier beacause of its fat tails. The difference between the configurations becomes more apparent when increasing v_0 and decreasing the confidence level α_2. When v_0 is chosen small and α_2 is large we can see that the distributional assumption matters less. This is in line with the discussion presented on convergence to the GMV portfolio.

Next, we apply the new portfolio on the hold out period of 100 weeks. We fix v_0 equal to 0.5. The out-of-sample metrics will be calculated in the following way: estimate the portfolio weights for a configuration of the CVoR portfolio using the in-sample data. We then construct the out-of-sample portfolio returns by using these



(a) Increasing α_2 (b) Increasing v_0

Figure 2: Illustration of the portfolio weights for different portfolio configurations (v_0, α_2).

portfolio weights. We aggregate the out-of-sample portfolio returns by a number of performance metrics such as the average return, variance and the Sharpe ratio. All performance metrics will be calculated using the empirical cumulative distribution function. The performance of the portfolios are displayed in Table 1. The table displays both nonparametric portfolios, obtained by the solution to the linear programming problem 3, and portfolios assuming elliptically contoured returns. The nonparametric portfolios are denoted by ECDF. In Table 1 we see that the portfolios obtained by assuming that the asset returns follow elliptically contoured distributions performs better than the nonparametric portfolios in terms of the out-of-sample expected return. The same holds true for the out-of-sample CVoR. The nonparametric portfolios produces lower out-of-sample CVaR, VaR and variance in comparison to the portfolios constructed using a t-distribution. It can be noted that the nonparametric portfolio results in the same value for each performance metric for each consequtive configuration (α_1, α_2). The value of α_2 seems to have no implication on the construction of the portfolio weight which can be described by the extreme quantile we are trying to approximate.

In connection to Figure 1 we previously noted that there exists a large loss in the AAPL stock in the out-of-sample period. If the same analysis is performed without that specific loss then the results for all portfolios are very different. These results

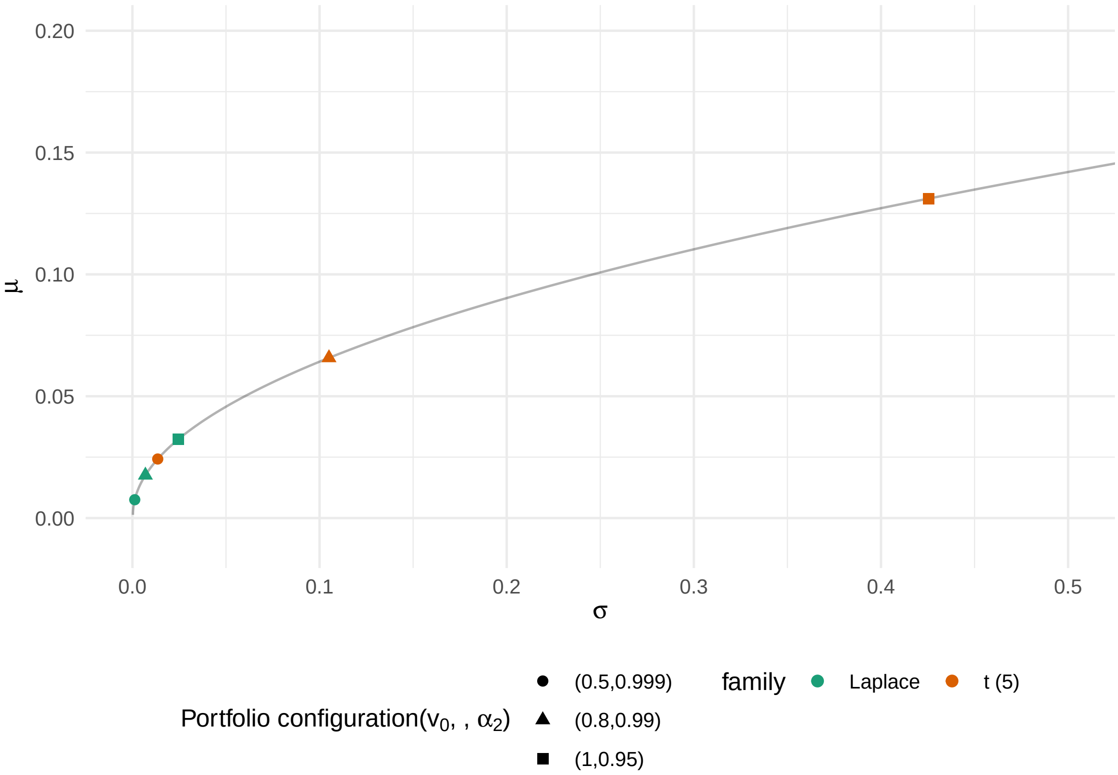


Figure 3: Illustrating the location of the CVoR portfolio on the efficient frontier for different portfolio configurations and distribution. Here results for the Laplace distribution are displayed together with the results for a t distribution with 5 degrees of freedom.

can be seen in Table 2. In this situation the nonparametric portfolios outperform the elliptically contoured ones, regardsless of performance measure.

5 Summary

In this paper we analyse an entirely quantile-based optimal portfolio choice problem. The resulting optimal portfolios are obtained under general distributional assumptions on the asset returns. Both sufficient and necessary conditions for the existence of the optimal portfolio, called the CvoR portfolio, are provided under different risk measures. A special emphasis is placed on the use of risk measures which are demanded by todays Basel and Solvency regulations. The portfolio is shown to be very flexible and provide good theoretical results as well as a straight forward implementation of numerical procedures.

Our empirical illustration shows an application of the CVoR portfolio when the asset return distribution are assumed (probably misleading) to be elliptically contoured as well as in a general case without imposing specific distributional assumption on the asset returns. The results of our application relies on the true underlying return distribution which we do not know. The process parameters are estimated by applying the empirical cumulative distribution function and imputing the unkown parameters. In doing so, we introduce estimation uncertainty which is not accounted for in the derived theoretical results. Also, the investigation of temporal independence in the underlying data-generating process has been neglected in the empirical application. This was done in order to keep the financial application short.

In our application the nonparametric portfolio results in the same weights for different configurations. A possible explanation to this observation could be the discrete space that we construct by using the empirical cumulative distribution function and that we use very large confidence values. The out-of-sample performance of the nonparametric portfolios are also in line with the discussion on purely data-driven portfolios presented in Lim et al. (2011), where the fragility of nonparametric portfolios using CVaR was discussed. That extends to a fragility of the nonparametric CVoR portfolio.

This paper provides an introduction and foundation of quantile-based portfolios. Surely, other quantile-based return measures except the CVoR could be used which we will consider for future research. As the CVoR is a modification of the CVaR and the CVaR is a special case of spectral risk measures we could consider any one on these forms. The use of these would imply a great deal of flexibility for investors which can then also rely on nice theoretical properties of the optimization problem.

6 Appendix

Proof of Lemma 1. By using equation (2) together with some abuse of notation we may rewrite (1) to

$$\begin{aligned} &\max_{\mathbf{w}} \min_{\gamma} \left\{ \gamma + \frac{1}{(1-\alpha_1)N} \sum_{k=1}^{N} [\mathbf{w}^\top \mathbf{X}_k - \gamma]^+ \right\} \\ &\quad s.t.\ \mathbf{w}^\top \mathbf{1} = 1 \\ &\qquad \min_{\xi} \left\{ \xi + \frac{1}{(1-\alpha_2)N} \sum_{k=1}^{N} [-\mathbf{w}^\top \mathbf{X}_k - \xi]^+ \right\} \le v_0. \end{aligned} \tag{14}$$

By Theorem 16 of Rockafellar and Uryasev (2002) we can rewrite (14) according to

$$
\begin{aligned}
\max_{\mathbf{w}} \min_{\gamma,\xi} & \left\{ \gamma + \frac{1}{(1-\alpha_1)N} \sum_{k=1}^{N} [\mathbf{w}^\top \mathbf{X}_k - \gamma]^+ \right\} \\
s.t.\ & \mathbf{w}^\top \mathbf{1} = 1 \\
& \left\{ \xi + \frac{1}{(1-\alpha_2)N} \sum_{k=1}^{N} [-\mathbf{w}^\top \mathbf{X}_k - \xi]^+ \right\} \leq v_0,
\end{aligned}
\tag{15}
$$

We start by optimizing over $-\mathbf{w}$ instead of $\mathbf{w}$ such that the objective only contains minimization procedures. Since all linear programming problems assume that the elements of the decision vector have positive support we also introduce the following variables. Let $\mathbf{w} = \mathbf{u} - \mathbf{v}$ where $\mathbf{u}$ take care of the positive part and $\mathbf{v}$ takes care of the negative if $u_i, v_i \geq 0$, $i = 1, 2, ..., p$. We perform the same operation for q and ξ by introducing $q = q_1 - q_2$ and $\xi = \xi_1 - \xi_2$ where $q_1, q_2, \xi_1, \xi_2 \geq 0$. For each observation $k = 1, ..., N$, introduce the auxilary variables $\eta_k, \theta_k \geq 0$ such that $-(\mathbf{u} - \mathbf{v})^\top \mathbf{X}_k - (q_1 - q_2) - \eta_k \leq 0$ and $(\mathbf{u} - \mathbf{v})^\top \mathbf{X}_k - (\xi_1 - \xi_2) - \theta_k \leq 0$. We then have that

$$
\begin{aligned}
\min_{\mathbf{z}} & \left\{ q_1 - q_2 + \frac{1}{(1-\alpha_1)N} \sum_{k=1}^{N} \eta_k \right\} \\
s.t.\ & (\mathbf{u} - \mathbf{v})^\top \mathbf{1} + 1 \leq 0 \\
& -(\mathbf{u} - \mathbf{v})^\top \mathbf{1} - 1 \leq 0 \\
& \xi_1 - \xi_2 + \frac{1}{(1-\alpha_2)N} \sum_{k=1}^{N} \theta_k \leq v_0, \\
& -(\mathbf{u} - \mathbf{v})^\top \mathbf{X}_k - (\gamma_1 - \gamma_2) - \eta_k \leq 0 \\
& (\mathbf{u} - \mathbf{v})^\top \mathbf{X}_k - (\xi_1 - \xi_2) - \theta_k \leq 0 \\
& u_1, u_2, ...u_p, v_1, v_2, ..., v_p, \gamma_1, \gamma_2, \xi_1, \xi_2 \geq 0
\end{aligned}
\tag{16}
$$

where $\mathbf{z} = (u_1, u_2, ...u_p, v_1, v_2, ..., v_p, q_1, q_2, \xi_1, \xi_2, \eta_1, \eta_2, ..., \eta_N, \theta_1, \theta_2, ..., \theta_N)$. Note that we rewrote the constraint that the weights should sum to one into two inequalities. By introducing the matrix and vectors $\mathbf{A}, \mathbf{b}$ and $\mathbf{c}$, the lemma follows. □

Proof of Lemma 2. Let $W = \{\mathbf{w} : \mathbf{w}^\top \mathbf{1} = 1, -\mathbf{w}^\top \boldsymbol{\mu} - k_{1-\alpha_2} \sqrt{\mathbf{w}^\top \boldsymbol{\Sigma} \mathbf{w}} \leq v_0\}$, i.e. the set of weights which fulfills the constraints of (5) and let $W_{v_0} = \{\mathbf{w} : \mathbf{w}^\top \mathbf{1} =$

$1, -\mathbf{w}^\top\boldsymbol{\mu} - k_{1-\alpha_2}\sqrt{\mathbf{w}^\top\boldsymbol{\Sigma}\mathbf{w}} = v_0\}$ denote its boundary. It holds that

$$\begin{aligned}
\mathbf{w}_{CVoR} &= \arg\max_{\mathbf{w}\in W}\left\{\mathbf{w}^\top\boldsymbol{\mu} + k_{\alpha_1}\sqrt{\mathbf{w}^\top\boldsymbol{\Sigma}\mathbf{w}}\right\}\\
&= \arg\max_{\mathbf{w}\in W}\left\{\mathbf{w}^\top\boldsymbol{\mu} - \frac{k_{\alpha_1}}{k_{1-\alpha_2}}\left(-k_{1-\alpha_2}\sqrt{\mathbf{w}^\top\boldsymbol{\Sigma}\mathbf{w}} - \mathbf{w}^T\boldsymbol{\mu}\right) - \frac{k_{\alpha_1}}{k_{1-\alpha_2}}\mathbf{w}^\top\boldsymbol{\mu}\right\}\\
&= \arg\max_{\mathbf{w}\in W}\left\{\left(1-\frac{k_{\alpha_1}}{k_{1-\alpha_2}}\right)\mathbf{w}^\top\boldsymbol{\mu} - \frac{k_{\alpha_1}}{k_{1-\alpha_2}} - \frac{k_{\alpha_1}}{k_{1-\alpha_2}}\left(-k_{1-\alpha_2}\sqrt{\mathbf{w}^\top\boldsymbol{\Sigma}\mathbf{w}} - \mathbf{w}^T\boldsymbol{\mu}\right)\right\}
\end{aligned}$$

where we $-\frac{k_{\alpha_1}}{k_{1-\alpha_2}} > 0$ since $\alpha_1, \alpha_2 \in (1/2, 1)$.

Assume that the statement of the lemma does not hold, i.e. there exists $v_1 < v_0$ such that for the solution $\mathbf{w}^*_{CVoR}$ of the optimization problem

$$\mathbf{w}^*_{CVoR} = \arg\max_{\mathbf{w}\in W\setminus W_{v_0}}\left\{\mathbf{w}^\top\boldsymbol{\mu} + k_{\alpha_1}\sqrt{\mathbf{w}^\top\boldsymbol{\Sigma}\mathbf{w}}\right\},$$

we have

$$-\mathbf{w}^{*\top}_{CVoR}\boldsymbol{\mu} - k_{1-\alpha_2}\sqrt{\mathbf{w}^{*\top}_{CVoR}\boldsymbol{\Sigma}\mathbf{w}^*_{CVoR}} = v_1.$$

Then, because $-\frac{k_{\alpha_1}}{k_{1-\alpha_2}} > 0$ we get

$$\begin{aligned}
&\left(1-\frac{k_{\alpha_1}}{k_{1-\alpha_2}}\right)\mathbf{w}^{*\top}_{CVoR}\boldsymbol{\mu} - \frac{k_{\alpha_1}}{k_{1-\alpha_2}}\left(-k_{1-\alpha_2}\sqrt{\mathbf{w}^{*\top}_{CVoR}\boldsymbol{\Sigma}\mathbf{w}^*_{CVoR}} - \mathbf{w}^{*\top}_{CVoR}\boldsymbol{\mu}\right)\\
= &\left(1-\frac{k_{\alpha_1}}{k_{1-\alpha_2}}\right)\mathbf{w}^{*\top}_{CVoR}\boldsymbol{\mu} - \frac{k_{\alpha_1}}{k_{1-\alpha_2}}v_1 < \left(1-\frac{k_{\alpha_1}}{k_{1-\alpha_2}}\right)\mathbf{w}^{*\top}_{CVoR}\boldsymbol{\mu} - \frac{k_{\alpha_1}}{k_{1-\alpha_2}}v_0\\
\leq &\max_{\mathbf{w}\in W}\left\{\left(1-\frac{k_{\alpha_1}}{k_{1-\alpha_2}}\right)\mathbf{w}^\top\boldsymbol{\mu} - \frac{k_{\alpha_1}}{k_{1-\alpha_2}}v_0\right\} = \left(1-\frac{k_{\alpha_1}}{k_{1-\alpha_2}}\right)\max_{\mathbf{w}\in W}\left\{\mathbf{w}^\top\boldsymbol{\mu}\right\} - \frac{k_{\alpha_1}}{k_{1-\alpha_2}}v_0.
\end{aligned}$$

Since $\mathbf{w}^\top\boldsymbol{\mu}$ is a linear function and W is a bounded set, then $\max_{\mathbf{w}\in W}\left\{\mathbf{w}^\top\boldsymbol{\mu}\right\}$ is attained in the boundary of W, that is in W_{v_0}. Consequently, the solution of $\max_{\mathbf{w}\in W}\left\{\mathbf{w}^\top\boldsymbol{\mu}\right\}$ satisfies the constraint $-\mathbf{w}^\top\boldsymbol{\mu} - k_{1-\alpha_2}\sqrt{\mathbf{w}^\top\boldsymbol{\Sigma}\mathbf{w}} = v_0$ and

$$\max_{\mathbf{w}\in W\setminus W_{v_0}}\left\{\mathbf{w}^\top\boldsymbol{\mu} + k_{\alpha_1}\sqrt{\mathbf{w}^\top\boldsymbol{\Sigma}\mathbf{w}}\right\} < \max_{\mathbf{w}\in W}\left\{\mathbf{w}^\top\boldsymbol{\mu} + k_{\alpha_1}\sqrt{\mathbf{w}^\top\boldsymbol{\Sigma}\mathbf{w}}\right\}$$

The last inequality contradicts that the statement that the solution of (5) is an interior point of W. □

Proof of Theorem 3. The Langragian of (6) is defined as

$$\mathcal{L}(\mathbf{w}, \lambda_1, \lambda_2) = \mathbf{w}^\top\boldsymbol{\mu} + \lambda_1\left(-\mathbf{w}^\top\boldsymbol{\mu} - k_{1-\alpha_2}\sqrt{\mathbf{w}^\top\boldsymbol{\Sigma}\mathbf{w}} - v_0\right) + \lambda_2(\mathbf{w}^\top\mathbf{1} - 1). \quad (17)$$

Computing the gradient, necessary derivatives and thereafter setting these to zero give us the following system of equations,

$$\begin{cases} \boldsymbol{\mu} - \lambda_1 \left(\boldsymbol{\mu} - \mathrm{k}_{\alpha_2} \frac{\boldsymbol{\Sigma}\mathbf{w}}{\sqrt{\mathbf{w}^\top \boldsymbol{\Sigma}\mathbf{w}}} \right) + \lambda_2 \mathbf{1} = 0 \\ -\mathbf{w}^\top \boldsymbol{\mu} - k_{1-\alpha_2} \sqrt{\mathbf{w}^\top \boldsymbol{\Sigma}\mathbf{w}} - v_0 = 0 \\ \mathbf{w}^\top \mathbf{1} - 1 = 0. \end{cases} \tag{18}$$

Since the Langrange parameters are arbitrary, let $\tilde{\lambda}_1 := \frac{\lambda_1 - 1}{\lambda_1 k_{\alpha_2}} \sqrt{\mathbf{w}^\top \boldsymbol{\Sigma}\mathbf{w}}$ and $\tilde{\lambda}_2 := -\frac{\lambda_2 \sqrt{\mathbf{w}^\top \boldsymbol{\Sigma}\mathbf{w}}}{\lambda_1 k_{\alpha_2}}$, where $k_{\alpha_2} = -k_{1-\alpha_2}$ by the symmetry of $\mathbf{X}$. From the first equation of (18) we have that

$$\mathbf{w} = \tilde{\lambda}_2 \boldsymbol{\Sigma}^{-1} \mathbf{1} + \tilde{\lambda}_1 \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} \tag{19}$$

and by using the second and third equations of (18), equation (19) can be rewritten as

$$\tilde{\lambda}_2 \mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1} + \tilde{\lambda}_1 \mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} = 1 \tag{20}$$

$$\tilde{\lambda}_2 \boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1} + \tilde{\lambda}_1 \boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} = k_{\alpha_2} \sqrt{\mathbf{w}^\top \boldsymbol{\Sigma}\mathbf{w}} - v_0. \tag{21}$$

Let $\mu_0 = \mathbf{w}^\top \boldsymbol{\mu}$. Since $\mu_0 = k_{\alpha_2} \sqrt{\mathbf{w}^\top \boldsymbol{\Sigma}\mathbf{w}} - v_0$ we can rewrite (20) and (21) as

$$\begin{aligned} \begin{pmatrix} \tilde{\lambda}_2 \\ \tilde{\lambda}_1 \end{pmatrix} &= \begin{pmatrix} \mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1} & \mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} \\ \boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1} & \boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ k_{\alpha_2} \sqrt{\mathbf{w}^\top \boldsymbol{\Sigma}\mathbf{w}} - v_0 \end{pmatrix} \\ &= \frac{1}{\mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1} \boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} - (\boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1})^2} \begin{pmatrix} \boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} & -\mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} \\ -\boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1} & \mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1} \end{pmatrix} \begin{pmatrix} 1 \\ \mu_0 \end{pmatrix} \\ &= \frac{1}{\mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1} \boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} - (\boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1})^2} \begin{pmatrix} \boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} - \mu_0 \mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} \\ \mu_0 \mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1} - \mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} \end{pmatrix} \\ &= \frac{1}{s} \begin{pmatrix} \frac{\boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}}{\mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1}} - \mu_0 R_{GMV} \\ \mu_0 - R_{GMV} \end{pmatrix} \end{aligned} \tag{22}$$

where $R_{GMV} = \boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1} / \mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1}$ and $s = \boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} - (\boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1})^2 / \mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1}$. Let $V_{GMV} = 1/\mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1}$, then

$$\begin{aligned} \frac{\boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}}{\mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1}} &= \frac{\boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}}{\mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1}} - \left(\frac{\boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1}}{\mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1}} \right)^2 + \left(\frac{\boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1}}{\mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1}} \right)^2 \\ &= \frac{1}{\mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1}} s + R_{GMV}^2 = V_{GMV} s + R_{GMV}^2. \end{aligned}$$

Hence,

$$\begin{pmatrix} \tilde{\lambda}_2 \\ \tilde{\lambda}_1 \end{pmatrix} = \frac{1}{s} \begin{pmatrix} V_{GMV}s + R_{GMV}^2 - \mu_0 R_{GMV} \\ \mu_0 - R_{GMV} \end{pmatrix}$$

which implies that by using the Langrange parameters in (19) we have following solution

$$\begin{aligned}
\mathbf{w} &= \tilde{\lambda}_2 \mathbf{\Sigma}^{-1}\mathbf{1} + \tilde{\lambda}_1 \mathbf{\Sigma}^{-1}\boldsymbol{\mu} && (23)\\
&= \mathbf{w}_{\text{GMV}} + \Big((\tilde{\lambda}_2 - V_{GMV})\mathbf{\Sigma}^{-1}\mathbf{1} + \tilde{\lambda}_1 \mathbf{\Sigma}^{-1}\boldsymbol{\mu} \Big) && (24)\\
&= \mathbf{w}_{\text{GMV}} + \left(\frac{R_{GMV}^2 - \mu_0 R_{GMV}}{s}\mathbf{\Sigma}^{-1}\mathbf{1} + \frac{\mu_0 - R_{GMV}}{s}\mathbf{\Sigma}^{-1}\boldsymbol{\mu} \right) && (25)\\
&= \mathbf{w}_{\text{GMV}} + \frac{\mu_0 - R_{GMV}}{s}\mathbf{\Sigma}^{-1}\boldsymbol{\mu} - \frac{\mu_0 - R_{GMV}}{s} R_{GMV}\mathbf{\Sigma}^{-1}\mathbf{1}. && (26)
\end{aligned}$$

Since $\mathbf{\Sigma}^{-1}\mathbf{1}R_{GMV} = \mathbf{\Sigma}^{-1}\mathbf{1}\mathbf{1}^\top\mathbf{\Sigma}^{-1}\boldsymbol{\mu}/\mathbf{1}^\top\mathbf{\Sigma}^{-1}\mathbf{1}$ we can further simplify the solution to

$$\mathbf{w}_{CVoR} = \mathbf{w}_{\text{GMV}} + \frac{\mu_0 - R_{GMV}}{s}\mathbf{Q}\boldsymbol{\mu}$$

where $\mathbf{Q} = \mathbf{\Sigma}^{-1} - \mathbf{\Sigma}^{-1}\mathbf{1}\mathbf{1}^\top\mathbf{\Sigma}^{-1}/\mathbf{1}^\top\mathbf{\Sigma}^{-1}\mathbf{1}$. Given that $\mu_0 > R_{GMV}$, the portfolio $\mathbf{w}_{CVoR}$ is mean-variance efficient. □

Proof of Theorem 4. By Theorem 3 we have that the optimal portfolio has the following form

$$\mathbf{w}_{\text{CVoR}} = \mathbf{w}_{\text{GMV}} + \frac{\mu_0 - R_{GMV}}{s}\mathbf{Q}\boldsymbol{\mu}. \tag{27}$$

The portfolio $\mathbf{w}_{\text{CVoR}}$ satisfies the VaR constraint given by (6). By using the closed form, we have that

$$\begin{aligned}
& -\mu_0 - k_{1-\alpha_2}\sqrt{\mathbf{w}_{\text{CVoR}}^\top \mathbf{\Sigma}\mathbf{w}_{\text{CVoR}}} = v_0 \\
\Leftrightarrow \quad & \sqrt{\mathbf{w}_{\text{CVoR}}^\top \mathbf{\Sigma}\mathbf{w}_{\text{CVoR}}} = -\frac{v_0 + \mu_0}{k_{1-\alpha_2}} \\
\Leftrightarrow \quad & \mathbf{w}_{\text{CVoR}}^\top \mathbf{\Sigma}\mathbf{w}_{\text{CVoR}} = \left(\frac{v_0 + \mu_0}{k_{1-\alpha_2}}\right)^2. && (28)
\end{aligned}$$

It holds that $\mathbf{w}_{\text{GMV}}\mathbf{\Sigma}\mathbf{Q}\boldsymbol{\mu} = \mathbf{0}$ and $\boldsymbol{\mu}^\top\mathbf{Q}\mathbf{\Sigma}\mathbf{Q}\boldsymbol{\mu} = s$. Therefore equation (28) can be further simplified to

$$\left(V_{GMV} + \frac{(\mu_0 - R_{GMV})^2}{s}\right) = \left(\frac{v_0 + \mu_0}{k_{1-\alpha_2}}\right)^2. \tag{29}$$

To solve (29) for μ_0, we need to solve a second degree polynom. Expand by the squares, we have that

$$
\begin{aligned}
&V_{GMV} + \frac{1}{s}\left(\mu_0^2 - 2R_{GMV}\mu_0 + R_{GMV}^2\right) = \frac{1}{k_{1-\alpha_2}^2}\left(\mu_0^2 + 2v_0\mu_0 + v_0^2\right) \\
\Leftrightarrow &\left(\frac{1}{s} - \frac{1}{k_{1-\alpha_2}^2}\right)\mu_0^2 - 2\left(\frac{R_{GMV}}{s} + \frac{v_0}{k_{1-\alpha_2}^2}\right)\mu_0 + \left(V_{GMV} + \frac{R_{GMV}^2}{s} - \frac{v_0^2}{k_{1-\alpha_2}^2}\right) = 0.
\end{aligned}
$$

Multiply by $sk_{1-\alpha_2}^2$ leads to

$$
a_1\mu_0^2 - 2a_2\mu_0 + a_3 = 0 \tag{30}
$$

where $a_1 = \left(k_{1-\alpha_2}^2 - s\right)$, $a_2 = \left(R_{GMV}k_{1-\alpha_2}^2 + sv_0\right)$ and $a_3 = \left(V_{GMV}k_{1-\alpha_2}^2 s + R_{GMV}^2 k_{1-\alpha_2}^2 - v_0^2 s\right)$

Assuming that $k_{1-\alpha_2}^2 > s$, the solution to (30) is given by $\mu_0 = (a_2 \pm \sqrt{a_2^2 - a_3 a_1})/a_1$ where

$$
\begin{aligned}
a_2^2 - a_3a_1 &= \left(R_{GMV}k_{1-\alpha_2}^2 + sv_0\right)^2 - \left(V_{GMV}k_{1-\alpha_2}^2 s + R_{GMV}^2 k_{1-\alpha_2}^2 - v_0^2 s\right)\left(k_{1-\alpha_2}^2 - s\right) \\
&= 2k_{1-\alpha_2}^2 R_{GMV}v_0 s - V_{GMV}k_{1-\alpha_2}^4 s + v_0^2 s k_{1-\alpha_2}^2 + V_{GMV}k_{1-\alpha_2}^2 s^2 + R_{GMV}^2 k_{1-\alpha_2}^2 s \\
&= k_{1-\alpha_2}^2 s(2R_{GMV}v_0 - V_{GMV}k_{1-\alpha_2}^2 + v_0^2 + V_{GMV}s + R_{GMV}^2) \\
&= k_{1-\alpha_2}^2 s\left((R_{GMV} + v_0)^2 + (s - k_{1-\alpha_2}^2)V_{GMV}\right)
\end{aligned}
$$

Therefore the roots are equal to

$$
\mu_0 = \frac{k_{1-\alpha_2}^2 R_{GMV} + sv_0 \pm \left(k_{1-\alpha_2}^2 s\left((R_{GMV} + v_0)^2 + (s - k_{1-\alpha_2}^2)V_{GMV}\right)\right)^{1/2}}{k_{1-\alpha_2}^2 - s}. \tag{31}
$$

Since we aim to maximize the expected return of the portfolio, the first root is optimal. Also, if $\mu_0 \in \mathbb{R}$ then the following needs to hold

$$
k_{1-\alpha_2}^2 s\left((R_{GMV} + v_0)^2 + (s - k_{1-\alpha_2}^2)V_{GMV}\right) \geq 0.
$$

The condition is equivalent to

$$
v_0 \geq -R_{GMV} + \sqrt{(k_{1-\alpha_2}^2 - s)}\sqrt{V_{GMV}}, \tag{32}
$$

and since $\sqrt{(k_{1-\alpha_2}^2 - s)} \leq \sqrt{k_{1-\alpha_2}^2} = -k_{1-\alpha_2}$, the inequality $v_0 \geq \mathrm{CVaR}_{\alpha_2}(X_{\mathbf{w}_{GMV}})$ holds if (32) does. The characteristics R_{CVoR}, V_{CVoR} can be easily calculated by using the closed form of the portfolio weights. □

Proof of Proposition 1. We first note that η is a composite function of the quantile function $k_{1-\alpha_2}^2 = k_{\alpha_2}^2$ which is increasing in α_2. We look upon η as a function of

$k = k^2_{1-\alpha_2}$ where k has support $\{k > \max\{s, 2\}\}$ and the support of v is $\{v \geq -R_{GMV} + \sqrt{kV_{GMV}}\}$ given by

$$\eta(k, v) = \frac{g_1(v) + \sqrt{g_2(k, v)}}{g_3(k)},$$

where $g_1(v) = (R_{GMV} + v)s \geq \sqrt{kV_{GMV}} \geq 0$, $g_2(k, v) = ks((R_{GMV} + v)^2 - (k - s)V_{GMV}) \geq 0$ and $g_3(k) = k - s \geq 0$ for all k and v from their supports.

First, we note that η is increasing in v, since both $g_1(v)$ and $g_2(k, v)$ are increasing in v. To show that η is decreasing in k we compute $g_3'(k) = 1$,

$$\frac{\partial g_2(k, v)}{\partial k} = s((R_{GMV} + v)^2 - (k - s)V) - ksV \leq 0$$

and, hence,

$$\begin{aligned}
\frac{\partial \eta(k, v)}{\partial k} &= -\frac{g_3'(k)}{(g_3(k))^2}\left(g_1(v) + \sqrt{g_2(k, v)}\right) + \frac{1}{2g_3(k)\sqrt{g_2(k, v)}}\frac{\partial g_2(k, v)}{\partial k} \\
&= -\frac{g_1(v) + \sqrt{g_2(k, v)}}{g_3(k)} + \frac{1}{2g_3(k)\sqrt{g_2(k, v)}}\frac{\partial g_2(k, v)}{\partial k} \\
&= \frac{1}{g_3(k)}\left(-g_1(v) - \left(1 - \frac{1}{2k}\right)\sqrt{g_2(k, v)} - \frac{ksV_{GMV}}{2\sqrt{g_2(k, v)}}\right) < 0,
\end{aligned}$$

which proves the proposition. □

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Table 1: Returns of the portfolio configuration for a 100 week holdout period and v_0 was set to 0.5. Elliptically contoured portfolios are indicated by the use of a distribution. All other are nonparametric portfolios.

Return	Variance	CVoR	CVaR	VaR	Sharpe ratio	α_2	α_1	Distribution
0.00915	0.61714	0.31226	0.62715	0.48331	0.01482	0.950	0.500	t(5)
0.00915	0.61714	0.40286	0.62715	0.48331	0.01482	0.950	0.600	t(5)
0.00915	0.61714	0.52908	0.62715	0.48331	0.01482	0.950	0.700	t(5)
0.00514	0.22902	0.19217	0.62045	0.34146	0.02246	0.990	0.500	t(5)
0.00514	0.22902	0.24754	0.62045	0.34146	0.02246	0.990	0.600	t(5)
0.00514	0.22902	0.32559	0.62045	0.34146	0.02246	0.990	0.700	t(5)
0.00234	0.06951	0.10808	0.35787	0.34110	0.03365	0.999	0.500	t(5)
0.00234	0.06951	0.13899	0.35787	0.34110	0.03365	0.999	0.600	t(5)
0.00234	0.06951	0.18301	0.35787	0.34110	0.03365	0.999	0.700	t(5)
0.00797	0.48348	0.27693	0.55608	0.42479	0.01649	0.950	0.500	t(10)
0.00797	0.48348	0.35719	0.55608	0.42479	0.01649	0.950	0.600	t(10)
0.00797	0.48348	0.46927	0.55608	0.42479	0.01649	0.950	0.700	t(10)
0.00509	0.22533	0.19066	0.61572	0.33874	0.02260	0.990	0.500	t(10)
0.00509	0.22533	0.24559	0.61572	0.33874	0.02260	0.990	0.600	t(10)
0.00509	0.22533	0.32302	0.61572	0.33874	0.02260	0.990	0.700	t(10)
0.00302	0.09979	0.12854	0.42175	0.40227	0.03027	0.999	0.500	t(10)
0.00302	0.09979	0.16536	0.42175	0.40227	0.03027	0.999	0.600	t(10)
0.00302	0.09979	0.21770	0.42175	0.40227	0.03027	0.999	0.700	t(10)
0.00740	0.42462	0.25983	0.52167	0.39645	0.01743	0.950	0.500	t(50)
0.00740	0.42462	0.33508	0.52167	0.39645	0.01743	0.950	0.600	t(50)
0.00740	0.42462	0.44031	0.52167	0.39645	0.01743	0.950	0.700	t(50)
0.00515	0.22914	0.19222	0.62061	0.34155	0.02245	0.990	0.500	t(50)
0.00515	0.22914	0.24761	0.62061	0.34155	0.02245	0.990	0.600	t(50)
0.00515	0.22914	0.32568	0.62061	0.34155	0.02245	0.990	0.700	t(50)
0.00357	0.12833	0.14511	0.47349	0.45182	0.02785	0.999	0.500	t(50)
0.00357	0.12833	0.18676	0.47349	0.45182	0.02785	0.999	0.600	t(50)
0.00357	0.12833	0.24580	0.47349	0.45182	0.02785	0.999	0.700	t(50)
0.00109	0.02835	0.07071	0.14657	0.11257	0.03852	0.950	0.500	Laplace
0.00109	0.02835	0.09095	0.14657	0.11257	0.03852	0.950	0.600	Laplace
0.00109	0.02835	0.11976	0.14657	0.11257	0.03852	0.950	0.700	Laplace
0.00022	0.01041	0.04490	0.15939	0.09597	0.02098	0.990	0.500	Laplace
0.00022	0.01041	0.05783	0.15939	0.09597	0.02098	0.990	0.600	Laplace
0.00022	0.01041	0.07602	0.15939	0.09597	0.02098	0.990	0.700	Laplace
-0.00037	0.00339	0.02846	0.10431	0.10100	-0.10913	0.999	0.500	Laplace
-0.00037	0.00339	0.03641	0.10431	0.10100	-0.10913	0.999	0.600	Laplace
-0.00037	0.00339	0.04711	0.10431	0.10100	-0.10913	0.999	0.700	Laplace
-0.00029	0.00081	0.01559	0.07492	0.03094	-0.36537	0.950	0.500	ECDF
-0.00067	0.00074	0.01763	0.06909	0.02826	-0.90715	0.950	0.600	ECDF
-0.00131	0.00086	0.01990	0.07657	0.02678	-1.51799	0.950	0.700	ECDF
-0.00029	0.00081	0.01559	0.21372	0.05095	-0.36537	0.990	0.500	ECDF
-0.00067	0.00074	0.01763	0.20924	0.04046	-0.90715	0.990	0.600	ECDF
-0.00131	0.00086	0.01990	0.23624	0.04365	-1.51799	0.990	0.700	ECDF
-0.00029	0.00081	0.01559	0.21372	0.19744	-0.36537	0.999	0.500	ECDF
-0.00067	0.00074	0.01763	0.20924	0.19236	-0.90715	0.999	0.600	ECDF
-0.00131	0.00086	0.01990	0.23624	0.21698	-1.51799	0.999	0.700	ECDF

Table 2: Returns of the portfolio configuration for a 99 week holdout period without the large loss in AAPL. The parameter v_0 was set to 0.5. Elliptically contoured portfolios are indicated by the use of a distribution. All other are nonparametric portfolios.

Return	Variance	CVoR	CVaR	VaR	Sharpe ratio	α_2	α_1	Distribution
-0.06324	0.09410	0.16707	0.62715	0.48420	-0.67207	0.950	0.500	t(5)
-0.06324	0.09410	0.22287	0.62715	0.48420	-0.67207	0.950	0.600	t(5)
-0.06324	0.09410	0.29179	0.62715	0.48420	-0.67207	0.950	0.700	t(5)
-0.03882	0.03610	0.10402	0.62045	0.34427	-1.07537	0.990	0.500	t(5)
-0.03882	0.03610	0.13827	0.62045	0.34427	-1.07537	0.990	0.600	t(5)
-0.03882	0.03610	0.18141	0.62045	0.34427	-1.07537	0.990	0.700	t(5)
-0.02171	0.01179	0.05985	0.35787	0.34127	-1.84159	0.999	0.500	t(5)
-0.02171	0.01179	0.07935	0.35787	0.34127	-1.84159	0.999	0.600	t(5)
-0.02171	0.01179	0.10422	0.35787	0.34127	-1.84159	0.999	0.700	t(5)
-0.05606	0.07421	0.14854	0.55608	0.42587	-0.75549	0.950	0.500	t(10)
-0.05606	0.07421	0.19800	0.55608	0.42587	-0.75549	0.950	0.600	t(10)
-0.05606	0.07421	0.25935	0.55608	0.42587	-0.75549	0.950	0.700	t(10)
-0.03851	0.03554	0.10323	0.61572	0.34153	-1.08353	0.990	0.500	t(10)
-0.03851	0.03554	0.13721	0.61572	0.34153	-1.08353	0.990	0.600	t(10)
-0.03851	0.03554	0.18002	0.61572	0.34153	-1.08353	0.990	0.700	t(10)
-0.02587	0.01647	0.07059	0.42175	0.40247	-1.57137	0.999	0.500	t(10)
-0.02587	0.01647	0.09366	0.42175	0.40247	-1.57137	0.999	0.600	t(10)
-0.02587	0.01647	0.12297	0.42175	0.40247	-1.57137	0.999	0.700	t(10)
-0.05259	0.06543	0.13957	0.52167	0.39763	-0.80376	0.950	0.500	t(50)
-0.05259	0.06543	0.18596	0.52167	0.39763	-0.80376	0.950	0.600	t(50)
-0.05259	0.06543	0.24364	0.52167	0.39763	-0.80376	0.950	0.700	t(50)
-0.03883	0.03612	0.10405	0.62061	0.34437	-1.07509	0.990	0.500	t(50)
-0.03883	0.03612	0.13831	0.62061	0.34437	-1.07509	0.990	0.600	t(50)
-0.03883	0.03612	0.18146	0.62061	0.34437	-1.07509	0.990	0.700	t(50)
-0.02924	0.02084	0.07930	0.47349	0.45204	-1.40353	0.999	0.500	t(50)
-0.02924	0.02084	0.10525	0.47349	0.45204	-1.40353	0.999	0.600	t(50)
-0.02924	0.02084	0.13817	0.47349	0.45204	-1.40353	0.999	0.700	t(50)
-0.01411	0.00530	0.04022	0.14657	0.11264	-2.66058	0.950	0.500	Laplace
-0.01411	0.00530	0.05325	0.14657	0.11264	-2.66058	0.950	0.600	Laplace
-0.01411	0.00530	0.06997	0.14657	0.11264	-2.66058	0.950	0.700	Laplace
-0.00878	0.00234	0.02683	0.15939	0.09661	-3.75135	0.990	0.500	Laplace
-0.00878	0.00234	0.03545	0.15939	0.09661	-3.75135	0.990	0.600	Laplace
-0.00878	0.00234	0.04662	0.15939	0.09661	-3.75135	0.990	0.700	Laplace
-0.00519	0.00108	0.01879	0.10431	0.10104	-4.80453	0.999	0.500	Laplace
-0.00519	0.00108	0.02447	0.10431	0.10104	-4.80453	0.999	0.600	Laplace
-0.00519	0.00108	0.03144	0.10431	0.10104	-4.80453	0.999	0.700	Laplace
0.00186	0.00035	0.01559	0.03833	0.02630	5.38862	0.950	0.500	ECDF
0.00143	0.00030	0.01763	0.03288	0.02530	4.76118	0.950	0.600	ECDF
0.00107	0.00030	0.01990	0.03465	0.02599	3.53639	0.950	0.700	ECDF
0.00186	0.00035	0.01559	0.04930	0.04024	5.38862	0.990	0.500	ECDF
0.00143	0.00030	0.01763	0.03876	0.03589	4.76118	0.990	0.600	ECDF
0.00107	0.00030	0.01990	0.04171	0.03824	3.53639	0.990	0.700	ECDF
0.00186	0.00035	0.01559	0.04930	0.04840	5.38862	0.999	0.500	ECDF
0.00143	0.00030	0.01763	0.03876	0.03847	4.76118	0.999	0.600	ECDF
0.00107	0.00030	0.01990	0.04171	0.04136	3.53639	0.999	0.700	ECDF