Mathematical Statistics
Stockholm University

Individual Ergodic Theorems for Perturbed Alternating Regenerative Processes

Dmitrii Silvestrov

Research Report 2018:9

ISSN 1650-0377

Postal address:

Mathematical Statistics
Dept. of Mathematics
Stockholm University
SE-106 91 Stockholm
Sweden

Internet:

http://www.math.su.se

Mathematical Statistics
Stockholm University
Research Report **2018:9**,
http://www.math.su.se

Individual Ergodic Theorems for Perturbed Alternating Regenerative Processes

Dmitrii Silvestrov[1]

February 2018

Abstract

The paper presents results of complete analysis and classification of individual ergodic theorems for perturbed alternating regenerative processes with semi-Markov modulation. New short, long and super-long time ergodic theorems for regularly and singular type perturbed alternating regenerative processes are presented.

Key words: Regenerative process, Alternating regenerative process, Semi-Markov modulation, Renewal equation, Regular perturbation, Singular Perturbation, Super-singular perturbation, Ergodic theorem, Individual ergodic theorem, Quasi-ergodic theorem, Time scaling, Time compression, Short time ergodic theorem, Long time ergodic theorem, Super-long time ergodic theorem.

AMS Mathematical Subject Classification 2010: 60K05, 60K15, 60K99.

---

[1]Department of Mathematics, Stockholm University, 106 91, Stockholm, Sweden,
E-mail: silvestrov@math.su.se

Individual Ergodic Theorems for Perturbed Alternating Regenerative Processes

Dmitrii Silvestrov[1]

25 February, 2018

Abstract. The paper presents results of complete analysis and classification of individual ergodic theorems for perturbed alternating regenerative processes with semi-Markov modulation. New short, long and super-long time ergodic theorems for regularly, singularly and super-singularly perturbed alternating regenerative processes are presented.

Key words: Regenerative process, Alternating regenerative process, Semi-Markov modulation, Renewal equation, Regular perturbation, Singular Perturbation, Super-singular perturbation, Ergodic theorem, Individual ergodic theorem, Quasi-ergodic theorem, Time scaling, Time compression, Short time ergodic theorem, Long time ergodic theorem, Super-long time ergodic theorem.

AMS Mathematical Subject Classification 2010: 60K05, 60K15, 60K99.

1. Introduction

The paper presents results of complete analysis and classification of individual ergodic theorems for perturbed alternating regenerative processes with semi-Markov modulation.

The alternating regenerative processes and related alternating renewal processes are popular models of stochastic processes, which have diverse applications to queuing, reliability, control and many other types of stochastic processes and systems. We refer here to papers and books, which contain basic materials about regenerative processes including their alternating variants and applications [4, 7, 9, 15, 17, 23, 28 – 32, 40, 53, 57].

Standard alternating models are constructed from sequences of "random blocks" of two types, say, 1 and 2. Each block consists of a "piece" of stochastic process of random duration. All blocks are independent. Blocks of each type have the same probabilistic characteristics. The corresponding

[1]Department of Mathematics, Stockholm University, SE-106 91 Stockholm, Sweden, E-mail: silvestrov@math.su.se

alternating regenerative process is constructed by sequential in time alternate connection of blocks of types 1 and 2 taken from the above mentioned sequences.

In the present paper, more general alternating regenerative processes are studied, where sequential alternate connection of the blocks is controlled by some binary switching random variables. The piece of stochastic process creating every block, its duration and the binary random variable, controlling the decision about switching/non-switching of block type at the end of time interval corresponding to this block, may be dependent. This let us speak about semi-Markov modulation for the corresponding alternating regenerative process.

If the above alternating regeneration process $\xi_\varepsilon(t), t \geq 0$ describes functioning of some stochastic system, it is naturally to interpret $\xi_\varepsilon(t)$ as the state of this system at instant t and the corresponding modulating semi-Markov process $\eta_\varepsilon(t)$ as the stochastic index, which shows that the system is in one of two possible regimes (for example, "working" or "not working") at instant t if, respectively, $\eta_\varepsilon(t) = 1$ or $\eta_\varepsilon(t) = 2$.

It is assumed that joint probabilistic characteristics of the alternating regenerative process $\xi_\varepsilon(t)$ and the corresponding semi-Markov process $\eta_\varepsilon(t)$ controlling switching of types depend on some perturbation parameter $\varepsilon \in [0, 1]$ and converge to the corresponding joint characteristics of the processes $\xi_0(t)$ and $\eta_0(t)$, as $\varepsilon \to 0$. This makes it possible to consider process $(\xi_\varepsilon(t), \eta_\varepsilon(t))$, for $\varepsilon \in (0, 1]$, as a perturbed version of the process $(\xi_0(t), \eta_0(t))$.

The object of our interest are individual ergodic theorems about asymptotic behaviour of joint distributions $\mathsf{P}_{\varepsilon,ij}(t, A) = \mathsf{P}_i\{\xi_\varepsilon(t) \in A,\ \eta_\varepsilon(t) = j\}$ for perturbed alternating regenerative process $\xi_\varepsilon(t)$ and modulating semi-Markov processes $\eta_\varepsilon(t)$, as time $t \to \infty$ and the perturbation parameter $\varepsilon \to 0$.

Models with tree different types of perturbation are considered. These types are determined by the asymptotic behaviour of transition probabilities $p_{\varepsilon,ij}, i, j = 1, 2$ of the embedded Markov chain $\eta_{\varepsilon,n}$ for the semi-Markov process $\eta_\varepsilon(t)$. These transition probabilities converge, as $\varepsilon \to 0$, to the corresponding transition probabilities of the limiting Markov chain $\eta_{0,n}$.

The first class constitutes regularly perturbed models, where the limiting embedded Markov chain $\eta_{0,n}$ is ergodic that, in this case, is equivalent to the assumption that $\max(p_{0,12}, p_{0,21}) > 0$.

In the case of regularly perturbed models, the corresponding individual ergodic theorems take forms of asymptotic relations $\mathsf{P}_{\varepsilon,ij}(t_\varepsilon, A) \to \pi_j(A)$ as $\varepsilon \to 0$, which holds for any $0 \leq t_\varepsilon \to \infty$ as $\varepsilon \to 0$. The corresponding limiting probabilities $\pi_j(A)$ do not depend on an initial state i of the modulating semi-Markov process.

Such theorems resemble well known ergodic theorems for unperturbed alternating regenerative processes and more general stochastic processes with semi-Markov modulation.

Here, works [4, 7, 9, 12, 17, 23, 24, 29, 30, 35, 36, 40 – 45, 53, 57] can be referred, where one can find the corresponding ergodic theorems for unperturbed regenerative and alternating regenerative processes $(\xi_0(t), \eta_0(t))$, and [1, 10, 11, 13, 14, 23, 25 – 27, 33, 34, 37 – 39, 41 – 46, 48 – 50, 58], where such theorems are given for some classes of regularly perturbed regenerative and alternating regenerative processes $(\xi_\varepsilon(t), \eta_\varepsilon(t))$.

The second and third classes constitute singularly and super-singularly perturbed models, where the limiting embedded Markov chain $\eta_{0,n}$ is not ergodic that is equivalent to the assumption that $\max(p_{0,12}, p_{0,21}) = 0$.

The individual ergodic theorems for such models are the main objects of studies in the present paper. They take much more interesting and complex forms, if to compare them with individual ergodic theorems for regularly perturbed alternating regenerative processes. In particular, the corresponding individual ergodic theorems for singularly and super-singularly perturbed models take forms of asymptotic relations $\mathsf{P}_{\varepsilon,ij}(t_\varepsilon, A) \to \pi_{ij}(t, A)$ as $\varepsilon \to 0$, which hold for any $0 \leq t_\varepsilon \to \infty$ as $\varepsilon \to 0$, which satisfy some time scaling relation, $t_\varepsilon/v_\varepsilon \to t \in [0, \infty]$ or $t_\varepsilon/w_\varepsilon \to t \in [0, \infty]$ as $\varepsilon \to 0$, with time scaling factors $v_\varepsilon = p_{\varepsilon,12}^{-1} + p_{\varepsilon,21}^{-1} > w_\varepsilon = (p_{\varepsilon,12} + p_{\varepsilon,21})^{-1} \to \infty$ as $\varepsilon \to 0$. The corresponding limiting probabilities $\pi_{ij}(t, A)$ may depend on parameter t and an initial state i of the modulating semi-Markov process. They take essentially different forms, for cases $t = 0$, $t \in (0, \infty)$ and $t = \infty$. We classify the corresponding theorems, respectively, as short, long and super-long time individual ergodic theorems.

Individual ergodic theorems for singularly and super-singularly perturbed alternating regenerative processes presented in the paper were not known before.

The main analytic tool used for obtaining ergodic theorems is based on results concerned generalisation of the renewal theorem to the model of perturbed renewal equation given in works [14, 37 – 39] and quasi-ergodic theorems for perturbed regenerative processes with regenerative lifetimes given in works [13, 14, 48, 49].

Here, works [2, 3, 5, 6, 8, 14, 16, 18 – 22, 51, 52, 54 – 56, 59, 60] can also be mentioned, where one can find results and bibliographies of works on limit and ergodic type theorems for singularly perturbed Markov type processes. The difference with some related results presented in these works, is that we operate, in general, with non-Markov regenerative type processes and do not exploit additive accumulation or phase merging phenomena.

We do prefer to use for getting individual ergodic type theorems, as we think, the most effective methods based on generalisations of the classical renewal theorem to model of perturbed renewal equation developed in the above mentioned works [13, 14, 37 – 39, 48, 49]. This let us get the corresponding ergodic theorems under minimal conditions. In the case of unperturbed and non-alternating regenerative processes, these conditions just reduce to the minimal conditions of the classical individual ergodic theorem for unperturbed regenerative processes yielded by the famous renewal theorem, which is given in its final form in [12].

The paper includes 7 sections. In Section 2, so-called quasi-ergodic theorems for perturbed regenerative processes with regenerative lifetimes, which play the role of basic analytical tool in our studies, and the model of perturbed alternating regenerative processes are presented, and comments concerning regularly, singularly and super-singularly perturbed alternating regenerative processes are given. In Section 3 – 6, short, long and super-long individual ergodic theorems for regularly, singularly and super-singularly perturbed alternating regenerative processes are presented. Section 7, contains a short summary of the results and a list of some directions for future development and improvement of results presented in the paper.

2. Perturbed regenerative and alternating regenerative processes

In this section, we present so-called quasi-ergodic theorems for perturbed regenerative processes with regenerative lifetimes, which play the role of basic analytical tool in our studies, introduce alternating regenerative processes, comment and compare models of regularly, singularly and super-singularly perturbed alternating regenerative processes and forms of the corresponding ergodic theorems, and describe the special procedure of aggregation for regeneration times, which play an important role in ergodic theorems for perturbed alternating regenerative processes.

2.1. Quasi-ergodic theorems for perturbed regenerative processes with regenerative lifetimes. The main tool, which we are going to use are ergodic theorems for perturbed regenerative processes with regenerative lifetimes, given in the book [14].

Let $\langle \Omega_\varepsilon, \mathcal{F}_\varepsilon, \mathsf{P}_\varepsilon \rangle$ be, for every $\varepsilon \in [0, 1]$, a probability space. We assume that all stochastic processes and random variables introduced below and indexed by parameter ε are defined on this probability space.

Let also, for every $n = 1, 2, \ldots$: **(a)** $\bar{\xi}_{\varepsilon,n} = \langle \xi_{\varepsilon,n}(t), t \geq 0 \rangle$ be a stochastic process with a phase space $\mathbb{X}$ (with the corresponding σ-algebra of measurable subsets $\mathcal{B}_{\mathbb{X}}$), measurable in the sense that $\xi_{\varepsilon,n}(t, \omega), (t, \omega) \in [0, \infty) \times \Omega$ is measurable function of (t, ω) (this means that $\{(t;\omega) \in A\} \in \mathcal{B}_+ \times \mathcal{F}_\varepsilon, A \in \mathcal{B}_{\mathbb{X}}$, where $\mathcal{B}_+ \times \mathcal{F}_\varepsilon$ is the minimal σ-algebra containing all products $B \times C, B \in \mathcal{B}_+, C \in \mathcal{F}_\varepsilon$, $\mathcal{B}_+$ is the σ-algebra of Borel subsets of $[0, \infty)$); **(b)** $\kappa_{\varepsilon,n}$ be a non-negative random variable; **(c)** $\mu_{\varepsilon,n}$ is a non-negative random variable. Further, we assume that: **(d)** random triplets $\langle \bar{\xi}_{\varepsilon,n} = \langle \xi_{\varepsilon,n}(t), t \geq 0\rangle, \kappa_{\varepsilon,n}, \mu_{\varepsilon,n}\rangle$, are mutually independent; **(e)** the joint distributions of random variables $\xi_{\varepsilon,n}(t_k), k = 1, \ldots, r$ and $\kappa_{\varepsilon,n}, \mu_{\varepsilon,n}$ do not depend on $n \geq 1$, for every $t_k \in [0, \infty), k = 1, \ldots, r, r \geq 1$.

Let us define regeneration times, $\tau_{\varepsilon,n} = \kappa_{\varepsilon,1} + \cdots + \kappa_{\varepsilon,n}, n = 1, 2, \ldots, \tau_{\varepsilon,0} = 0$, a standard regenerative process,

$$\xi_\varepsilon(t) = \xi_{\varepsilon,n}(t - \tau_{\varepsilon,n-1}), \text{ for } t \in [\tau_{\varepsilon,n-1}, \tau_{\varepsilon,n}), \ n = 1, 2, \ldots, \tag{1}$$

and a regenerative lifetime,

$$\mu_\varepsilon = \sum_{k=1}^{\nu_\varepsilon - 1} \kappa_{\varepsilon,k} + \mu_{\varepsilon,\nu_\varepsilon} \mathrm{I}(\nu_\varepsilon < \infty), \tag{2}$$

where

$$\nu_\varepsilon = \min(n \geq 1 : \mu_{\varepsilon,n} < \kappa_{\varepsilon,n}).$$

We exclude instant regenerations and, thus, assume that the following condition holds:

A: $\mathsf{P}\{\kappa_{\varepsilon,1} > 0\} = 1$, for every $\varepsilon \in [0, 1]$.

Condition **A** obviously implies that random variables $\tau_{\varepsilon,n} \stackrel{\mathsf{P}}{\longrightarrow} \infty$ as $n \to \infty$, for every $\varepsilon \in [0, 1]$, and, thus, the regenerative process $\xi_\varepsilon(t)$ is well defined on the time interval $[0, \infty)$.

Let us introduce distribution functions $F_\varepsilon(t) = \mathsf{P}\{\tau_{\varepsilon,1} \leq t, \mu_\varepsilon \geq \tau_{\varepsilon,1}\} = \mathsf{P}\{\kappa_{\varepsilon,1} \leq t, \mu_{\varepsilon,1} \geq \kappa_{\varepsilon,1}\}, t \geq 0$ and stopping probabilities $f_\varepsilon = 1 - F_\varepsilon(\infty) = \mathsf{P}\{\mu_\varepsilon < \tau_{\varepsilon,1}\} = \mathsf{P}\{\mu_{\varepsilon,1} < \kappa_{\varepsilon,1}\}$. We also assume that the following condition holds:

B: **(a)** $F_\varepsilon(\cdot) \Rightarrow F_0(\cdot)$ as $\varepsilon \to 0$, **(b)** $F_0(t)$ is a proper non-arithmetic distribution function.

Here and henceforth symbol $\varepsilon \to 0$ is used to show that $0 < \varepsilon \to 0$.

Condition **B** obviously implies that the stopping probabilities,

$$f_\varepsilon \to f_0 = 0 \text{ as } \varepsilon \to 0. \tag{3}$$

Let us introduce expectations $e_\varepsilon = \int_0^\infty s F_\varepsilon(ds)$. We also assume that the following condition holds:

C: **(a)** $e_\varepsilon < \infty$, for $\varepsilon \in [0, 1]$, **(b)** $e_\varepsilon \to e_0$ as $\varepsilon \to 0$.

The object of our interest are probabilities $P_\varepsilon(t, A) = \mathsf{P}\{\xi_\varepsilon(t) \in A, \mu_\varepsilon > t\}$, $A \in \mathcal{B}_{\mathbb{X}}, t \geq 0$. These probabilities are, for every $A \in \mathcal{B}_{\mathbb{X}}$, a measurable function of $t \geq 0$, which is the unique bounded solution for the following renewal equation,

$$P_\varepsilon(t, A) = q_\varepsilon(t, A) + \int_0^t P_\varepsilon(t - s, A) F_\varepsilon(ds), \ t \geq 0, \tag{4}$$

where $q_\varepsilon(t, A) = \mathsf{P}\{\xi_\varepsilon(t) \in A, \tau_{\varepsilon,1} \wedge \mu_\varepsilon > t\} = \mathsf{P}\{\xi_{\varepsilon,1}(t) \in A, \tau_{\varepsilon,1} \wedge \mu_{\varepsilon,1} > t\}$, $A \in \mathcal{B}_{\mathbb{X}}, t \geq 0$.

We also impose the following condition on the functions $q_\varepsilon(t, A)$:

D: There exist a non-empty class of sets $\Gamma \subseteq \mathcal{B}_{\mathbb{X}}$ such that, for every $A \in \Gamma$, the asymptotic relation, $\lim_{u \to 0} \overline{\lim}_{0 \leq \varepsilon \to 0} \sup_{-(u \wedge s) \leq v \leq u} |q_\varepsilon(s + v, A) - q_0(s, A)| = 0$, holds almost everywhere with respect to the Lebesgue measure $m(ds)$ on $[0, \infty)$.

The class Γ appearing in condition **D** contains the phase space $\mathbb{X}$ and is closed with respect to the operation of union for not intersecting sets, the operation of difference for sets connected by relation of inclusion, and the complement operation. The detailed comments are given in Subsection 2.5.

Conditions **A** – **D** imply that process $\xi_0(t), t \geq 0$ is ergodic and the following asymptotic relation holds, for $A \in \Gamma$,

$$P_0(t, A) \to \pi_0(A) \text{ as } t \to \infty, \tag{5}$$

where $\pi_0(A)$ is the corresponding stationary distribution given by the following relation,

$$\pi_0(A) = \frac{1}{e_0} \int_0^\infty q_0(s, A) m(ds), \ A \in \mathcal{B}_{\mathbb{X}}. \tag{6}$$

Now we are prepared to formulate the basic, so-called quasy-ergodic theorem, for the perturbed regenerative processes with regenerative lifetimes given in book [14]. It is also worth to note that this theorem is the direct corollary of the version renewal theorem for perturbed renewal equation given in papers [37 – 39].

Theorem 1. *Let conditions* **A** – **D** *hold. Then, for every* $A \in \Gamma$, *and any* $0 \leq t_\varepsilon \to \infty$ *as* $\varepsilon \to 0$ *such that* $f_\varepsilon t_\varepsilon \to t \in [0, \infty]$ *as* $\varepsilon \to 0$,

$$P_\varepsilon(t_\varepsilon, A) \to e^{-t/e_0} \pi_0(A) \text{ as } \varepsilon \to 0. \tag{7}$$

Let us now assume that the model assumption **(e)** formulated above holds only for $n \geq 2$. In this case, the process $\xi_\varepsilon(t), t \geq 0$ is usually referred as a regenerative process with transition period $[0, \tau_{\varepsilon,1})$.

We also shall use the extension of Theorem 1 on the model of perturbed regenerative processes with transition period. In this case, the shifted process $\xi^{(1)}_\varepsilon(t) = \xi_\varepsilon(\tau_{\varepsilon,1} + t), t \geq 0$ is a standard regenerative process, with regeneration times $\tau^{(1)}_{\varepsilon,n} = \kappa_{\varepsilon,2} + \cdots + \kappa_{\varepsilon,n+1}, n = 1, 2, \ldots, \tau^{(1)}_{\varepsilon,0} = 0$ and the the corresponding shifted regenerative lifetime $\mu^{(1)}_\varepsilon = \sum_{k=1}^{\nu^{(1)}_\varepsilon - 1} \kappa_{\varepsilon,1+k} + \mu_{\varepsilon,1+\nu^{(1)}_\varepsilon} \mathrm{I}(\nu^{(1)}_\varepsilon < \infty)$, where $\nu^{(1)}_\varepsilon = \min(n \geq 1 : \mu_{\varepsilon,1+n} < \kappa_{\varepsilon,1+n})$.

All quantities appearing in conditions **A** – **D**, the renewal equation (4) and relation (6) should be defined using shifted sequence of triplets $\langle \bar{\xi}_{\varepsilon,2} = \langle \xi_{\varepsilon,2}(t), t \geq 0 \rangle, \kappa_{\varepsilon,2}, \mu_{\varepsilon,2} \rangle$. It is also natural to index the above mentioned quantities by the upper index $^{(1)}$, for example, to use notation $P^{(1)}_\varepsilon(t, A) = \mathsf{P}\{\xi^{(1)}_\varepsilon(t) \in A, \mu^{(1)}_\varepsilon > t\}$, etc. Probabilities $P^{(1)}_\varepsilon(t, A)$ satisfy the renewal equation (4). Theorem 1 presents, in this case, the corresponding ergodic relation for these probabilities.

Probabilities $P_\varepsilon(t, A) = \mathsf{P}\{\xi_\varepsilon(t) \in A, \mu_\varepsilon > t\}$, defined for the initial regenerative process with transition period, are, for every $A \in \mathcal{B}_{\mathbb{X}}$, connected with probabilities $P^{(1)}_\varepsilon(t_\varepsilon, A)$ by the following renewal type transition relation,

$$P_\varepsilon(t, A) = \tilde{q}_\varepsilon(t, A) + \int_0^t P^{(1)}_\varepsilon(t - s, A) \tilde{F}_\varepsilon(ds), \ t \geq 0, \tag{8}$$

where $\tilde{q}_\varepsilon(t, A) = \mathsf{P}\{\xi_\varepsilon(t) \in A, \tau_{\varepsilon,1} \wedge \mu_\varepsilon > t\} = \mathsf{P}\{\xi_{\varepsilon,1}(t) \in A, \tau_{\varepsilon,1} \wedge \mu_{\varepsilon,1} > t\}$, $A \in \mathcal{B}_{\mathbb{X}}, t \geq 0$ and $\tilde{F}_\varepsilon(t) = \mathsf{P}\{\tau_{\varepsilon,1} \leq t, \mu_{\varepsilon,1} \geq \tau_{\varepsilon,1}\}, t \geq 0$ are the corresponding characteristics related to the transition period.

We admit that the transition period can be of zero duration and, thus, the distribution function $\tilde{F}_\varepsilon(t)$ can possess an atom in zero or even be concentrated at zero, for $\varepsilon \in [0, 1]$.

Let us additionally assume that the following condition holds:

E: $\tilde{F}_\varepsilon(\cdot) \Rightarrow \tilde{F}_0(\cdot)$ as $\varepsilon \to 0$, where $\tilde{F}_0(t)$ is a proper distribution function.

Let also $\tilde{f}_\varepsilon = \mathsf{P}\{\mu_{\varepsilon,1} < \tau_{\varepsilon,1}\} = 1 - \tilde{F}_\varepsilon(\infty)$. Condition **E** obviously implies that the stopping probabilities for transition period, $\tilde{f}_\varepsilon \to \tilde{f}_0 = 0$ as $\varepsilon \to 0$.

It is also useful to note that $\tilde{q}_\varepsilon(t, A) \leq \mathsf{P}\{\tau_{\varepsilon,1} \wedge \mu_{\varepsilon,1} > t\} = \mathsf{P}\{\tau_{\varepsilon,1} > t, \mu_{\varepsilon,1} \geq \tau_{\varepsilon,1}\} + \mathsf{P}\{\tau_{\varepsilon,1} \wedge \mu_{\varepsilon,1} > t, \mu_{\varepsilon,1} < \tau_{\varepsilon,1}\} \leq \mathsf{P}\{\mu_{\varepsilon,1} \geq \tau_{\varepsilon,1}\} - \mathsf{P}\{\tau_{\varepsilon,1} \leq t, \mu_{\varepsilon,1} \geq \tau_{\varepsilon,1}\} + \mathsf{P}\{\mu_{\varepsilon,1} < \tau_{\varepsilon,1}\} = \tilde{F}_\varepsilon(\infty) - \tilde{F}_\varepsilon(t) + \tilde{f}_\varepsilon$. This relation and condition **E** imply that $\tilde{q}_\varepsilon(t_\varepsilon, A) \to 0$ as $\varepsilon \to 0$, for any $0 \leq t_\varepsilon \to \infty$ as $\varepsilon \to 0$.

The following quasi-ergodic theorem for perturbed regenerative processes with transition period, is also given in book [14].

Theorem 2. *Let conditions* **A** – **E** *hold. Then, for every* $A \in \Gamma$, *and any* $0 \leq t_\varepsilon \to \infty$ *as* $\varepsilon \to 0$ *such that* $f_\varepsilon t_\varepsilon \to t \in [0, \infty]$ *as* $\varepsilon \to 0$,

$$P_\varepsilon(t_\varepsilon, A) \to e^{-t/e_0} \pi_0(A) \text{ as } \varepsilon \to 0. \tag{9}$$

In the case of standard regenerative processes, Theorem 2 just reduces to Theorem 1. Indeed, condition **E** can be omitted since it is implied by condition **B**. The ergodic relation (9) reduces to the ergodic relation (7).

Let us also introduce modified regenerative lifetimes $\mu_{\varepsilon,-} = \sum_{k=1}^{\nu_\varepsilon - 1} \kappa_{\varepsilon,k}$, and $\mu_{\varepsilon,+} = \sum_{k=1}^{\nu_\varepsilon} \kappa_{\varepsilon,k}$ and consider probabilities $P_{\varepsilon,\pm}(t, A) = \mathsf{P}\{\xi_\varepsilon(t) \in A, \mu_{\varepsilon,\pm} > t\}, A \in \mathcal{B}_{\mathbb{X}}, t \geq 0$.

Obviously, $\mu_{\varepsilon,-} \leq \mu_\varepsilon \leq \mu_{\varepsilon,+}$ and, thus, $P_{\varepsilon,-}(t, A) \leq P_\varepsilon(t, A) \leq P_{\varepsilon,+}(t, A)$, for any $A \in \mathcal{B}_{\mathbb{X}}, t \geq 0$.

The following theorem is a useful modification of Theorem 2.

Theorem 3. *Let conditions* **A** – **E** *hold. Then, for every* $A \in \Gamma$, *and any* $0 \leq t_\varepsilon \to \infty$ *as* $\varepsilon \to 0$ *such that* $f_\varepsilon t_\varepsilon \to t \in [0, \infty]$ *as* $\varepsilon \to 0$,

$$P_{\varepsilon,\pm}(t_\varepsilon, A) \to e^{-t/e_0}\pi_0(A) \text{ as } \varepsilon \to 0. \tag{10}$$

Proof. Conditions **A** – **C** imply that the asymptotic relation, $f_\varepsilon \kappa_{\varepsilon,\nu_\varepsilon} \mathrm{I}(\nu_\varepsilon < \infty) \xrightarrow{P} 0$ as $\varepsilon \to 0$, holds. The asymptotic relation (10) is an obvious corollary of this asymptotic relation and the ergodic relation (9) given in Theorem 2. □

2.2. One- and multi-dimensional distributions for perturbed regenerative processes. Individual ergodic theorems formulated in Theorems 1 – 3 present ergodic relations for one-dimensional distributions $P_\varepsilon(t, A) = \mathsf{P}\{\xi_\varepsilon(t) \in A, \mu_\varepsilon > t\}$ for regenerative processes with regenerative lifetimes.

It possible to weaken the model assumption **(e)** formulated in Subsection 2.1. This assumption concerns multi-dimensional joint distributions of random variables $\xi_{\varepsilon,n}(t_k), k = 1, \ldots, r, \kappa_{\varepsilon,n}$ and $\mu_{\varepsilon,n}$.

It can be replaced by the weaker assumption that the joint distributions of random variables $\xi_{\varepsilon,n}(t), \kappa_{\varepsilon,n}$ and $\mu_{\varepsilon,n}$ do not depend on $n \geq 1$, for every $t \geq 0$.

The process $\xi_\varepsilon(t), t \geq 0$ will still process the corresponding weaken, say, one-dimensional regenerative property, which, in fact, means that one-dimensional distributions $P_\varepsilon(t, A) = \mathsf{P}\{\xi_\varepsilon(t) \in A, \mu_\varepsilon > t\}, t \geq 0$ satisfy the renewal equations (4).

Formulations of conditions **A** – **E** as well as propositions of Theorems 1 – 3 still remain to be valid.

2.3. Ergodic theorems for standard regenerative processes. We would like to mention the important case, where stopping probability $f_\varepsilon = 0, \varepsilon \in [0, 1]$. In this case, the regenerative stopping time $\mu_\varepsilon = \infty$ with probability 1. Also, $f_\varepsilon t_\varepsilon \to 0$ as $\varepsilon \to 0$, for any $0 \leq t_\varepsilon \to \infty$ as $\varepsilon \to 0$.

Probability $P_\varepsilon(t, A) = \mathsf{P}\{\xi_\varepsilon(t) \in A\}$ is a one-dimensional distribution for process $\xi_\varepsilon(t)$. Theorems 1 – 3 present in this case usual individual ergodic theorems for perturbed regenerative processes $\xi_\varepsilon(t)$.

It is also worth to mention the case of unperturbed regenerative process $\xi_0(t), t \geq 0$. Conditions $\mathbf{A} - \mathbf{D}$ reduce in this case to the the minimal conditions of the individual ergodic theorem for regenerative processes, which directly follows from the renewal theorem given in its final form in [12]: (a) $F_0(\cdot)$ is a non-arithmetic distribution function without an atom in zero; (b) $e_0 = \int_0^\infty sF_0(ds) < \infty$; (c) function $q_0(s, A), s \geq 0$ is, for $A \in \Gamma$, continuous almost everywhere with respect to the Lebesgue measure $m(ds)$ on $[0, \infty)$.

Note that $q_0(s, A) \leq 1 - F_0(s), s \geq 0$ and, thus, under the above condition (b), condition (c) is equivalent to the assumption of direct Riemann integrability of the free term in the renewal equation (4), imposed on this term in the renewal theorem given in [12].

Also, condition $\mathbf{E}$ just reduces to the assumption that (d) $\tilde{F}_0(\cdot)$ is a proper distribution function.

The corresponding individual ergodic theorem takes in this case the form of the asymptotic relation (5), i.e., $P_0(t, A) \to \pi_0(A)$ as $t \to \infty$, for $A \in \Gamma$.

2.4. Perturbed alternating regenerative processes. Let, for every $i = 1, 2, n = 1, 2, \ldots$: **(f)** $\bar{\xi}_{\varepsilon,i,n} = \langle \xi_{\varepsilon,i,n}(t), t \geq 0\rangle$ be a measurable stochastic process with a phase space $\mathbb{X}$; **(g)** $\kappa_{\varepsilon,i,n}$ be a non-negative random variable; **(h)** $\eta_{\varepsilon,i,n}$ and η_ε are binary random variables taking values in the space $\mathbb{Y} = \{1, 2\}$. Further, we assume that: **(i)** triplets $\langle \bar{\xi}_{\varepsilon,i,n} = \langle \xi_{\varepsilon,i,n}(t), t \geq 0\rangle, \kappa_{\varepsilon,i,n}, \eta_{\varepsilon,i,n}\rangle, i = 1, 2, n = 1, 2, \ldots$ and the random variable η_ε are mutually independent; **(j)** the joint distributions of random variables $\xi_{\varepsilon,i,n}(t_k), k = 1, \ldots, r$ and $\kappa_{\varepsilon,i,n}, \eta_{\varepsilon,i,n}$ do not depend on $n \geq 1$, for every $i = 1, 2$ and $t_k \in [0, \infty), k = 1, \ldots, r, r \geq 1$.

Here, the measurability assumption for processes $\bar{\xi}_{\varepsilon,i,n}$ is absolutely analogous to those formulated in the model assumption **(a)** for processes $\bar{\xi}_{\varepsilon,n}$.

Let us define recurrently stochastic sequences of switching binary random indices $\eta_{\varepsilon,n}, n = 0, 1, \ldots$ and regeneration times $\tau_{\varepsilon,n}, n = 0, 1, \ldots$ by the following recurrent relations, $\eta_{\varepsilon,n} = \eta_{\varepsilon,\eta_{\varepsilon,n-1},n}, n = 1, 2, \ldots, \eta_{\varepsilon,0} = \eta_\varepsilon$ and $\tau_{\varepsilon,n} = \kappa_{\varepsilon,\eta_{\varepsilon,0},1} + \cdots + \kappa_{\varepsilon,\eta_{\varepsilon,n-1},n}, n = 1, 2, \ldots, \tau_{\varepsilon,0} = 0$, and the modulated alternating regenerative process $(\xi_\varepsilon(t), \eta_\varepsilon(t)), t \geq 0$ by the following recurrent relations,

$$\begin{aligned} &\xi_\varepsilon(t) = \xi_{\varepsilon,\eta_{\varepsilon,n-1},n}(t - \tau_{\varepsilon,n-1}) \text{ and } \eta_\varepsilon(t) = \eta_{\varepsilon,n-1}, \\ &\qquad \text{for } t \in [\tau_{\varepsilon,n-1}, \tau_{\varepsilon,n}), n = 1, 2, \ldots. \end{aligned} \tag{11}$$

We exclude instant regenerations and, thus, assume that the following condition holds:

F: $\mathsf{P}\{\kappa_{\varepsilon,i,1} > 0\} = 1, i = 1, 2$, for every $\varepsilon \in [0, 1]$.

This condition obviously implies that $\tau_{\varepsilon,n} \stackrel{\mathsf{P}}{\longrightarrow} \infty$ as $n \to \infty$, for every $\varepsilon \in [0, 1]$, and thus, the above alternating regenerative process is well defined on the time interval $[0, \infty)$.

Now, let us formulate conditions, which make it possible to consider $(\xi_0(t), \eta_0(t)), t \geq 0$ as an unperturbed process and $(\xi_\varepsilon(t), \eta_\varepsilon(t)), t \geq 0$ as its perturbed version, for $\varepsilon \in (0, 1]$.

The above model assumptions **(f)** – **(j)** imply that the modulating index sequence $\eta_{\varepsilon,n}, n = 0, 1, \dots$ is a homogeneous Markov chain with the phase space $\mathbb{Y} = \{1, 2\}$, the initial distribution $\bar{p}_\varepsilon = \langle p_{\varepsilon,i} = \mathsf{P}\{\eta_{\varepsilon,0} = i\}, i = 1, 2\rangle$, and transition probabilities, $p_{\varepsilon,ij} = \mathsf{P}\{\eta_{\varepsilon,1} = j/\eta_{\varepsilon,0} = i\} = \mathsf{P}\{\eta_{\varepsilon,i,1} = j\},\ i, j = 1, 2$. We assume that the following condition holds:

G: **(a)** $p_{\varepsilon,ij} = 0, \varepsilon \in (0, 1]$ or $p_{\varepsilon,ij} > 0, \varepsilon \in (0, 1]$, for $i, j = 1, 2$; **(b)** $p_{\varepsilon,ij} \to p_{0,ij}$ as $\varepsilon \to 0$, for $i, j = 1, 2$.

The above model assumptions **(f)** – **(j)** also imply that the modulating index process $\eta_\varepsilon(t), t \geq 0$ is a semi-Markov process with the phase space $\mathbb{Y}$ and transition probabilities, $Q_{\varepsilon,ij}(t) = \mathsf{P}\{\tau_{\varepsilon,1} \leq t, \eta_{\varepsilon,1} = j/\eta_{\varepsilon,0} = i\} = \mathsf{P}\{\kappa_{\varepsilon,i,1} \leq t, \eta_{\varepsilon,i,1} = j\},\ t \geq 0, i, j = 1, 2$. Also, let us introduce conditional distribution functions $F_{\varepsilon,ij}(t) = Q_{\varepsilon,ij}(t)/p_{\varepsilon,ij}, t \geq 0$ defined for $i, j \in \mathbb{Y}$ such that $p_{\varepsilon,ij} > 0, \varepsilon \in (0, 1]$.

We also assume that the following condition holds:

H: **(a)** $Q_{\varepsilon,ij}(\cdot) \Rightarrow Q_{0,ij}(\cdot)$ as $\varepsilon \to 0$, for $i, j = 1, 2$, **(b)** $Q_{0,ij}(t) = 0, t \geq 0$ if $p_{0,ij} = 0$ or $F_{0,ij}(t) = Q_{0,ij}(t)/p_{0,ij}, t \geq 0$ is a non-arithmetic distribution function if $p_{0,ij} > 0$.

Remark 1. Conditions of convergence **G (b)** and **H (a)** can be reformulated in terms of Laplace transforms $\phi_{\varepsilon,ij}(s) = \int_0^\infty e^{-st} Q_{\varepsilon,ij}(dt), s \geq 0, i, j = 1, 2$. These conditions are equivalent to the assumption that $\phi_{\varepsilon,ij}(s) \to \phi_{0,ij}(s)$ as $\varepsilon \to 0$, for $s \geq 0$ and $i, j = 1, 2$.

Let us introduce expectations, $e_{\varepsilon,ij} = \mathsf{E}_i \tau_{\varepsilon,1} \mathrm{I}(\eta_{\varepsilon,1} = j) = \mathsf{E}\kappa_{\varepsilon,i,1} \mathrm{I}(\eta_{\varepsilon,i,1} = j)$ $= \int_0^\infty s Q_{\varepsilon,ij}(ds),\ i, j = 1, 2$ and $e_{\varepsilon,i} = \mathsf{E}_i \tau_{\varepsilon,1} = \mathsf{E}\kappa_{\varepsilon,i,1} = e_{\varepsilon,i1} + e_{\varepsilon,i2}, i = 1, 2$.

Here and henceforth, we use notations P_i and E_i for conditional probabilities and expectations under condition $\eta_\varepsilon(0) = \eta_\varepsilon = i$.

We also impose the following condition of convergence for the above expectations:

I: **(a)** $e_{\varepsilon,ij} < \infty$, for every $\varepsilon \in [0, 1]$ and $i = 1, 2$; **(b)** $e_{\varepsilon,ij} \to e_{0,ij}$ as $\varepsilon \to 0$, for $i = 1, 2$.

The object of our interest is the joint distributions,

$$P_{\varepsilon,ij}(t, A) = \mathsf{P}_i\{\xi_\varepsilon(t) \in A, \eta_\varepsilon(t) = j\}, A \in \mathcal{B}_{\mathbb{X}}, i, j = 1, 2, t \geq 0. \quad (12)$$

Probabilities $P_{\varepsilon,ij}(t,A)$ are, for every $A \in \mathcal{B}_{\mathbb{X}}, j = 1,2$, a measurable functions of $t \geq 0$, which are the unique bounded solution for the following system of renewal type equations,

$$
\begin{aligned}
P_{\varepsilon,ij}(t,A) &= \delta(i,j) q_{\varepsilon,i}(t,A) \\
&\quad + \sum_{k=1}^{2} \int_0^t P_{\varepsilon,kj}(t-s,A) Q_{\varepsilon,ik}(ds), \ t \geq 0, i = 1,2. \qquad (13)
\end{aligned}
$$

where $q_{\varepsilon,i}(t,A) = \mathsf{P}_i\{\xi_\varepsilon(t) \in A, \eta_\varepsilon(t) = i, \tau_{\varepsilon,1} > t\} = \mathsf{P}\{\xi_{\varepsilon,i,1}(t) \in A, \kappa_{\varepsilon,i,1} > t\}$, $A \in \mathcal{B}_{\mathbb{X}}$, $t \geq 0, i,j = 1,2$.

Finally, we also impose the following condition on functions $q_{\varepsilon,i}(t,A)$:

- **J**: There exists a non-empty class of sets $\Gamma \subseteq \mathcal{B}_{\mathbb{X}}$ such that, for every $A \in \Gamma$, the asymptotic relations, $\lim_{u\to 0} \overline{\lim}_{0 \leq \varepsilon \to 0} \sup_{-(u\wedge s) \leq v \leq u} |q_{\varepsilon,i}(s+v,A) - q_{0,i}(s,A)| = 0$, $i = 1,2$, hold almost everywhere with respect to the Lebesgue measure $m(ds)$ on $[0,\infty)$.

As for condition **D**, the class Γ appearing in condition **J** contains the phase space $\mathbb{X}$ and is closed with respect to the operation of union for not intersecting sets, the operation of difference for sets connected by relation of inclusion, and the complement operation. The corresponding comments are given below, in Subsection 2.5.

Let us also consider, for $i = 1,2$, the standard regenerative process $\xi_{\varepsilon,i}(t), t \geq 0$ with regeneration times $\tau_{\varepsilon,i,n} = \kappa_{\varepsilon,i,1} + \cdots + \kappa_{\varepsilon,i,n}, n = 1,2,\ldots$, $\tau_{\varepsilon,i,0} = 0$, defined by the following recurrent relations, $\xi_{\varepsilon,i}(t) = \xi_{\varepsilon,i,n}(t - \tau_{\varepsilon,i,n-1})$, for $t \in [\tau_{\varepsilon,i,n-1}, \tau_{\varepsilon,in}), n = 1,2,\ldots$.

Conditions **F** – **J** imply that, for every $i = 1,2$, all conditions of Theorem 1 hold for regenerative process $\xi_{\varepsilon,i}(t)$, with the corresponding stopping probabilities $f_{\varepsilon,i} = 0, \varepsilon \in [0,1]$.

Thus, for every $i = 1,2$, the following ergodic relation holds, for $A \in \Gamma$ and any $0 \leq t_\varepsilon \to \infty$ as $\varepsilon \to 0$,

$$
\mathsf{P}\{\xi_{\varepsilon,i}(t_\varepsilon) \in A\} \to \pi_{0,i}(A) \text{ as } \varepsilon \to 0, \qquad (14)
$$

where the probabilities $\pi_{0,i}(A)$ are corresponding stationary probabilities for the regenerative process $\xi_{0,i}(t)$ given by the following relation,

$$
\pi_{0,i}(A) = \frac{1}{e_{0,i}} \int_0^\infty q_{0,i}(s,A) m(ds), \ A \in \mathcal{B}_{\mathbb{X}}. \qquad (15)
$$

2.5. Stricture of class Γ. Note that functions $q_\varepsilon(s,A)$ and $q_{\varepsilon,i}(s,A)$ appearing, respectively, in conditions **D** and **J** are finite measures as functions of $A \in \mathcal{B}_{\mathbb{X}}$.

This, in obvious way, implies that the class Γ appearing in condition **D** or **J** is closed with respect to the operation of union for non-intersecting sets, i.e., if the convergence relation given in condition **D** or **J** holds for sets A' and A'' such that $A' \cap A'' = \emptyset$, then this relation also holds for set $A = A' \cup A''$.

The class Γ appearing in condition **D** or **J** also is closed with respect to the operation of differences for sets connected by relation of inclusion, i.e., if the convergence relation given in condition **D** or **J** holds for sets A' and A'' such that $A' \subseteq A''$, then this relation also holds for set $A = A'' \setminus A'$.

Also, the class of sets Γ appearing in condition **D** or **J** includes the phase space $\mathbb{X}$ under assumption that, respectively, condition **B** holds or conditions **G** and **H** hold.

Let us check this, for example, for the case of condition **D**. Indeed, $q_\varepsilon(t, \mathbb{X}) = \mathsf{P}\{\tau_{\varepsilon,1} \wedge \mu_{\varepsilon,1} > t\} = \mathsf{P}\{\tau_{\varepsilon,1} > t, \mu_{\varepsilon,1} \geq \tau_{\varepsilon,1}\} + \mathsf{P}\{\tau_{\varepsilon,1} \wedge \mu_{\varepsilon,1} > t, \mu_{\varepsilon,1} < \tau_{\varepsilon,1}\}$. Probability $\mathsf{P}\{\tau_{\varepsilon,1} > t, \mu_{\varepsilon,1} \geq \tau_{\varepsilon,1}\} = \mathsf{P}\{\mu_{\varepsilon,1} \geq \tau_{\varepsilon,1}\} - \mathsf{P}\{\tau_{\varepsilon,1} \leq t, \mu_{\varepsilon,1} \geq \tau_{\varepsilon,1}\} = F_\varepsilon(\infty) - F_\varepsilon(t)$. Condition **B** implies that $F_\varepsilon(\infty) - F_\varepsilon(t_\varepsilon) \to 1 - F_0(t)$ as $\varepsilon \to 0$, for any $t_\varepsilon \to t$ as $\varepsilon \to 0$ and $t \in \mathbb{C}(F_0)$, where $\mathbb{C}(F_0)$ is the set of continuity points for the distribution function $F_0(\cdot)$. Also, $\mathsf{P}\{\tau_{\varepsilon,1} \wedge \mu_{\varepsilon,1} > t_\varepsilon, \mu_{\varepsilon,1} < \tau_{\varepsilon,1}\} \leq \mathsf{P}\{\mu_{\varepsilon,1} < \tau_{\varepsilon,1}\} = f_\varepsilon \to 0$ as $\varepsilon \to 0$. The above relations imply that $q_\varepsilon(t_\varepsilon, \mathbb{X}) \to q_0(t, \mathbb{X}) = 1 - F_0(t)$ as $\varepsilon \to 0$, for any $t_\varepsilon \to t \in \mathbb{C}(F_0)$ as $\varepsilon \to 0$. Since $\overline{\mathbb{C}}(F_0) = [0, \infty) \setminus \mathbb{C}(F_0)$ is at most a countable set, $m(\overline{\mathbb{C}}(F_0)) = 0$. These relations imply, by Lemma 1 given in Subsection 4.3, that the asymptotic relation appearing in condition **D** holds for $\Gamma = \mathbb{X}$.

Finally, the above remarks imply that class Γ appearing in condition **D** or **J** is also closed with respect to the complement operation, i.e., if the convergence relation given in condition **D** holds for set A, then it also holds for set $\overline{A}$.

Let us, for example, consider the model, where the phase space $\mathbb{X} = \{1, 2, \ldots, m\}$ is a finite set and $\mathcal{B}_\mathbb{X}$ is the σ-algebra of all subsets of $\mathbb{X}$. In this case, it is natural to assume that the corresponding locally uniform convergence relation appearing in condition **J** holds for all one-point sets $A = \{j\}, j \in \mathbb{X}$. This will obviously imply that this convergence relation also holds for any subset $A \subseteq \mathbb{X}$ that means that, in this case, class $\Gamma = \mathcal{B}_\mathbb{X}$.

2.6. Regularly, singularly and super-singularly perturbed alternating regenerative processes. The aim of the present paper is to give a detailed analysis of individual ergodic theorems for probabilities $P_{\varepsilon,ij}(t, A)$ that is to describe possible variants of their asymptotic behaviour as $t \to \infty$ and $\varepsilon \to 0$.

We shall see that the asymptotic behaviour of transition probabilities $p_{\varepsilon,ij}, i, j = 1, 2$ for the Markov chains $\eta_{\varepsilon,n}$ plays an important role in these ergodic theorems. Note that, according to condition **G**, these transition probabilities converge to the corresponding transition probabilities $p_{0,ij}, i, j = 1, 2$ of the Markov chain $\eta_{0,n}$, as $\varepsilon \to 0$.

There are three classes of perturbed alternating regenerative processes, with essentially different ergodic properties.

The first class includes so-called "regularly" perturbed alternating regenerative processes, for which the limiting Markov chain $\eta_{0,n}$ is ergodic that, in this case, is equivalent to the assumption that at least one of its transition probabilities $p_{0,12}$ and $p_{0,21}$ is positive.

Here, parameter $\beta = p_{0,12}/p_{0,21}$ plays the key role. Obviously, **(a)** $\beta \in (0, \infty)$, if $p_{0,12}, p_{0,21} > 0$, **(b)** $\beta = 0$, if $p_{0,12} = 0, p_{0,21} > 0$, and **(c)** we should count $\beta = \infty$, if $p_{0,12} > 0, p_{0,21} = 0$. In case **(a)**, the phase space $\mathbb{Y}$ is one class of communicative states and the corresponding stationary probabilities $\alpha_1(\beta) = \frac{p_{0,21}}{p_{0,12}+p_{0,21}} = \frac{1}{1+\beta}$ and $\alpha_2(\beta) = \frac{p_{0,12}}{p_{0,12}+p_{0,21}} = \frac{1}{1+\beta^{-1}}$. In case **(b)**, the phase space $\mathbb{Y}$ consists of the absorbing state 1 and the transient state 2. In this case, $\alpha_1(0) = 1$ and $\alpha_2(0) = 0$. Analogously, in case **(c)**, the phase space $\mathbb{Y}$ consists of the absorbing state 2 and the transient state 1. In this case $\alpha_1(\infty) = 0$ and $\alpha_2(\infty) = 1$.

In ergodic theorems for perturbed alternating regenerative processes, the asymptotic stability of stationary probabilities for Markov chains $\eta_{\varepsilon,n}$ play the key role. In the case of regularly perturbed models, condition **G** obviously implies that the Markov chain $\eta_{\varepsilon,n}$ is ergodic, for every $\varepsilon \in [0, 1]$. Its stationary probabilities are determined by parameter $\beta_\varepsilon = p_{\varepsilon,12}/p_{\varepsilon,21}$, namely, $\alpha_1(\beta_\varepsilon) = \frac{p_{\varepsilon,21}}{p_{\varepsilon,12}+p_{\varepsilon,21}} = \frac{1}{1+\beta_\varepsilon}$ and $\alpha_2(\beta_\varepsilon) = \frac{p_{\varepsilon,12}}{p_{\varepsilon,12}+p_{\varepsilon,21}} = \frac{1}{1+\beta_\varepsilon^{-1}}$. Condition **G** implies that $\beta_\varepsilon \to \beta$ as $\varepsilon \to 0$ and, in sequel, $\alpha_1(\beta_\varepsilon) \to \alpha_1(\beta)$ and $\alpha_2(\beta_\varepsilon) \to \alpha_2(\beta)$ as $\varepsilon \to 0$.

We shall see that ergodic theorems for regularly perturbed alternating processes have a form of asymptotic relation, $P_{\varepsilon,ij}(t_\varepsilon, A) \to \pi^{(\beta)}_{0,j}(A)$ as $\varepsilon \to 0$, which holds for $A \in \Gamma, i, j = 1, 2$ and any $0 \leq t_\varepsilon \to \infty$ as $\varepsilon \to 0$,

The limiting probabilities $\pi^{(\beta)}_{0,j}(A)$ depend on parameter $\beta \in [0, \infty]$, but they do not depend on an initial state $i \in \mathbb{Y}$. The forms of ergodic theorems are analogous to those, which are known for unperturbed alternating regenerative processes.

The second and the third classes include so-called "singularly" and "super-singularly" perturbed alternating regenerative processes, for which the limiting Markov chain $\eta_{0,n}$ is not ergodic that is equivalent to the assumption that both transition probabilities $p_{0,12}$ and $p_{0,21}$ equal 0.

According condition **G**, four cases are possible. The case **(d)** $0 < p_{\varepsilon,12}, p_{\varepsilon,21} \to 0$ as $\varepsilon \to 0$, corresponds to singularly perturbed alternating regenerative processes. Three cases, where **(e)** $p_{\varepsilon,12} = 0, \varepsilon \in [0, 1]$ and $0 < p_{\varepsilon,21} \to 0$ as $\varepsilon \to 0$, or **(f)** $0 < p_{\varepsilon,12} \to 0$ as $\varepsilon \to 0$ and $p_{\varepsilon,21} = 0, \varepsilon \in [0, 1]$, or **(g)**

$p_{\varepsilon,12}, p_{\varepsilon,21} = 0, \varepsilon \in [0,1]$, correspond to super-singularly perturbed alternating regenerative processes.

In case **(d)**, the asymptotic stability for stationary probabilities $\alpha_j(\beta_\varepsilon), j = 1,2$ is provided by the following additional condition that should be assumed to hold for some $\beta \in [0,\infty]$:

$\mathbf{K}_\beta$: $\beta_\varepsilon = p_{\varepsilon,12}/p_{\varepsilon,21} \to \beta$ as $\varepsilon \to 0$.

Condition $\mathbf{G}$ implies that the Markov chain $\eta_{\varepsilon,n}$ is ergodic, for $\varepsilon \in (0,1]$. Its stationary probabilities are determined by parameter $\beta_\varepsilon = p_{\varepsilon,12}/p_{\varepsilon,21}$, namely, $\alpha_1(\beta_\varepsilon) = \frac{p_{\varepsilon,21}}{p_{\varepsilon,12}+p_{\varepsilon,21}} = \frac{1}{1+\beta_\varepsilon}$ and $\alpha_2(\beta_\varepsilon) = \frac{p_{\varepsilon,12}}{p_{\varepsilon,12}+p_{\varepsilon,21}} = \frac{1}{1+\beta_\varepsilon^{-1}}$. Conditions $\mathbf{G}$ and $\mathbf{K}_\beta$ imply that $\beta_\varepsilon \to \beta$ and, in sequel, $\alpha_1(\beta_\varepsilon) \to \alpha_1(\beta) = \frac{1}{1+\beta}$ and $\alpha_2(\beta_\varepsilon) \to \alpha_2(\beta) = \frac{1}{1+\beta^{-1}}$ as $\varepsilon \to 0$.

In case **(e)**, $\beta_\varepsilon = p_{\varepsilon,12}/p_{\varepsilon,21} = 0$, for $\varepsilon \in [0,1]$, and, thus, condition $\mathbf{K}_0$ holds. Condition $\mathbf{G}$ implies that the Markov chain $\eta_{\varepsilon,n}$ is ergodic, for $\varepsilon \in (0,1]$ and its stationary probabilities $\alpha_{\varepsilon,1}(0) = 1, \alpha_{\varepsilon,2}(0) = 0$, for $\varepsilon \in (0,1]$. Obviously, relations $\alpha_{\varepsilon,1}(0) \to \alpha_1(0) = 1$ and $\alpha_{\varepsilon,2}(0) \to \alpha_2(0) = 0$ as $\varepsilon \to 0$ also hold. Analogously, In the case **(f)**, $\beta_\varepsilon = p_{\varepsilon,12}/p_{\varepsilon,21} = \infty$, for $\varepsilon \in [0,1]$, and, thus, condition $\mathbf{K}_\infty$ holds. Condition $\mathbf{G}$ implies that the Markov chain $\eta_{\varepsilon,n}$ is ergodic, for $\varepsilon \in (0,1]$ and its stationary probabilities $\alpha_{\varepsilon,1}(\infty) = 0, \alpha_{\varepsilon,2}(\infty) = 1$, for $\varepsilon \in (0,1]$. Obviously, relations $\alpha_{\varepsilon,1}(\infty) \to \alpha_1(\infty) = 0$ and $\alpha_{\varepsilon,2}(\infty) \to \alpha_2(\infty) = 1$ as $\varepsilon \to 0$ also hold.

Ergodic theorems for singularly and super-singularly perturbed alternating processes have much more complex and interesting forms than for regularly perturbed alternating regenerative processes.

Functions $v_\varepsilon = p_{\varepsilon,12}^{-1} + p_{\varepsilon,21}^{-1}, \varepsilon \in (0,1]$ and $w_\varepsilon = (p_{\varepsilon,12} + p_{\varepsilon,21})^{-1}, \varepsilon \in (0,1]$ play important roles of so-called "time scaling" factor, respectively, for singularly and super-singularly perturbed models. In the case **(d)**, $0 < w_\varepsilon < v_\varepsilon < \infty$, for $\varepsilon \in (0,1]$ and $w_\varepsilon, v_\varepsilon \to \infty$ as $\varepsilon \to 0$. In the cases **(e)** and **(f)**, $0 < w_\varepsilon < v_\varepsilon = \infty$, for $\varepsilon \in (0,1]$ and $w_\varepsilon \to \infty$ as $\varepsilon \to 0$.

The main individual ergodic theorems for singularly perturbed alternating regenerative processes have forms of asymptotic relations, $P_{\varepsilon,ij}(t_\varepsilon, A) \to \pi_{0,ij}^{(\beta)}(t, A)$ as $\varepsilon \to 0$, holding under assumption that condition $\mathbf{K}_\beta$ holds for some $\beta \in [0,\infty]$, for $A \in \Gamma, i,j = 1,2$, and any $0 \leq t_\varepsilon \to \infty$ as $\varepsilon \to 0$ such that $t_\varepsilon/v_\varepsilon \to t \in [0,\infty]$ as $\varepsilon \to 0$.

The asymptotic behaviour of probabilities $P_{\varepsilon,ij}(t_\varepsilon, A)$ can differ for different asymptotic time zones determined by the asymptotic relation $t_\varepsilon/v_\varepsilon \to t \in [0,\infty]$. The corresponding limiting probabilities $\pi_{0,ij}^{(\beta)}(t, A)$ may depend on $t \in [0,\infty]$, parameter $\beta \in 0,\infty]$, appearing in condition $\mathbf{K}_\beta$, and, also, on the initial state $i \in \mathbb{Y}$, if $t \in [0,\infty)$. It is natural to classify the corresponding theorems as super-long, long and short time ergodic theorem, respectively, for cases $t = \infty, t \in (0, \infty)$ and $t = 0$, for which the corresponding limiting probabilities take different analytical forms.

The corresponding ergodic theorems for super-singularly perturbed alternating regenerative processes have forms of analogous asymptotic relations, $P_{\varepsilon,ij}(t_\varepsilon, A) \to \dot{\pi}^{(\beta)}_{0,ij}(t, A)$ as $\varepsilon \to 0$, holding under assumption that condition $\mathbf{K}_0$ or $\mathbf{K}_\infty$ holds, for $A \in \Gamma, i, j = 1, 2$ and any $0 \leq t_\varepsilon \to \infty$ as $\varepsilon \to 0$ such that $t_\varepsilon / w_\varepsilon \to t \in [0, \infty]$ as $\varepsilon \to 0$.

In this case, the asymptotic behaviour of probabilities $P_{\varepsilon,ij}(t_\varepsilon, A)$ also can differ for different asymptotic time zones determined by the asymptotic relation $t_\varepsilon / w_\varepsilon \to t \in [0, \infty]$. The corresponding limiting probabilities $\dot{\pi}^{(\beta)}_{0,ij}(t, A)$ may depend on $t \in [0, \infty]$, parameter β, taking in this case one of two values 0 or ∞, and, also, on the initial state $i \in \mathbb{Y}$, if $t \in [0, \infty)$. As for singularly perturbed models, it is natural to classify the corresponding theorems as super-long, long and short time ergodic theorem, respectively, for cases $t = \infty, t \in (0, \infty)$ and $t = 0$, for which the corresponding limiting probabilities take different analytical forms.

Ergodic theorems for singularly perturbed models for the cases, where condition $\mathbf{K}_0$ or $\mathbf{K}_\infty$ is assumed to hold, can be compared with ergodic theorems for super-singularly perturbed models, respectively, for the cases **(e)** or **(f)**. Indeed, as was mentioned above, condition $\mathbf{K}_0$ or $\mathbf{K}_\infty$ holds, respectively, in the case **(e)** or **(f)**.

In cases **(e)** and **(f)**, i.e., for super-singularly perturbed models, $v_\varepsilon = \infty$, while $0 < w_\varepsilon < \infty$, for $\varepsilon \in (0, 1]$. The only factor w_ε can be used as a time scaling factor. In the case **(d)**, i.e., for singularly perturbed models, $0 < w_\varepsilon < v_\varepsilon < \infty$, for $\varepsilon \in (0, 1]$. The question arises if w_ε can be used as a time scaling factor instead of v_ε. The answer is in some sense affirmative, if condition $\mathbf{K}_\beta$ holds for some $\beta \in (0, \infty)$. Indeed, in this case, $w_\varepsilon / v_\varepsilon \to \beta(1+\beta)^{-2} \in (0, \infty)$ as $\varepsilon \to 0$. The asymptotic relations, $t_\varepsilon / v_\varepsilon \to t$ as $\varepsilon \to 0$, and, $t_\varepsilon / w_\varepsilon \to t$ as $\varepsilon \to 0$, generate, in fact, in some sense equivalent asymptotic time zones. However, the answer for the above question is negative, if condition $\mathbf{K}_0$ or $\mathbf{K}_\infty$ holds. Indeed, in this case, $w_\varepsilon / v_\varepsilon \to 0$ as $\varepsilon \to 0$. The asymptotic relations, $t_\varepsilon / v_\varepsilon \to t$ as $\varepsilon \to 0$, and, $t_\varepsilon / w_\varepsilon \to t$ as $\varepsilon \to 0$, generate essentially different asymptotic time zones, in the corresponding ergodic theorems. This, actually, makes it possible to get, under the assumption that condition $\mathbf{K}_0$ or $\mathbf{K}_\infty$ holds, additional ergodic relations for singularly perturbed processes, similar to those given above for super-singularly perturbed processes, for asymptotic time zones generated by relation $t_\varepsilon / w_\varepsilon \to t$ as $\varepsilon \to 0$,

The extremal case, **(g)** $p_{\varepsilon,12}, p_{\varepsilon,21} = 0, \varepsilon \in [0, 1]$, corresponds to absolutely singular perturbed alternating regenerative processes. This case is not covered by condition $\mathbf{K}_\beta$. However, in this case the modulating process $\eta_\varepsilon(t) = \eta_\varepsilon(0), t \geq 0$. Respectively, the process $\xi_\varepsilon(t), t \geq 0$ coincides with the standard regenerative process $\xi_{\varepsilon,i}(t), t \geq 0$, if $\eta_\varepsilon(0) = i$. The corresponding ergodic theorem for process $\xi_{\varepsilon,i}(t)$ is given by Theorem 1 for its particular case described in Subsection 2.3.

In conclusion, let us make some comments concerning ergodic theorems for probabilities $P_{\varepsilon,\bar{p}_\varepsilon,j}(t, A) = \mathsf{P}\{\xi_\varepsilon(t) \in A, \eta_\varepsilon(t) = j\} = p_{\varepsilon,1}P_{\varepsilon,1j}(t, A) + p_{\varepsilon,2}P_{\varepsilon,2j}(t, A)$.

In models, where the corresponding limits for probabilities $P_{\varepsilon,ij}(t_\varepsilon, A)$ do not depend of the initial state i, for example, for regularly perturbed alternating regenerative processes, probabilities $P_{\varepsilon,\bar{p}_\varepsilon,j}(t_\varepsilon, A)$ converge to the same limits for any initial distributions $\bar{p}_\varepsilon = \langle p_{\varepsilon,1}, p_{\varepsilon,2}\rangle$.

However, in models, where the corresponding limits for the probabilities $P_{\varepsilon,ij}(t_\varepsilon, A)$ may depend of the initial state i, for example, for some singularly or super-singularly perturbed alternating regenerative processes, probabilities $P_{\varepsilon,\bar{p}_\varepsilon,j}(t, A)$ converge to some limits under an additional condition of asymptotic stability for initial distributions:

L: $p_{\varepsilon,i} \to p_{0,i}$ as $\varepsilon \to 0$, for $i = 1, 2$.

If, for example, condition **L** holds and, $P_{\varepsilon,ij}(t_\varepsilon, A) \to \pi^{(\beta)}_{0,ij}(t, A)$ as $\varepsilon \to 0$, for $i = 1, 2$, then,

$$P_{\varepsilon,\bar{p}_\varepsilon,j}(t_\varepsilon, A) \to \pi^{(\beta)}_{0,\bar{p}_0,j}(t, A) = p_{0,1}\pi^{(\beta)}_{0,1j}(t, A) + p_{0,2}\pi^{(\beta)}_{0,2j}(t, A) \,\text{as}\, \varepsilon \to 0. \quad (16)$$

2.7. Aggregation of regeneration times. The alternating regenerative process $(\xi_\varepsilon(t), \eta_\varepsilon(t)), t \geq 0$ is a standard regenerative process with regeneration times $\tau_{\varepsilon,0}, \tau_{\varepsilon,1}, \tau_{\varepsilon,2}, \ldots$ if and only if the joint distributions of random variables $\xi_{\varepsilon,i,n}(t_k), k = 1, \ldots, r$ and $\kappa_{\varepsilon,i,n}, \eta_{\varepsilon,i,n}$ do not depend on $n \geq 1$, for every $t_k \in [0, \infty), k = 1, \ldots, r, r \geq, i = 1, 2$.

However, it is possible to construct new aggregated regeneration times such that the process $(\xi_\varepsilon(t), \eta_\varepsilon(t)), t \geq 0$ becomes a standard regenerative process with these new regeneration times.

Let us define stopping times for Markov chain $\eta_{\varepsilon,n}$ that are, $\hat{\theta}_\varepsilon[r] = \min(k > r : \eta_{\varepsilon,k} = \eta_{\varepsilon,r})$, which is the first after r return time to the state $\eta_{\varepsilon,r}$, $\tilde{\theta}_\varepsilon[r] = \min(k > r : \eta_{\varepsilon,k} \neq \eta_{\varepsilon,r})$, which is the first after r time of change of state $\eta_{\varepsilon,r}$, and $\check{\theta}_\varepsilon[r] = \min(k > \tilde{\theta}_\varepsilon[r] : \eta_{\varepsilon,k} = \eta_{\varepsilon,r})$, which is the first after $\tilde{\theta}_\varepsilon[r]$ return time to the state $\eta_{\varepsilon,r}$. Obviously, the above return times are connected by the inequality $r < \hat{\theta}_\varepsilon[r] < \check{\theta}_\varepsilon[r]$, for $r = 0, 1, \ldots$.

Let us also $\hat{\nu}_{\varepsilon,0} = 0, \hat{\nu}_{\varepsilon,n} = \hat{\theta}[\hat{\nu}_{\varepsilon,n-1}], n = 1, 2, \ldots$, and $\check{\nu}_{\varepsilon,0} = 0, \check{\nu}_{\varepsilon,n} = \check{\theta}[\check{\nu}_{\varepsilon,n-1}], n = 1, 2, \ldots$ be the corresponding sequential return times to the state $\eta_{\varepsilon,0}$ by the Markov chain $\eta_{\varepsilon,n}$.

Let also us consider sequential return times $\hat{\tau}_{\varepsilon,n} = \tau_{\varepsilon,\hat{\nu}_{\varepsilon,n}}, n = 0, 1, \ldots$ and $\check{\tau}_{\varepsilon,n} = \tau_{\varepsilon,\check{\nu}_{\varepsilon,n}}, n = 0, 1, \ldots$ to the state $\eta_\varepsilon(0)$ by the semi-Markov process $\eta_\varepsilon(t)$.

Process $(\xi_\varepsilon(t), \eta_\varepsilon(t)), t \geq 0$ is a regenerative process with regeneration times $\hat{\tau}_{\varepsilon,n}, n = 0, 1, \ldots$. It also is a regenerative process with regeneration times $\check{\tau}_{\varepsilon,n}, n = 0, 1, \ldots$.

We can also consider shifted sequences of stopping times $\hat{\nu}'_{\varepsilon,0} = 0, \hat{\nu}'_{\varepsilon,1} = \tilde{\theta}_\varepsilon[0], \hat{\nu}'_{\varepsilon,n} = \hat{\theta}[\hat{\nu}'_{\varepsilon,n-1}], n = 2, 3, \ldots$ and $\check{\nu}'_{\varepsilon,0} = 0, \check{\nu}'_{\varepsilon,1} = \tilde{\theta}_\varepsilon[0], \check{\nu}'_{\varepsilon,n} = \check{\theta}[\check{\nu}'_{\varepsilon,n-1}]$, $n = 2, 3, \ldots$, and the corresponding continuous time stopping times $\hat{\tau}'_{\varepsilon,n} = \tau_{\varepsilon,\hat{\nu}'_{\varepsilon,n}}, n = 0, 1, \ldots$ and $\check{\tau}'_{\varepsilon,n} = \tau_{\varepsilon,\check{\nu}'_{\varepsilon,n}}, n = 0, 1, \ldots$.

If $\eta_\varepsilon(0) = 1$, then the stopping times $\hat{\tau}_{\varepsilon,n}, n = 1, 2, \ldots$ and $\check{\tau}_{\varepsilon,n}, n = 1, 2, \ldots$ are return times to the state 1 for the semi-Markov process $\eta_\varepsilon(t)$. As far as the shifted stopping times $\hat{\tau}'_{\varepsilon,n}$ and $\check{\tau}'_{\varepsilon,n}$ are concerned, $\hat{\tau}'_{\varepsilon,1} = \check{\tau}'_{\varepsilon,1}$ is the first hitting time to state 2, while $\hat{\tau}_{\varepsilon,n}, n = 2, 3, \ldots$ and $\check{\tau}_{\varepsilon,n}, n = 2, 3, \ldots$ are return times to the state 2 for the semi-Markov process $\eta_\varepsilon(t)$.

If $\eta_\varepsilon(0) = 2$, then the stopping times $\hat{\tau}_{\varepsilon,n}, n = 1, 2, \ldots$ and $\check{\tau}_{\varepsilon,n}, n = 1, 2, \ldots$ are return times to the state 2 for the semi-Markov process $\eta_\varepsilon(t)$. As far as the shifted stopping times $\hat{\tau}'_{\varepsilon,n}$ and $\check{\tau}'_{\varepsilon,n}$ are concerned, $\hat{\tau}'_{\varepsilon,1} = \check{\tau}'_{\varepsilon,1}$ is the first hitting time to state 1, while $\hat{\tau}_{\varepsilon,n}, n = 2, 3, \ldots$ and $\check{\tau}_{\varepsilon,n}, n = 2, 3, \ldots$ are return times to the state 1 for the semi-Markov process $\eta_\varepsilon(t)$.

Process $(\xi_\varepsilon(t), \eta_\varepsilon(t)), t \geq 0$ is a regenerative process with the transition period $[0, \hat{\tau}'_{\varepsilon,1})$ and the regeneration times $\hat{\tau}'_{\varepsilon,n}, n = 0, 1, \ldots$. It also is a regenerative process with the transition period $[0, \check{\tau}_{\varepsilon,1})$ and the regeneration times $\check{\tau}'_{\varepsilon,n}, n = 0, 1, \ldots$.

We shall see that regeneration times $\hat{\tau}_{\varepsilon,n}$ and $\hat{\tau}'_{\varepsilon,n}$ work well for models with regular perturbations. However, these regeneration times do not work well for the models with singular and super-singular perturbations. Here, the regeneration times $\check{\tau}_{\varepsilon,n}$ and $\check{\tau}'_{\varepsilon,n}$ should be used.

3. Ergodic theorems for regularly perturbed alternating regenerative processes

In this section, we present individual ergodic theorems for regularly perturbed alternating regenerative processes. These theorems are, in fact, rather simple examples illustrating applications of results generalising the renewal theorem to the model of perturbed renewal equation [37 – 39] and individual ergodic theorems for perturbed regenerative processes throughly presented in [14]. Other related references are given in the introduction.

3.1. Perturbed standard alternating regenerative processes. Let us consider regularly perturbed standard alternating regenerative processes, where, additionally to **F** – **J**, the following condition holds:

$\mathbf{M}_1$: $p_{\varepsilon,12}$, $p_{\varepsilon,21} = 1$, for $\varepsilon \in [0, 1]$.

In this case, the Markov chain $\eta_{0,n}$ is ergodic. Obviously, parameter $\beta = 1$, and its stationary probabilities are, $\alpha_1(1) = \alpha_2(1) = \frac{1}{2}$.

Conditions $\mathbf{F} - \mathbf{I}$ and $\mathbf{M}_1$ imply that the semi-Markov process $\eta_0(t)$ is ergodic. Its stationary probabilities have the form,

$$\rho_1(1) = e_{0,1}/(e_{0,1} + e_{0,2}),\ \rho_2(1) = e_{0,2}/(e_{0,1} + e_{0,2}). \tag{17}$$

The corresponding stationary probabilities for the alternating regenerative process $(\xi_0(t), \eta_0(t))$ have the form,

$$\pi^{(1)}_{0,j}(A) = \rho_j(1)\pi_{0,j}(A),\ A \in \mathcal{B}_{\mathbb{X}}, j = 1, 2. \tag{18}$$

The ergodic theorem for perturbed standard alternating regenerative processes takes the following form.

Theorem 4. *Let conditions* $\mathbf{F}$ – $\mathbf{J}$ *and* $\mathbf{M}_1$ *hold. Then, for every* $A \in \Gamma, i, j = 1, 2$, *and any* $0 \leq t_\varepsilon \to \infty$ *as* $\varepsilon \to 0$,

$$P_{\varepsilon,ij}(t_\varepsilon, A) \to \pi^{(1)}_{0,j}(A) \text{ as } \varepsilon \to 0. \tag{19}$$

Proof. In the case, where condition $\mathbf{M}_1$ holds, the stopping times $\tilde{\theta}[r] = r + 1$ and $\hat{\theta}[r] = \check{\theta}[r] = r + 2$, for $r = 0, 1, \ldots$.

Thus, the regenerations times $\hat{\tau}_{\varepsilon,n} = \check{\tau}_{\varepsilon,n} = \tau_{\varepsilon,2n}, n = 0, 1, \ldots$ and $\hat{\tau}'_{\varepsilon,0} = \check{\tau}'_{\varepsilon,0} = 0, \hat{\tau}'_{\varepsilon,1} = \check{\tau}'_{\varepsilon,1} = \tau_{\varepsilon,1}, \hat{\tau}'_{\varepsilon,n} = \check{\tau}'_{\varepsilon,n} = \tau_{\varepsilon,2n-1}, n = 2, 3, \ldots$.

Here and henceforth, we use the same symbol for equalities or inequalities which hold for random variables, for all $\omega \in \Omega$ or almost sure, since this difference does not affect the corresponding probabilities and expectations.

Therefore, the standard alternating regenerative process $(\xi_\varepsilon(t), \eta_\varepsilon(t)), t \geq 0$ is a standard regenerative process with regeneration times $\tau_{\varepsilon,0}, \tau_{\varepsilon,2}, \tau_{\varepsilon,4}, \ldots$. It also can be considered as a regenerative process with transition period $[0, \tau_{\varepsilon,1})$ and regenerative times $\tau_{\varepsilon,0}, \tau_{\varepsilon,1}, \tau_{\varepsilon,3}, \tau_{\varepsilon,5}, \ldots$.

Regenerative lifetimes are not involved. We can use the Theorems 1 – 3, for the model with stopping probabilities $f_\varepsilon = 0, \varepsilon \in [0, 1]$.

First, let us analyse the asymptotic behaviour of probabilities $P_{\varepsilon,11}(t, A)$. In this case, we do prefer to consider $(\xi_\varepsilon(t), \eta_\varepsilon(t)), t \geq 0$ as the standard regenerative process with regeneration times $\tau_{\varepsilon,0}, \tau_{\varepsilon,2}, \ldots$.

The renewal type equation (4) takes for probabilities $P_{\varepsilon,11}(t, A)$ the following form,

$$P_{\varepsilon,11}(t, A) = q^{(2)}_{\varepsilon,1}(t, A) + \int_0^t P_{\varepsilon,11}(t - s, A)Q^{(2)}_{\varepsilon,11}(ds), t \geq 0, \tag{20}$$

where $q^{(2)}_{\varepsilon,1}(t, A) = \mathsf{P}_1\{\xi_\varepsilon(t) \in A, \eta_\varepsilon(t) = 1, \tau_{\varepsilon,2} > t\}, t \geq 0$ and $Q^{(2)}_{\varepsilon,11}(t) = \mathsf{P}_1\{\tau_{\varepsilon,2} \leq t\}, t \geq 0$.

In this case, $\eta_\varepsilon(t) = 1$, for $t \in [0, \tau_{\varepsilon,1})$, and $\eta_\varepsilon(t) = 2$, for $t \in [\tau_{\varepsilon,1}, \tau_{\varepsilon,2})$. Therefore, for every $A \in \mathcal{B}_{\mathbb{X}}, t \geq 0$,

$$\begin{aligned} q^{(2)}_{\varepsilon,1}(t, A) &= \mathsf{P}_1\{\xi_\varepsilon(t) \in A, \eta_\varepsilon(t) = 1, \tau_{\varepsilon,2} > t\} \\ &= \mathsf{P}_1\{\xi_\varepsilon(t) \in A, \tau_{\varepsilon,1} > t\} = q_{\varepsilon,1}(t, A), \end{aligned} \tag{21}$$

Also, for $t \geq 0$,

$$Q^{(2)}_{\varepsilon,11}(t) = \mathsf{P}_1\{\tau_{\varepsilon,2} \leq t\} = Q_{\varepsilon,12}(t) * Q_{\varepsilon,21}(t), \tag{22}$$

and, thus,

$$e^{(2)}_{\varepsilon,11} = \mathsf{E}_1\tau_{\varepsilon,2} = e_{\varepsilon,12} + e_{\varepsilon,21}. \tag{23}$$

Note that condition $\mathbf{M}_1$ implies that expectations $e_{\varepsilon,11}, e_{\varepsilon,22} = 0$ and, therefore, $e_{\varepsilon,12} + e_{\varepsilon,21} = e_{\varepsilon,1} + e_{\varepsilon,2}$.

Condition **F** obviously implies that condition **A** holds. Relation (22) and conditions **G**, **H**, and $\mathbf{M}_1$ imply that condition **B (a)** holds. Relation (22) and condition **H (b)** implies that condition **B (b)** holds. Relation (23) and condition **I** imply that condition **C** holds. Relation (21) and condition **J** imply that condition **D** holds. As was mentioned above, in this case, $f_\varepsilon \equiv 0$. Thus, all conditions of Theorem 1 holds, and the ergodic relation given in this theorem takes place for probabilities $P_{\varepsilon,11}(t_\varepsilon, A)$. In this case, it takes the form of relation (19), where one should choose $i, j = 1$.

Second, let us analyse the asymptotic behaviour of probabilities $P_{\varepsilon,21}(t, A)$. In this case, we do prefer to consider $(\xi_\varepsilon(t), \eta_\varepsilon(t)), t \geq 0$ as the regenerative process with transition period $[0, \tau_{\varepsilon,1})$ and regenerative times $\tau_{\varepsilon,0}, \tau_{\varepsilon,1}, \tau_{\varepsilon,3}, \ldots$.

The shifted process $(\xi_\varepsilon(\tau_{\varepsilon,1} + t), \eta_\varepsilon(\tau_{\varepsilon,1} + t)), t \geq 0$ is a standard regenerative process. If $\eta_\varepsilon(0) = 2$, then $\eta_\varepsilon(\tau_{\varepsilon,1}) = 1$. That is why, probabilities $P_{\varepsilon,11}(t, A)$ play for the above shifted regenerative process the role of probabilities $P^{(1)}_\varepsilon(t, A)$ defined in Subsection 2.1.

The distribution function for the duration of the transition period $[0, \tau_{\varepsilon,1})$ has, in this case, the following form,

$$\mathsf{P}_2\{\tau_{\varepsilon,1} \leq t\} = Q_{\varepsilon,21}(t),\ t \geq 0. \tag{24}$$

Relation (24) and conditions **H**, $\mathbf{M}_1$ imply that condition **E** holds. Thus, all conditions of Theorem 2 hold, and the corresponding ergodic relation for probabilities $P_{\varepsilon,11}(t_\varepsilon, A)$ also holds for probabilities $P_{\varepsilon,21}(t_\varepsilon, A)$.

Due to the symmetricity of conditions **F** – **J** and $\mathbf{M}_1$ with respect to the indices $i, j = 1, 2$, the ergodic relations, analogous to the mentioned above ergodic relations for probabilities $P_{\varepsilon,11}(t_\varepsilon, A)$ and $P_{\varepsilon,21}(t_\varepsilon, A)$, also take place for probabilities $P_{\varepsilon,22}(t_\varepsilon, A)$ and $P_{\varepsilon,12}(t_\varepsilon, A)$. The only, stationary probabilities $\pi^{(1)}_{0,1}(A)$ should be replaced by stationary probabilities $\pi^{(1)}_{0,2}(A)$ in the corresponding ergodic relations. □

3.2. Regularly perturbed alternating regenerative processes. Let us now consider alternating regenerative processes with a regular perturbation model, where additionally to $\mathbf{F} - \mathbf{J}$, the following condition holds:

$\mathbf{M}_2$: $p_{0,12},\, p_{0,21} > 0$.

Note that condition $\mathbf{M}_1$ is a particular case of condition $\mathbf{M}_2$, and, thus, any standard alternating regenerative process also is a regularly alternating regenerative process.

In this case, the Markov chain $\eta_{0,n}$ is ergodic. Obviously, parameter $\beta = p_{0,12}/p_{0,21} \in (0, \infty)$, and the stationary probabilities for the above Markov chain are, $\alpha_1(\beta) = \frac{1}{1+\beta}$ and $\alpha_2(\beta) = \frac{1}{1+\beta^{-1}}$.

Conditions $\mathbf{F} - \mathbf{I}$ and $\mathbf{M}_2$ imply that the semi-Markov process $\eta_0(t)$ is ergodic. Its stationary probabilities have the form,

$$\rho_1(\beta) = \frac{e_{0,1}\alpha_1(\beta)}{e_{0,1}\alpha_1(\beta) + e_{0,2}\alpha_2(\beta)},\ \rho_2(\beta) = \frac{e_{0,2}\alpha_2(\beta)}{e_{0,1}\alpha_1(\beta) + e_{0,2}\alpha_2(\beta)}. \tag{25}$$

The corresponding stationary probabilities for the alternating regenerative process $\xi_0(t)$ has the form, for $\beta \in (0, \infty)$,

$$\pi^{(\beta)}_{0,j}(A) = \rho_j(\beta)\pi_{0,j}(A),\ A \in \mathcal{B}_{\mathbb{X}}, j = 1, 2. \tag{26}$$

The ergodic theorem for perturbed alternating regenerative processes takes the following form.

Theorem 5. *Let conditions* $\mathbf{F} - \mathbf{J}$ *hold and, also, condition* $\mathbf{M}_2$ *holds and parameter* $p_{0,12}/p_{0,21} = \beta \in (0, \infty)$. *Then, for every* $A \in \Gamma, i, j = 1, 2$, *and any* $0 \leq t_\varepsilon \to \infty$ *as* $\varepsilon \to 0$,

$$P_{\varepsilon,ij}(t_\varepsilon, A) \to \pi^{(\beta)}_{0,j}(A) \text{ as } \varepsilon \to 0. \tag{27}$$

Proof. As was pointed out in Section 4, process $(\xi_\varepsilon(t), \eta_\varepsilon(t))$ is a regenerative process with regeneration times with regeneration times $\hat{\tau}_{\varepsilon,n}, n = 0, 1, \ldots$. It is also a regenerative process with the transition period $[0, \hat{\tau}'_{\varepsilon,1})$ and the regeneration times $\hat{\tau}'_{\varepsilon,n}, n = 0, 1, \ldots$.

Again, regenerative lifetimes are not involved. We can use the Theorems 1 – 3, for the model with stopping probabilities $f_\varepsilon = 0, \varepsilon \in [0, 1]$.

First, let us analyse the asymptotic behaviour of probabilities $P_{\varepsilon,11}(t, A)$. In this case, we do prefer to consider $(\xi_\varepsilon(t), \eta_\varepsilon(t)), t \geq 0$ as the standard regenerative process with regeneration times $\hat{\tau}_{\varepsilon,n}, n = 0, 1, \ldots$.

The renewal type equation (4) for probabilities $P_{\varepsilon,11}(t, A)$ takes, in this case, the following form,

$$P_{\varepsilon,11}(t, A) = \hat{q}_{\varepsilon,1}(t, A) + \int_0^t P_{\varepsilon,11}(t - s, A)\hat{Q}_{\varepsilon,11}(ds), t \geq 0, \tag{28}$$

where $\hat{q}_{\varepsilon,1}(t, A) = \mathsf{P}_1\{\xi_\varepsilon(t) \in A, \eta_\varepsilon(t) = 1, \hat{\tau}_{\varepsilon,1} > t\}, t \geq 0$ and $\hat{Q}_{\varepsilon,11}(t) = \mathsf{P}_1\{\hat{\tau}_{\varepsilon,1} \leq t\}, t \geq 0$.

If $\eta_\varepsilon(0) = 1$, then $\eta_\varepsilon(t) = 1$ for $t \in [0, \tau_{\varepsilon,1})$. Also, $\hat{\tau}_{\varepsilon,1} = \tau_{\varepsilon,1}$, if $\eta_{\varepsilon,1} = 1$, and $\eta_\varepsilon(t) = 2$, for $t \in [\tau_{\varepsilon,1}, \hat{\tau}_{\varepsilon,1})$, if $\eta_{\varepsilon,1} = 2$. Therefore, for every $A \in \mathcal{B}_{\mathbb{X}}, t \geq 0$,

$$\begin{aligned} \hat{q}_{\varepsilon,1}(t, A) &= \mathsf{P}_1\{\xi_\varepsilon(t) \in A, \eta_\varepsilon(t) = 1, \hat{\tau}_{\varepsilon,1} > t\} \\ &= \mathsf{P}_1\{\xi_\varepsilon(t) \in A, \tau_{\varepsilon,1} > t, \eta_{\varepsilon,1} = 1\} \\ &\quad + \mathsf{P}_1\{\xi_\varepsilon(t) \in A, \tau_{\varepsilon,1} > t, \eta_{\varepsilon,1} = 2\} \\ &= \mathsf{P}_1\{\xi_\varepsilon(t) \in A, \tau_{\varepsilon,1} > t\} = q_{\varepsilon,1}(t, A). \end{aligned} \tag{29}$$

In this case, $\hat{Q}_{\varepsilon,11}(t)$ is the distribution function of the first return time to state 1 for semi-Markov process $\eta_\varepsilon(t)$. It can be expressed in terms of convolutions of transition probabilities for this semi-Markov process,

$$\hat{Q}_{\varepsilon,11}(t) = Q_{\varepsilon,11}(t) + Q_{\varepsilon,12}(t) * \sum_{n=0}^{\infty} Q^{*n}_{\varepsilon,22}(t) * Q_{\varepsilon,21}(t), t \geq 0. \tag{30}$$

Relation (30) takes the following equivalent form in terms of Laplace transforms,

$$\begin{aligned} \hat{\phi}_{\varepsilon,11}(s) &= \int_0^\infty e^{-st} \hat{Q}_{\varepsilon,11}(dt) \\ &= \phi_{\varepsilon,11}(s) + \phi_{\varepsilon,12}(s) \sum_{n=0}^{\infty} \phi^n_{\varepsilon,22}(s) \phi_{\varepsilon,21}(s) \\ &= \phi_{\varepsilon,11}(s) + \phi_{\varepsilon,12}(s) \frac{1}{1 - \phi_{\varepsilon,22}(s)} \phi_{\varepsilon,21}(s), \ s \geq 0. \end{aligned} \tag{31}$$

Relation (30) also implies that random variable $\hat{\nu}_{\varepsilon,1}$ has a so-called burned geometric distribution that is,

$$\hat{\nu}_{\varepsilon,1} = \begin{cases} 1 & \text{with probability } p_{\varepsilon,11}, \\ n & \text{with probability } p_{\varepsilon,12}\, p^{n-2}_{\varepsilon,22}\, p_{\varepsilon,21}, \text{for } n \geq 2. \end{cases}$$

This fact and conditions **G**, **H**, and $\mathbf{M}_2$ imply, in an obvious way, that expectation $\hat{e}_{\varepsilon,11} = \mathsf{E}_1 \hat{\tau}_{\varepsilon,1} < \infty$. It can be easily computed, for example, using the derivative of the Laplace transform $\hat{\phi}_{\varepsilon,11}(s)$ at zero,

$$\begin{aligned} \hat{e}_{\varepsilon,11} = \mathsf{E}_1 \hat{\tau}_{\varepsilon,1} = -\hat{\phi}'_{\varepsilon,11}(0) &= e_{\varepsilon,11} + e_{\varepsilon,12} \frac{1}{1 - p_{\varepsilon,22}} p_{\varepsilon,21} \\ &+ p_{\varepsilon,12} \frac{e_{\varepsilon,22}}{(1 - p_{\varepsilon,22})^2} p_{\varepsilon,21} + p_{\varepsilon,12} \frac{1}{1 - p_{\varepsilon,22}} e_{\varepsilon,21} \\ &= \frac{e_{\varepsilon,1} p_{\varepsilon,21} + e_{\varepsilon,2} p_{\varepsilon,12}}{p_{\varepsilon,21}} = \frac{e_{\varepsilon,1} \alpha_1(\beta_\varepsilon) + e_{\varepsilon,2} \alpha_2(\beta_\varepsilon)}{\alpha_1(\beta_\varepsilon)}. \end{aligned} \tag{32}$$

Obviously, $\hat{\tau}_{\varepsilon,n} \geq \tau_{\varepsilon,n}$, for $n = 0, 1, \ldots$. Thus, condition **F** implies that condition **A** holds. Relations (30), (31) and conditions **G**, **H**, and $\mathbf{M}_2$ imply that Laplace transforms $\hat{\phi}_{\varepsilon,11}(s) \to \hat{\phi}_{0,11}(s)$ as $\varepsilon \to 0$, for $s \geq 0$. Thus, by Remark 1, condition **B (a)** holds. Also, relation (30) and condition **H (b)** implies that condition **B (b)** holds. Relation (32) and conditions **H** and **I** imply that condition **C** holds. Relation (29) and condition **J** imply that condition **D** holds. As was mentioned above, in this case, $f_\varepsilon \equiv 0$. Thus, all conditions of Theorem 1 hold, and the ergodic relation given in this theorem takes place for probabilities $P_{\varepsilon,11}(t_\varepsilon, A)$. In this case, it takes the form of relation (27), where one should choose $i, j = 1$, i.e., for every $A \in \Gamma$, and any $0 \leq t_\varepsilon \to \infty$ as $\varepsilon \to 0$,

$$\begin{aligned} P_{\varepsilon,11}(t_\varepsilon, A) &\to \frac{\alpha_1(\beta)}{e_{0,1}\alpha_1(\beta) + e_{0,2}\alpha_2(\beta)} \int_0^\infty q_{0,1}(s, A)m(ds) \\ &= \frac{e_{0,1}\alpha_1(\beta)}{e_{0,1}\alpha_1(\beta) + e_{0,2}\alpha_2(\beta)} \frac{1}{e_{0,1}} \int_0^\infty q_{0,1}(s, A)m(ds) \\ &= \rho_j(\beta)\pi_{0,j}(A) = \pi_{0,j}^{(\beta)}(A) \text{ as } \varepsilon \to 0. \end{aligned} \tag{33}$$

Second, let us analyse the asymptotic behaviour of probabilities $P_{\varepsilon,21}(t, A)$. In this case, we do prefer to consider $(\xi_\varepsilon(t), \eta_\varepsilon(t)), t \geq 0$ as the regenerative process with transition period $[0, \hat{\tau}'_{\varepsilon,1})$ and regenerative times $\hat{\tau}'_{\varepsilon,0}, \hat{\tau}'_{\varepsilon,1} = \tilde{\tau}_{\varepsilon,1}, \hat{\tau}'_{\varepsilon,2}, \hat{\tau}'_{\varepsilon,3}, \ldots$.

The shifted process $(\xi_\varepsilon(\hat{\tau}'_{\varepsilon,1} + t), \eta_\varepsilon(\hat{\tau}'_{\varepsilon,1} + t)), t \geq 0$ is a standard regenerative process. If $\eta_\varepsilon(0) = 2$, then $\eta_\varepsilon(\tilde{\tau}_{\varepsilon,1}) = 1$. That is why, probabilities $P_{\varepsilon,11}(t, A)$ play for this process the role of probabilities $P_\varepsilon^{(1)}(t, A)$ pointed out in Subsection 2.1.

The distribution function for the duration of the transition period $[0, \tilde{\tau}_{\varepsilon,1})$ has, in this case, the following form,

$$\mathsf{P}_2\{\tilde{\tau}_{\varepsilon,1} \leq t\} = \tilde{Q}_{\varepsilon,21}(t) = \sum_{n=0}^{\infty} Q_{\varepsilon,22}^{*n}(t) * Q_{\varepsilon,21}(t), t \geq 0. \tag{34}$$

This relation takes the following equivalent form in terms of Laplace transforms,

$$\begin{aligned} \tilde{\phi}_{\varepsilon,21}(s) &= \int_0^\infty e^{-st}\tilde{Q}_{\varepsilon,21}(dt) \\ &= \sum_{n=0}^{\infty} \phi_{\varepsilon,22}^n(s)\phi_{\varepsilon,21}(s) = \frac{\phi_{\varepsilon,21}(s)}{1 - \phi_{\varepsilon,22}(s)},\ s \geq 0. \end{aligned} \tag{35}$$

Relations (34), (35) and conditions **G**, **H**, and $\mathbf{M}_2$ imply that Laplace transforms $\tilde{\phi}_{\varepsilon,21}(s) \to \tilde{\phi}_{0,21}(s)$ as $\varepsilon \to 0$, for $s \geq 0$. Thus, by Remark 1, condition **E** holds. All conditions of Theorem 2 hold, and the corresponding ergodic relation for probabilities $P_{\varepsilon,11}(t_\varepsilon, A)$ also holds for probabilities $P_{\varepsilon,21}(t_\varepsilon, A)$.

Due to the symmetricity of conditions **F** – **J** and $\mathbf{M}_2$ with respect to the indices $i, j = 1, 2$, the ergodic relations, analogous to the mentioned above ergodic relations for probabilities $P_{\varepsilon,11}(t_\varepsilon, A)$ and $P_{\varepsilon,21}(t_\varepsilon, A)$, also take place for probabilities $P_{\varepsilon,22}(t_\varepsilon, A)$ and $P_{\varepsilon,12}(t_\varepsilon, A)$. The only, the stationary probabilities $\pi^{(\beta)}_{0,1}(A)$ should be replaced by stationary probabilities $\pi^{(\beta)}_{0,2}(A)$ in the corresponding ergodic relations. □

Remark 2. Theorem 4 is a particular case of Theorem 5. In this case, the ergodic relation (27) takes the form of ergodic relation (19).

3.3. Semi-regularly perturbed alternating regenerative processes. Let us now consider alternating regenerative processes with the semi-regular perturbation model, where additionally to **F** – **J**, the following condition holds:

$\mathbf{M}_3$: **(a)** $p_{0,12} = 0, p_{0,21} > 0$ or **(b)** $p_{0,12} > 0, p_{0,21} = 0$.

In this case, the Markov chain $\eta_{0,n}$ is ergodic. Obviously, parameter $\beta = p_{0,12}/p_{0,21} = 0$, and the stationary probabilities for the above Markov chain are, $\alpha_1(0) = 1, \alpha_2(0) = 0$, if condition $\mathbf{M}_3$ **(a)** holds. While $\beta = p_{0,12}/p_{0,21} = \infty$, and the stationary probabilities for the above Markov chain are, $\alpha_1(\infty) = 0, \alpha_2(\infty) = 1$, if condition $\mathbf{M}_3$ **(b)** holds.

Conditions **F** – **J** and $\mathbf{M}_3$ imply that the semi-Markov process $\eta_0(t)$ is ergodic. Its stationary probabilities have the form, $\rho_1(0) = e_{0,1}\alpha_1(0)/(e_{0,1}\alpha_1(0)+ e_{0,2}\alpha_2(0)) = 1$, $\rho_2(0) = \varepsilon_{0,2}\alpha_2(0)/(e_{0,1}\alpha_1(0) + e_{0,2}\alpha_2(0)) = 0$, if condition $\mathbf{M}_3$ **(a)** holds. While, $\rho_1(\infty) = e_{0,1}\alpha_1(\infty)/(e_{0,1}\alpha_1(\infty) + e_{0,2}\alpha_2(\infty)) = 0$, $\rho_2(\infty) = e_{0,2}\alpha_2(\infty)/(e_{0,1}\alpha_1(\infty) + e_{0,2}\alpha_2(\infty)) = 1$, if condition $\mathbf{M}_3$ **(b)** holds.

The corresponding stationary probabilities for the alternating regenerative process $(\xi_0(t), \eta_0(t))$ have the form, $\pi^{(\beta)}_{0,j}(A) = \rho_j(\beta)\pi_{0,j}(A), A \in \mathcal{B}_{\mathbb{X}}, j = 1, 2$, for $\beta = 0$ and $\beta = \infty$, i.e.,

$$\pi^{(0)}_{0,j}(A) = \begin{cases} \pi_{0,1}(A) & \text{for } j = 1, \\ 0 & \text{for } j = 2, \end{cases} \tag{36}$$

and

$$\pi^{(\infty)}_{0,j}(A) = \begin{cases} 0 & \text{for } j = 1, \\ \pi_{0,2}(A) & \text{for } j = 2. \end{cases} \tag{37}$$

The ergodic theorems for perturbed alternating regenerative processes take the following forms.

Theorem 6. *Let conditions* **F** – **J** *and* $\mathbf{M}_3$ **(a)** *hold. Then, for every* $A \in \Gamma, i, j = 1, 2$, *and any* $0 \le t_\varepsilon \to \infty$ *as* $\varepsilon \to 0$,

$$P_{\varepsilon,ij}(t_\varepsilon, A) \to \pi^{(0)}_{0,j}(A) \text{ as } \varepsilon \to 0. \tag{38}$$

Theorem 7. *Let conditions* **F** – **J** *and* $\mathbf{M}_3$ **(b)** *hold. Then, for every* $A \in \Gamma, i, j = 1, 2$, *and any* $0 \le t_\varepsilon \to \infty$ *as* $\varepsilon \to 0$,

$$P_{\varepsilon,ij}(t_\varepsilon, A) \to \pi^{(\infty)}_{0,j}(A) \text{ as } \varepsilon \to 0. \tag{39}$$

Proof. Process $(\xi_\varepsilon(t), \eta_\varepsilon(t))$ is a standard regenerative process with regeneration times $\hat{\tau}_{\varepsilon,n}, n = 0, 1, \ldots$. It also is a regenerative process with transition period $[0, \hat{\tau}'_{\varepsilon,1})$ and regenerative times $\hat{\tau}'_{\varepsilon,n}, n = 0, 1, \ldots$.

Again, regenerative stopping is not involved. We can use the Theorems 1 – 3, for the model with stopping probabilities $f_\varepsilon = 0, \varepsilon \in [0, 1]$.

Let us consider the case, where condition $\mathbf{M}_3$ **(a)** holds.

Let us analyse the asymptotic behaviour for probabilities $P_{\varepsilon,1j}(t, A), j = 1, 2$. In this case, we prefer to consider $(\xi_\varepsilon(t), \eta_\varepsilon(t)), t \ge 0$ as the standard regenerative process with regeneration times $\hat{\tau}_{\varepsilon,0}, \hat{\tau}_{\varepsilon,1}, \hat{\tau}_{\varepsilon,2}, \ldots$.

First, let us analyse the asymptotic behaviour of probabilities $P_{\varepsilon,11}(t, A)$.

The renewal type equation (4) takes for probabilities $P_{\varepsilon,1j}(t, A)$ the following form, for $j = 1, 2$,

$$P_{\varepsilon,1j}(t, A) = \hat{q}_{\varepsilon,1j}(t, A) + \int_0^t P_{\varepsilon,1j}(t - s, A)\hat{Q}_{\varepsilon,11}(ds), t \ge 0, \tag{40}$$

where $\hat{q}_{\varepsilon,1j}(t, A) = \mathsf{P}_1\{\xi_\varepsilon(t) \in A, \eta_\varepsilon(t) = j, \hat{\tau}_{\varepsilon,1} > t\}, t \ge 0, j = 1, 2$ and $\hat{Q}_{\varepsilon,11}(t) = \mathsf{P}_1\{\hat{\tau}_{\varepsilon,1} \le t\}, t \ge 0$.

In the case of probabilities $P_{\varepsilon,11}(t, A)$, we can repeat all calculations made in relations (28) – (32), given in the proof of Theorem 5. These relations, in fact, take simpler forms.

Analogously to relation (29), one can get, for every $A \in \mathcal{B}_{\mathbb{X}}, t \ge 0$,

$$\hat{q}_{\varepsilon,11}(t, A) = \mathsf{P}_1\{\xi_\varepsilon(t) \in A, \tau_{\varepsilon,1} > t\} = q_{\varepsilon,1}(t, A). \tag{41}$$

Also, as was pointed out in comments related to relation (30), $\hat{Q}_{\varepsilon,11}(t)$ is the distribution function of the first return time to state 1 for semi-Markov process $\eta_\varepsilon(t)$, and the following formula, analogous to (31), takes place for its Laplace transform,

$$\begin{aligned} \hat{\phi}_{\varepsilon,11}(s) &= \int_0^\infty e^{-st}\hat{Q}_{\varepsilon,11}(dt) \\ &= \phi_{\varepsilon,11}(s) + \phi_{\varepsilon,12}(s)\frac{1}{1 - \phi_{\varepsilon,22}(s)}\phi_{\varepsilon,21}(s),\ s \ge 0. \end{aligned} \tag{42}$$

Also, the following formula, analogous to (32), takes place for expectations,

$$\hat{e}_{\varepsilon,11} = \mathsf{E}_1 \hat{\tau}_{\varepsilon,1} = -\hat{\phi}'_{\varepsilon,11}(0) = \frac{e_{\varepsilon,1}\alpha_1(\beta_\varepsilon) + e_{\varepsilon,2}\alpha_2(\beta_\varepsilon)}{\alpha_1(\beta_\varepsilon)}. \tag{43}$$

Conditions **G**, **H**, and $\mathbf{M}_3$ **(a)** imply that, in relation (42), either Laplace transform $\phi_{\varepsilon,12}(s) = 0$, for $s \geq 0$, if $p_{\varepsilon,12} = 0$, for $\varepsilon \in [0,1]$, or $\phi_{\varepsilon,12}(s) \to 0$ as $\varepsilon \to 0$, for $s \geq 0$, if $0 < p_{\varepsilon,12} \to 0$ as $\varepsilon \to 0$. This implies that the Laplace transforms $\hat{\phi}_{\varepsilon,11}(s) \to \hat{\phi}_{0,11}(s) = \phi_{0,11}(s)$ as $\varepsilon \to 0$, for $s \geq 0$. Thus, by Remark 1, condition **B (a)** holds, with the corresponding limiting distribution function $Q_{0,11}(t)$. According to condition **H**, condition **B (b)** holds for the distribution function $Q_{0,11}(t)$. Analogously, in relation (43), either expectation $e_{\varepsilon,12} = 0$, if $p_{\varepsilon,12} = 0$, for $\varepsilon \in [0,1]$, or $e_{\varepsilon,12} \to 0$ as $\varepsilon \to 0$, if $0 < p_{\varepsilon,12} \to 0$ as $\varepsilon \to 0$. It follows from this remark and conditions **G** – **I** that the expectations $\hat{e}_{\varepsilon,11} \to \hat{e}_{0,11} = e_{0,11}$ as $\varepsilon \to 0$. Note also that condition $\mathbf{M}_3$ **(a)** implies that expectation $e_{0,11} = e_{0,1}$. Thus, condition **C** holds, with the corresponding limiting expectation $e_{0,11} = e_{0,1}$. Relation (41) and condition **J** imply that condition **D** holds. As was mentioned above, in this case, $f_\varepsilon \equiv 0$. Thus, all conditions all conditions of Theorem 1 hold, and the ergodic relation given in this theorem takes place for probabilities $P_{\varepsilon,11}(t_\varepsilon, A)$. In this case, it takes the form of relation (38), where one should choose $i, j = 1$.

Second, let us analyse the asymptotic behaviour of probabilities $P_{\varepsilon,12}(t, A)$.

Holding of conditions **A** – **C** was pointed above.

If $\eta_\varepsilon(0) = 1$, then $\eta_\varepsilon(t) = 1$ for $t \in [0, \tau_{\varepsilon,1})$, and $\hat{\tau}_{\varepsilon,1} = \tau_{\varepsilon,1}$, if $\eta_{\varepsilon,1} = 1$. Also, $\eta_\varepsilon(t) = 2$, for $t \in [\tau_{\varepsilon,1}, \hat{\tau}_{\varepsilon,1})$, if $\eta_{\varepsilon,1} = 2$. Therefore, for every $A \in \mathcal{B}_{\mathbb{X}}, t \geq 0$,

$$\begin{aligned}\hat{q}_{\varepsilon,12}(t, A) &= \mathsf{P}_1\{\xi_\varepsilon(t) \in A, \eta_\varepsilon(t) = 2, \hat{\tau}_{\varepsilon,1} > t\} \\ &\leq \mathsf{P}_1\{\xi_\varepsilon(t) \in A, \eta_\varepsilon(t) = 2, \hat{\tau}_{\varepsilon,1} > t, \tau_{\varepsilon,1} \leq t, \eta_{\varepsilon,1} = 2\} \\ &\leq \mathsf{P}_1\{\tau_{\varepsilon,1} \leq t, \eta_{\varepsilon,1} = 2\} \leq p_{\varepsilon,12}.\end{aligned} \tag{44}$$

Since, $p_{\varepsilon,12} = 0$, for $\varepsilon \in [0,1]$ or $p_{\varepsilon,12} \to 0$ as $\varepsilon \to 0$, condition **D** holds for function $\hat{q}_{\varepsilon,12}(t, A)$ with the corresponding limiting function $\hat{q}_{0,12}(t, A) = 0, t \geq 0$, for every $A \in \mathcal{B}_{\mathbb{X}}$. Thus, all conditions of Theorem 1 hold, and the ergodic relation given in this theorem takes place for probabilities $P_{\varepsilon,12}(t_\varepsilon, A)$. In this case, it takes the form of relation (38), where one should choose $i = 1, j = 2$.

Third, let us analyse asymptotic behaviour of probabilities $P_{\varepsilon,2j}(t, A), j = 1, 2$. In this case, we do prefer to consider $(\xi_\varepsilon(t), \eta_\varepsilon(t)), t \geq 0$ as the regenerative process with transition period $[0, \hat{\tau}'_{\varepsilon,1})$ and regenerative times $\hat{\tau}'_{\varepsilon,0}, \hat{\tau}'_{\varepsilon,1} = \tilde{\tau}_{\varepsilon,1}, \hat{\tau}'_{\varepsilon,2}, \hat{\tau}'_{\varepsilon,3}, \ldots$.

The shifted process $(\xi_\varepsilon(\hat{\tau}'_{\varepsilon,1} + t), \eta_\varepsilon(\hat{\tau}'_{\varepsilon,1} + t)), t \geq 0$ is a standard regenerative process. If $\eta_\varepsilon(0) = 2$, then $\eta_\varepsilon(\tilde{\tau}_{\varepsilon,1}) = 1$. That is why, probabilities $P_{\varepsilon,1j}(t, A)$ play for this process the role of probabilities $P^{(1)}_\varepsilon(t, A)$ defined out in Section 2.

The distribution function $\mathsf{P}_2\{\tilde{\tau}_{\varepsilon,1} \leq t\} = \tilde{Q}_{\varepsilon,21}(t)$ and the Laplace transform $\tilde{\phi}_{\varepsilon,12}(s) = \int_0^\infty e^{-st}\tilde{Q}_{\varepsilon,12}(dt)$ for the duration of the transition period $[0, \tilde{\tau}_{\varepsilon,1})$ are given, respectively in relations (34) and (35). These relations and conditions **G**, **H** and $\mathbf{M}_3$ **(a)** imply that Laplace transforms $\tilde{\phi}_{\varepsilon,12}(s) \to \tilde{\phi}_{0,12}(s)$ as $\varepsilon \to 0$, for $s \geq 0$. Thus, by Remark 1, condition **E** holds. All conditions of Theorem 2 hold, and the corresponding ergodic relation for probabilities $P_{\varepsilon,1j}(t_\varepsilon, A), j = 1, 2$ also holds for probabilities $P_{\varepsilon,2j}(t_\varepsilon, A), j = 1, 2$.

Due to symmetricity of conditions **F** – **J** with respect to the indices $i, j = 1, 2$ the corresponding asymptotic analysis for probabilities $P_{\varepsilon,ij}(t_\varepsilon, A), i, j = 1, 2$ (under assumption of holding condition $\mathbf{M}_3$ **(b)**) is analogous to the above asymptotic analysis for probabilities $P_{\varepsilon,ij}(t_\varepsilon, A), i.j = 1, 2$ (under assumption of holding condition $\mathbf{M}_3$ **(a)**). The corresponding ergodic relation (39) takes place for the above probabilities, under assumption of holding condition $\mathbf{M}_3$ **(b)**. $\square$

4. Super-long and long time ergodic theorems for singularly perturbed alternating regenerative processes

In this section, we present super-long and long time individual ergodic theorems for singularly perturbed alternating regenerative processes. We also present in this section the special procedure of time scaling for perturbed regenerative processes. It is essentially used in the corresponding proofs.

4.1. Time scaling for perturbed regenerative processes. Let return back to the model of perturbed regenerative processes with regenerative lifetimes introduced in Subsection 2.1. So, let $\xi_\varepsilon(t), t \geq 0$ be, for every $\varepsilon \in [0, 1]$, a regeneration process with regeneration times $\tau_{\varepsilon,n}, n = 0, 1, \ldots$ and a regenerative lifetime μ_ε constructed using the triplets $\langle \bar{\xi}_{\varepsilon,n} = \langle \xi_{\varepsilon,n}(t), t \geq 0\rangle, \kappa_{\varepsilon,n}, \mu_{\varepsilon,n}\rangle$ introduced in Subsection 2.1.

Let also $v_\varepsilon, \varepsilon \in (0, 1]$ be a positive function. We also choose some $v_0 \in [0, \infty]$.

In some cases, it can be useful to replace, for every $\varepsilon \in (0, 1]$, the above triplet by new one, $\langle \bar{\xi}_{\varepsilon,v_\varepsilon,n} = \langle \xi_{\varepsilon,v_\varepsilon,n}(t) = \xi_{\varepsilon,n}(tv_\varepsilon), t \geq 0\rangle, \kappa_{\varepsilon,v_\varepsilon,n} = v_\varepsilon^{-1}\kappa_{\varepsilon,n}, \mu_{\varepsilon,v_\varepsilon,n} = v_\varepsilon^{-1}\mu_{\varepsilon,n}\rangle$.

Respectively, the above regenerative process $\xi_\varepsilon(t), t \geq 0$ will be, for every $\varepsilon \in (0, 1]$, transformed in the new process $\xi_{\varepsilon,v_\varepsilon}(t) = \xi_\varepsilon(tv_\varepsilon), t \geq 0$. Obviously. $\xi_{\varepsilon,v_\varepsilon}(t), t \geq 0$ is also a regenerative process, with new regenerative times

$\tau_{\varepsilon,v_\varepsilon,n} = v_\varepsilon^{-1}\tau_{\varepsilon,n}, n = 0,1,\ldots$ and new lifetime $\mu_{\varepsilon,v_\varepsilon} = v_\varepsilon^{-1}\mu_\varepsilon$.

We also should introduce some limiting triplet $\langle \bar{\xi}_{0,v_0,n} = \langle \xi_{0,v_0,n}(t),\ t \geq 0\rangle, \kappa_{0,v_0,n}, \mu_{0,v_0,n}\rangle$, which possess the corresponding properties described in Subsection 2.1, and the corresponding limiting regenerative process $\xi_{0,v_0}(t) = \xi_{0,v_0,n}(t - \tau_{0,v_0,n-1})$ for $t \in [\tau_{0,v_0,n-1}, \tau_{0,v_0,n})$, $n = 1,2,\ldots$, regeneration times $\tau_{0,v_0,n} = \kappa_{0,v_0,1} + \cdots + \kappa_{0,v_0,n}, n = 1,2,\ldots, \tau_{0,v_0,0} = 0$, and a regenerative lifetime, $\mu_{0,v_0} = \kappa_{0,v_0,1} + \cdots + \kappa_{0,v_0,\nu_{0,v_0}-1} + \mu_{0,v_0,\nu_{0,v_0}}$, where $\nu_{0,v_0} = \min(n \geq 1 : \mu_{0,v_0,n} < \kappa_{0,v_0,n})$.

In such model, we can assume the the corresponding conditions **A** – **D** hold for the transformed regenerative processes $\xi_{\varepsilon,v_\varepsilon}(t), t \geq 0$, their regeneration times $\tau_{\varepsilon,v_\varepsilon,n}, n = 0,1,\ldots$ and lifetimes $\mu_{\varepsilon,v_\varepsilon}$.

It worth to note that the probabilities $P_{\varepsilon,v_\varepsilon}(t,A) = \mathsf{P}\{\xi_{\varepsilon,v_\varepsilon}(t) \in A, \mu_{\varepsilon,v_\varepsilon} > t\} = P_\varepsilon(tv_\varepsilon, A) = \mathsf{P}\{\xi_\varepsilon(tv_\varepsilon) \in A, \mu_\varepsilon > tv_\varepsilon\}, t \geq 0$, for $\varepsilon \in (0,1]$.

The basic renewal equation (4) for probabilities $P_{\varepsilon,v_\varepsilon}(t,A)$ takes, for $\varepsilon \in (0,1]$, the following form, for $A \in \mathcal{B}_{\mathbb{X}}$,

$$P_{\varepsilon,v_\varepsilon}(t,A) = q_{\varepsilon,v_\varepsilon}(t,A) + \int_0^t P_{\varepsilon,v_\varepsilon}(t-s,A)F_{\varepsilon,v_\varepsilon}(ds),\ t \geq 0, \tag{45}$$

where $q_{\varepsilon,v_\varepsilon}(t,A) = \mathsf{P}\{\xi_{\varepsilon,v_\varepsilon}(t) \in A, \tau_{\varepsilon,v_\varepsilon,1} \wedge \mu_{\varepsilon,v_\varepsilon} > t\} = q_\varepsilon(tv_\varepsilon, A) = \mathsf{P}\{\xi_\varepsilon(tv_\varepsilon) \in A, \tau_{\varepsilon,1} \wedge \mu_\varepsilon > tv_\varepsilon\}$ and $F_{\varepsilon,v_\varepsilon}(t) = \mathsf{P}\{\tau_{\varepsilon,v_\varepsilon,1} \leq t, \mu_{\varepsilon,v_\varepsilon} \geq \tau_{\varepsilon,v_\varepsilon,1}\} = \mathsf{P}\{\tau_{\varepsilon,1} \leq tv_\varepsilon, v_\varepsilon^{-1}\mu_\varepsilon \geq v_\varepsilon^{-1}\tau_{\varepsilon,1}\}$.

We shall see in the next section that the above scaling of time transformation can be effectively used in ergodic theorems for singularly perturbed alternating regenerative processes, where aggregated regeneration times can be stochastically unbounded as $\varepsilon \to 0$. In such models, we shall use time scaling factors $0 < v_\varepsilon \to v_0 = \infty$ as $\varepsilon \to 0$, and refer to v_ε as to time compression factors.

4.2. Singularly perturbed alternating regenerative processes. Let us now consider the alternating regenerative processes with the singular perturbation model, where additionally to **F** – **J**, the following condition holds:

$\mathbf{N}_1$: $0 < p_{\varepsilon,12} \to p_{0,12} = 0$ as $\varepsilon \to 0$ and $0 < p_{\varepsilon,21} \to p_{0,21} = 0$ as $\varepsilon \to 0$.

The case, where condition $\mathbf{N}_1$ holds, is the most interesting. Here, we should also assume that probabilities $p_{\varepsilon,12}$ and $p_{\varepsilon,21}$ are asymptotically comparable in the sense that the condition $\mathbf{K}_\beta$ holds for some $\beta \in [0,\infty]$.

Let us define function $v_\varepsilon = p_{\varepsilon,12}^{-1} + p_{\varepsilon,21}^{-1}$. Obviously, $0 < v_\varepsilon \to v_0 = \infty$ as $\varepsilon \to 0$. Also, $p_{\varepsilon,12}^{-1}/v_\varepsilon \to (1+\beta)^{-1}$ and $p_{\varepsilon,21}^{-1}/v_\varepsilon \to (1+\beta^{-1})^{-1}$ as $\varepsilon \to 0$.

As was pointed out in Section 4, process $(\xi_\varepsilon(t), \eta_\varepsilon(t))$ is a regenerative process with regeneration times $\hat{\tau}_{\varepsilon,n}, n = 0,1,\ldots$. It is also a regenerative process with the transition period $[0, \hat{\tau}'_{\varepsilon,1})$ and regeneration times $\hat{\tau}'_{\varepsilon,n}, n = 0, 1, \ldots$.

Unfortunately, the model with aggregated regeneration times $\hat{\tau}_{\varepsilon,n}$ does not work in this case. Indeed, conditions **G**, **H** and $\mathbf{N}_1$ implies that $\phi_{\varepsilon,12}(s) \to \phi_{0,12}(s) = 0$ as $\varepsilon \to 0$, for $s \geq 0$ and, thus, using relation (31), we get $\hat{\phi}_{\varepsilon,11}(s) = \phi_{\varepsilon,11}(s) + \phi_{\varepsilon,12}(s)\frac{1}{1-\phi_{\varepsilon,22}(s)}\phi_{\varepsilon,21}(s) \to \phi_{0,11}(s)$ as $\varepsilon \to 0$, for $s \geq 0$. Thus, the distributions of regeneration times $\hat{Q}_{\varepsilon,11}(\cdot) \Rightarrow \hat{Q}_{0,11}(\cdot) = Q_{0,11}(\cdot)$ as $\varepsilon \to 0$. At the same time, conditions **G** – **I**, $\mathbf{N}_1$ and relation (32) imply that, in this case, $\hat{e}_{\varepsilon,11} = \frac{e_{\varepsilon,1}p_{\varepsilon,21}+e_{\varepsilon,2}p_{\varepsilon,12}}{p_{\varepsilon,21}} \to e_{0,1} + e_{0,2}\beta = e_{0,11} + e_{0,22}\beta$ as $\varepsilon \to 0$. This makes it impossible to use Theorems 1 – 3, which require convergence of expectations for regeneration times to the first moment of the corresponding limiting distribution for regeneration times. In the above case, $\hat{e}_{0,11} = e_{0,11} \neq e_{0,11} + e_{0,22}\beta$, if $\beta > 0$.

Fortunately, we can use an alternative model with aggregated regeneration times $\check{\tau}_{\varepsilon,n}$ introduced in Subsection 4.2. Process $(\xi_\varepsilon(t), \eta_\varepsilon(t))$ is a regenerative process with regeneration times $\check{\tau}_{\varepsilon,n}, n = 0, 1, \ldots$. It is also a regenerative process with the transition period $[0, \check{\tau}'_{\varepsilon,1})$ and regeneration times $\check{\tau}'_{\varepsilon,n}, n = 0, 1, \ldots$.

Let us analyse the asymptotic behaviour for probabilities $P_{\varepsilon,11}(t, A)$. In this case, we do prefer to consider $(\xi_\varepsilon(t), \eta_\varepsilon(t)), t \geq 0$ as the standard regenerative process with regeneration times $\check{\tau}_{\varepsilon,n}, n = 0, 1, \ldots$.

The renewal equation (4) for probabilities $P_{\varepsilon,11}(t, A)$ takes, in this case, the following form,

$$P_{\varepsilon,11}(t, A) = \check{q}_{\varepsilon,1}(t, A) + \int_0^t P_{\varepsilon,11}(t-s, A)\check{Q}_{\varepsilon,11}(ds), t \geq 0, \tag{46}$$

where $\check{q}_{\varepsilon,1}(t, A) = \mathsf{P}_1\{\xi_\varepsilon(t) \in A, \eta_\varepsilon(t) = 1, \check{\tau}_{\varepsilon,1} > t\}, t \geq 0$ and $\check{Q}_{\varepsilon,11}(t) = \mathsf{P}_1\{\check{\tau}_{\varepsilon,1} \leq t\}, t \geq 0$.

If $\eta_\varepsilon(0) = 1$, then $\eta_\varepsilon(t) = 1$ for $t \in [0, \tilde{\tau}_{\varepsilon,1})$, and $\eta_\varepsilon(t) = 2$, for $t \in [\tilde{\tau}_{\varepsilon,1}, \check{\tau}_{\varepsilon,1})$. Therefore, for every $A \in \mathcal{B}_\mathbb{X}, t \geq 0$,

$$\begin{aligned}\check{q}_{\varepsilon,1}(t, A) &= \mathsf{P}_1\{\xi_\varepsilon(t) \in A, \eta_\varepsilon(t) = 1, \check{\tau}_{\varepsilon,1} > t\} \\ &= \mathsf{P}_1\{\xi_\varepsilon(t) \in A, \eta_\varepsilon(t) = 1, \tilde{\tau}_{\varepsilon,1} > t\} = \tilde{q}_{\varepsilon,1}(t, A).\end{aligned} \tag{47}$$

In this case, $\check{Q}_{\varepsilon,11}(t)$ is the distribution function of the first return time to state 1 after first hitting to state 2, for the semi-Markov process $\eta_\varepsilon(t)$. It can be expressed in terms of convolutions of transition probabilities for this semi-Markov process,

$$\check{Q}_{\varepsilon,11}(t) = \tilde{Q}_{\varepsilon,12}(t) * \tilde{Q}_{\varepsilon,21}(t), t \geq 0. \tag{48}$$

where, for $i, j \in \mathbb{Y}, i \neq j$,

$$\tilde{Q}_{\varepsilon,ij}(t) = \sum_{n=0}^{\infty} Q^{*n}_{\varepsilon,ii}(t) * Q_{\varepsilon,ij}(t), \ t \geq 0, \tag{49}$$

According relation (48), the distribution function $\check{Q}_{\varepsilon,11}(t)$ of return time $\check{\tau}_{\varepsilon,1}$ is the convolution of two distribution functions, $\tilde{Q}_{\varepsilon,12}(t)$ and $\tilde{Q}_{\varepsilon,21}(t)$. This means that return time $\check{\tau}_{\varepsilon,1}$ is the sum of two independent random variables $\tilde{\tau}_{\varepsilon,1}$ and $\check{\tau}_{\varepsilon,1} - \tilde{\tau}_{\varepsilon,1}$, which have the distribution functions, respectively, $\tilde{Q}_{\varepsilon,12}(t)$ and $\tilde{Q}_{\varepsilon,21}(t)$. The former one is the distribution of the first hitting time of state 2 from state 1, the latter one is the distribution of the first hitting time of state 1 from state 2, for the semi-Markov process $\eta_\varepsilon(t)$.

Remind that we assume that $\eta_\varepsilon(0) = \eta_\varepsilon = 1$. In this case, (a) the return time $\tilde{\tau}_{\varepsilon,1}$ is a random sum, $\tilde{\tau}_{\varepsilon,1} = \sum_{n=1}^{\theta_\varepsilon[0]} \kappa_{\varepsilon,1,n}$, where (b) the random index, $\theta_\varepsilon[0] = \min(n \geq 1 : \eta_{\varepsilon,1,n} = 1)$ has the geometric distribution with parameter $p_{\varepsilon,12}$, i.e., it takes value n with probability $p^{n-1}_{\varepsilon,11} p_{\varepsilon,12}$, for $n = 1, 2, \ldots$.

Relation (b) and condition $\mathbf{N}_1$ imply that random variables,

$$p_{\varepsilon,12}\theta_\varepsilon[0] \xrightarrow{d} \zeta \text{ as } \varepsilon \to 0, \tag{50}$$

where ζ is a random variable exponentially distributed, with parameter 1.

Random variables $\kappa_{\varepsilon,1,n}, n = 1, 2, \ldots$ are i.i.d. random variables with the distribution function $F_{\varepsilon,1}(t) = \mathsf{P}_1\{\kappa_{\varepsilon,1,1} \leq t\} = Q_{\varepsilon,11}(t) + Q_{\varepsilon,12}(t), t \geq 0$. Conditions $\mathbf{H}$ and $\mathbf{I}$ imply that (c) distributions $F_{\varepsilon,1}(\cdot) \Rightarrow F_{0,1}(\cdot)$ as $\varepsilon \to 0$ and (d) expectations $e_{\varepsilon,1} = \mathsf{E}_1\kappa_{\varepsilon,1,1} = \int_0^\infty sF_{\varepsilon,1}(ds) \to e_{0,1} = \int_0^\infty sF_{0,1}(ds)$ as $\varepsilon \to 0$.

Relations (c) and (d) imply that, for any integer-valued function $0 \leq n_\varepsilon \to \infty$ as $\varepsilon \to 0$,

$$n_\varepsilon^{-1} \sum_{k=1}^{n_\varepsilon} \kappa_{\varepsilon,1,n} \xrightarrow{d} e_{0,1} \text{ as } \varepsilon \to 0. \tag{51}$$

Indeed, let $0 < s_k \to \infty$ as $k \to \infty$ be a sequence of continuity points for the distribution function $F_{0,1}(t)$. The above relations (c) and (d) obviously imply that, for any $t > 0$,

$$\begin{aligned} \overline{\lim_{\varepsilon \to 0}} \int_{tn_\varepsilon}^\infty sF_{\varepsilon,1}(ds) &\leq \overline{\lim_{\varepsilon \to 0}} \int_{s_k}^\infty sF_{\varepsilon,1}(ds) = \overline{\lim_{\varepsilon \to 0}}(e_{\varepsilon,1} - \int_0^{s_k} sF_{\varepsilon,1}(ds)) \\ &= e_{0,1} - \int_0^{s_k} sF_{0,1}(ds) \to 0 \text{ as } k \to \infty, \end{aligned} \tag{52}$$

and, thus, the following relation holds, for any $t > 0$,

$$\overline{\lim_{\varepsilon \to 0}} \int_{tn_\varepsilon}^\infty sF_{\varepsilon,1}(ds) = 0. \tag{53}$$

Relation (53) implies that, for any $t > 0$,

$$n_\varepsilon \mathsf{P}_1\{n_\varepsilon^{-1}\kappa_{\varepsilon,1,1} > t\} = n_\varepsilon(1 - F_{\varepsilon,1}(tn_\varepsilon)) \\ \leq t^{-1}\int_{tn_\varepsilon}^{\infty} sF_{\varepsilon,1}(ds) \to 0 \text{ as } \varepsilon \to 0. \tag{54}$$

Also, relations (d) and (53) implies that, for any $t > 0$,

$$n_\varepsilon \mathsf{E}_1 n_\varepsilon^{-1}\kappa_{\varepsilon,1,1}\mathrm{I}(n_\varepsilon^{-1}\kappa_{\varepsilon,1,1} \leq t) = \int_0^{tn_\varepsilon} sdF_{\varepsilon,1}(ds) \to e_{0,1} \text{ as } \varepsilon \to 0. \tag{55}$$

Relations (54) and (55) imply, by the criterion of central convergence, that relation (51) holds.

The random index $\theta_\varepsilon[0]$ and the random variables $\kappa_{\varepsilon,1,n}, n = 1, 2, \ldots$ are dependent. Nevertheless, since the limit in relation (51) is non-random, relations (50) and (51) imply that stochastic processes,

$$(p_{\varepsilon,12}\theta_\varepsilon[0], \sum_{n \leq tp_{\varepsilon,12}^{-1}} \kappa_{\varepsilon,1,n}, t \geq 0 \xrightarrow{d} (\zeta, te_{0,1}), t \geq 0 \text{ as } \varepsilon \to 0. \tag{56}$$

By well known results about convergence of randomly stopped stochastic processes (for example, Theorem 2.2.1 [47]), representation (a) and relation (56) imply that random variables,

$$p_{\varepsilon,12}\tilde{\tau}_{\varepsilon,1} \xrightarrow{d} e_{0,1}\zeta \text{ as } \varepsilon \to 0. \tag{57}$$

Since, $p_{\varepsilon,12}^{-1}/v_\varepsilon \to (1+\beta)^{-1}$ as $\varepsilon \to 0$, relation (k) implies the following relation,

$$\tilde{Q}_{\varepsilon,v_\varepsilon,12}(\cdot) \Rightarrow \tilde{Q}_{0,v_0,12}(\cdot) \text{ as } \varepsilon \to 0, \tag{58}$$

where $\tilde{Q}_{0,v_0,12}(t) = \mathsf{P}\{e_{0,1}\frac{1}{1+\beta}\zeta \leq t\}, t \geq 0$ is the distribution function of an exponentially distributed random variable, with parameter $e_{0,1}^{-1}(1+\beta)$.

Since the random variable $\check{\tau}_{\varepsilon,1} - \tilde{\tau}_{\varepsilon,1}$ has distribution function $\tilde{Q}_{\varepsilon,21}(t)$, one can, in the way absolutely analogous with relation (57), prove the following relation,

$$p_{\varepsilon,21}(\check{\tau}_{\varepsilon,1} - \tilde{\tau}_{\varepsilon,1}) \xrightarrow{d} e_{0,2}\zeta \text{ as } \varepsilon \to 0, \tag{59}$$

and, in sequel,

$$\tilde{Q}_{\varepsilon,v_\varepsilon,21}(\cdot) \Rightarrow \tilde{Q}_{0,v_0,21}(\cdot) \text{ as } \varepsilon \to 0, \tag{60}$$

where $\tilde{Q}_{0,v_0,21}(t) = \mathsf{P}\{e_{0,2}\frac{1}{1+\beta^{-1}}\zeta \leq t\}, t \geq 0$ is the distribution function of an exponentially distributed random variable, with parameter $e_{0,2}^{-1}(1+\beta^{-1})$.

Let $\check{Q}_{\varepsilon,v_\varepsilon,11}(t) = \check{Q}_{\varepsilon,11}(tv_\varepsilon) = \mathsf{P}_1\{\check{\tau}_{\varepsilon,1}/v_\varepsilon \leq t\}, t \geq 0$ be the distribution function of the normalised return time $v_\varepsilon^{-1}\check{\tau}_{\varepsilon,1}$.

Relations (48), (58) and (60) imply that,

$$\check{Q}_{\varepsilon,v_\varepsilon,11}(\cdot) \Rightarrow \check{Q}_{0,v_0,11}(\cdot) \text{ as } \varepsilon \to 0, \tag{61}$$

where $\check{Q}_{0,v_0,11}(t) = \mathsf{P}\{e_{0,1}\frac{1}{1+\beta}\zeta_1 + e_{0,2}\frac{1}{1+\beta^{-1}}\zeta_2 \leq t\}, t \geq 0$ is the distribution function of the linear combination of two independent random variables ζ_1 and ζ_2, exponentially distributed, with parameter 1.

Note that in that cases $\beta = 0$ or $\beta = \infty$, respectively, the second or the first random variable in the above sum vanishes in zero. In this case, $\check{Q}_{0,v_0,11}(t)$ is an exponential distribution function with parameter, respectively, $e_{0,1}^{-1}$ or $e_{0,2}^{-1}$.

Also, that above representation (a) for the random variable $\tilde{\tau}_{\varepsilon,1}$, as the random sum, implies that,

$$\begin{aligned}
\tilde{e}_{\varepsilon,v_\varepsilon,12} &= \int_0^\infty s\tilde{Q}_{\varepsilon,v_\varepsilon,12}(ds) = v_\varepsilon^{-1}\mathsf{E}\sum_{n=1}^{\theta_\varepsilon[0]} \kappa_{\varepsilon,1,n} \\
&= v_\varepsilon^{-1}\mathsf{E}\sum_{n=1}^{\infty} \kappa_{\varepsilon,1,n}\mathrm{I}(\theta_\varepsilon[0] > n-1) \\
&= v_\varepsilon^{-1}\mathsf{E}\sum_{n=1}^{\infty} \kappa_{\varepsilon,1,n}\mathrm{I}(\eta_{\varepsilon,1,k} = 1, 1 \leq k \leq n-1) \\
&= v_\varepsilon^{-1}\sum_{n=1}^{\infty} \mathsf{E}\kappa_{\varepsilon,1,n}\mathsf{E}\mathrm{I}(\eta_{\varepsilon,1,k} = 1, 1 \leq k \leq n-1) \\
&= v_\varepsilon^{-1}\sum_{n=1}^{\infty} e_{\varepsilon,1}p_{\varepsilon,11}^{n-1} = \frac{e_{\varepsilon,1}}{v_\varepsilon p_{\varepsilon,12}}.
\end{aligned} \tag{62}$$

Analogous formula also takes place,

$$\tilde{e}_{\varepsilon,v_\varepsilon,21} = v_\varepsilon^{-1}\mathsf{E}(\check{\tau}_{\varepsilon,1} - \tilde{\tau}_{\varepsilon,1}) = \frac{e_{\varepsilon,2}}{v_\varepsilon p_{\varepsilon,21}}. \tag{63}$$

Relations (62) and (63) imply the following relation,

$$\begin{aligned}
\check{e}_{\varepsilon,v_\varepsilon,11} &= \tilde{e}_{\varepsilon,v_\varepsilon,12} + \tilde{e}_{\varepsilon,v_\varepsilon,21} = e_{\varepsilon,1}\frac{1}{v_\varepsilon p_{\varepsilon,12}} + e_{\varepsilon,2}\frac{1}{v_\varepsilon p_{\varepsilon,21}} \\
&\to e_{0,1}\frac{1}{1+\beta} + e_{0,2}\frac{1}{1+\beta^{-1}} = \check{e}_{0,v_0,11} = \int_0^\infty s\check{Q}_{0,v_0,11}(ds).
\end{aligned} \tag{64}$$

The above remarks prompt us how to apply the scaling of time transformation with compression function v_ε, described in Subsection 5.1, to the regenerative process $(\xi_\varepsilon(t), \eta_\varepsilon(t)), t \geq 0$ with regeneration times $\check{\tau}_{\varepsilon,n}, n = 0, 1, \ldots$.

So, let us consider, for every $\varepsilon \in (0,1]$, the compressed in time version of the regenerative process $(\xi_\varepsilon(t), \eta_\varepsilon(t)), t \geq 0$ with regeneration times $\check{\tau}_{\varepsilon,n}, n = 0, 1, \ldots$. It is the regenerative process $(\xi_{\varepsilon,v_\varepsilon}(t), \eta_{\varepsilon,v_\varepsilon}(t)), t \geq 0 = (\xi_\varepsilon(tv_\varepsilon), \eta_\varepsilon(tv_\varepsilon)), t \geq 0$ with regeneration times $\tau_{\varepsilon,v_\varepsilon,n} = v_\varepsilon^{-1}\check{\tau}_{\varepsilon,n}, n = 0, 1, \ldots$.

The renewal type equation (4) takes for probabilities $P_{\varepsilon,v_\varepsilon,11}(t, A) = P_{\varepsilon,11}(tv_\varepsilon, A)$ the following form,

$$P_{\varepsilon,v_\varepsilon,11}(t, A) = \check{q}_{\varepsilon,v_\varepsilon,1}(t, A) + \int_0^t P_{\varepsilon,v_\varepsilon,11}(t-s, A)\check{Q}_{\varepsilon,v_\varepsilon,11}(ds), t \geq 0, \tag{65}$$

where $\check{q}_{\varepsilon,v_\varepsilon,1}(t, A) = \mathsf{P}_1\{\xi_{\varepsilon,v_\varepsilon,}(t) \in A, \eta_{\varepsilon,v_\varepsilon}(t) = 1, \check{\tau}_{\varepsilon,v_\varepsilon,1} > t\} = \check{q}_{\varepsilon,1}(tv_\varepsilon, A) = \mathsf{P}_1\{\xi_\varepsilon(tv_\varepsilon) \in A, \eta_\varepsilon(tv_\varepsilon) = 1, v_\varepsilon^{-1}\check{\tau}_{\varepsilon,1} > t\}, t \geq 0$ and $\check{Q}_{\varepsilon,v_\varepsilon,11}(t) = \mathsf{P}_1\{\check{\tau}_{\varepsilon,v_\varepsilon,1} \leq t\} = \check{Q}_{\varepsilon,11}(tv_\varepsilon) = \mathsf{P}_1\{v_\varepsilon^{-1}\check{\tau}_{\varepsilon,1} \leq t\}, t \geq 0$.

We shall define the corresponding limiting regenerative process $(\xi_{0,v_0}(t), \eta_{0,v_0}(t)), t \geq 0$ and the regeneration times $\check{\tau}_{0,v_0,n}, n = 0, 1, \ldots$ in the next subsection, after computing the corresponding limits for functions $\check{q}_{\varepsilon,v_\varepsilon,1}(t, A)$ and distribution functions $\check{Q}_{\varepsilon,v_\varepsilon,11}(t)$.

4.3. Locally uniform convergence of functions and convergence of Lebesgue integrals in the scheme of series. In this subsection, we formulate two useful propositions concerned locally uniform convergence of functions and convergence of Lebesgue integrals in the scheme of series. The proofs can be found, for example, in book [14]. We slightly modify these propositions for the case, where the corresponding functions and measures are defined on a half-line.

Let $f_\varepsilon(s)$ be, for every $\varepsilon \in [0,1]$, a real-valued bounded Borel functions defined on $\mathbb{R}_+ = [0, \infty)$. We use the symbol $f_\varepsilon(s) \stackrel{\mathrm{U}}{\longrightarrow} f_0(s)$ as $\varepsilon \to 0$ to indicate that functions $f_\varepsilon(\cdot)$ converge to function $f_0(\cdot)$ locally uniformly at a point $s \in [0, \infty)$ as $\varepsilon \to 0$. This means that,

$$\lim_{0<u\to 0} \overline{\lim_{\varepsilon\to 0}} \sup_{-(u\wedge s)\leq v\leq u} |f_\varepsilon(s+v) - f_0(s)| = 0. \tag{66}$$

Lemma 1. *Functions $f_\varepsilon(s) \stackrel{\mathrm{U}}{\longrightarrow} f_\varepsilon(s)$ as $\varepsilon \to 0$ if and only if* $(\boldsymbol{\alpha})$ *$f_\varepsilon(s_\varepsilon) \to f_0(s)$ as $\varepsilon \to 0$, for any $0 \leq s_\varepsilon \to s$ as $\varepsilon \to 0$.*

Let $\mathcal{B}_+$ denote the Borel σ-algebra on $\mathbb{R}_+$ and let $\mu_\varepsilon(A)$ be, for every $\varepsilon \in [0,1]$, a finite measure on $\mathcal{B}_+$. We use the symbol $\mu_\varepsilon(\cdot) \Rightarrow \mu_0(\cdot)$ as $\varepsilon \to 0$ to indicate that the measures $\mu_\varepsilon(A)$ weakly converge to a measure $\mu_0(A)$ as $\varepsilon \to 0$. This means that, for all $0 \leq v < \infty$ such that the limiting measure has not an atom in the point v,

$$\mu_\varepsilon([0, v]) \to \mu_0([0, v]) \text{ as } \varepsilon \to 0. \tag{67}$$

Lemma 2. *Let the following conditions hold*: **(α)** $\mu_\varepsilon(\cdot) \Rightarrow \mu_0(\cdot)$ *as* $\varepsilon \to 0$; **(β)** $\mu_\varepsilon(\mathbb{R}_+) \to \mu_0(\mathbb{R}_+)$ *as* $\varepsilon \to 0$; **(γ)** $\overline{\lim}_{\varepsilon\to 0}\sup_{s\in\mathbb{R}_+}|f_\varepsilon(s)| < \infty$; **($\delta$)** $f_\varepsilon(s) \stackrel{\mathrm{U}}{\longrightarrow} f_0(s)$ *as* $\varepsilon \to 0$, *for* $s \in S$, *where* S *is some subset of* $\mathcal{B}_+$ *such that* $\mu_0(\overline{S}) = 0$. *Then,*

$$\int_{\mathbb{R}_+} f_\varepsilon(s)\mu_\varepsilon(ds) \to \int_{\mathbb{R}_+} f_0(s)\mu_0(ds) \text{ as } \varepsilon \to 0. \tag{68}$$

4.4. Super-long time ergodic theorems for singularly perturbed alternating regenerative processes. In this subsection, we describe the asymptotic behaviour for probabilities $P_{\varepsilon,ij}(t_\varepsilon, A)$ for so-called "super-long" times $0 \leq t_\varepsilon \to \infty$ as $\varepsilon \to 0$ satisfying the following relation,

$$t_\varepsilon/v_\varepsilon \to \infty \text{ as } \varepsilon \to 0. \tag{69}$$

The corresponding limits for stationary probabilities for perturbed semi-Markov processes $\eta_{\varepsilon,v_\varepsilon}(t)$ take, for $\beta \in [0,\infty]$, the following form,

$$\rho_1(\beta) = e_{0,1}\alpha_1(\beta)/e(\beta),\ \rho_2(\beta) = e_{0,2}\alpha_2(\beta)/e(\beta), \tag{70}$$

where

$$\alpha_1(\beta) = (1+\beta)^{-1},\ \alpha_2(\beta) = (1+\beta^{-1})^{-1}, \tag{71}$$

and

$$e(\beta) = e_{0,1}\alpha_1(\beta) + e_{0,2}\alpha_2(\beta). \tag{72}$$

Note that $\rho_1(\beta), \rho_2(\beta) \in (0,1)$, if $\beta \in (0,\infty)$, while $\rho_1(\beta) = 1, \rho_2(\beta) = 0$, if $\beta = 0$, and $\rho_1(\beta) = 0, \rho_2(\beta) = 1$, if $\beta = \infty$.

The corresponding limiting probabilities for singularly perturbed alternating regenerative processes take the following form,

$$\pi_{0,j}^{(\beta)}(A) = \rho_j(\beta)\pi_{0,j}(A),\ A \in \mathcal{B}_{\mathbb{X}}, j = 1,2. \tag{73}$$

It is useful to note that the above limiting probabilities coincide with the corresponding limiting probabilities for regularly perturbed alternating regenerative processes with parameter $\beta = p_{0,12}/p_{0,21}$ given in relations (26), (36), and (37).

The following theorem takes place.

Theorem 8. *Let conditions* **F** – **J**, $\mathbf{N}_1$ *hold and, also, condition* $\mathbf{K}_\beta$ *holds for some* $\beta \in [0,\infty]$. *Then, for every* $A \in \Gamma, i,j = 1,2$, *and* $0 \leq t_\varepsilon \to \infty$ *as* $\varepsilon \to 0$ *such that* $t_\varepsilon/v_\varepsilon \to \infty$ *as* $\varepsilon \to 0$,

$$P_{\varepsilon,ij}(t_\varepsilon, A) \to \pi_{0,j}^{(\beta)}(A) \text{ as } \varepsilon \to 0. \tag{74}$$

Proof. We are going to prove that all conditions of Theorem 1 hold for the regenerative processes $(\xi_{\varepsilon,v_\varepsilon}(t), \eta_{\varepsilon,v_\varepsilon}(t)) = (\xi_\varepsilon(tv_\varepsilon), \eta_\varepsilon(tv_\varepsilon)), t \geq 0$ with regeneration times $\check{\tau}_{\varepsilon,v_\varepsilon,n} = v_\varepsilon^{-1}\check{\tau}_{\varepsilon,v_\varepsilon,n}, n = 0, 1, \ldots$ and, also, that all conditions of Theorem 2 hold for the regenerative processes $(\xi_{\varepsilon,v_\varepsilon}(t), \eta_{\varepsilon,v_\varepsilon}(t)) = (\xi_\varepsilon(tv_\varepsilon), \eta_\varepsilon(tv_\varepsilon)), t \geq 0$ with the transition period $[0, \check{\tau}'_{\varepsilon,v_\varepsilon,1})$ and regeneration times $\check{\tau}'_{\varepsilon,v_\varepsilon,n} = v_\varepsilon^{-1}\check{\tau}'_{\varepsilon,v_\varepsilon,n}, , n = 0, 1, \ldots$.

Note that, the regenerative lifetimes are not involved. The corresponding stopping probabilities $f_{\varepsilon,v_\varepsilon} = 0, \varepsilon \in (0, 1]$.

First, let us analyse the asymptotic behaviour of probabilities $\check{q}_{\varepsilon,v_\varepsilon,1}(t, A)$. Here, we can use the quasi-stationary ergodic relation given in Theorem 3.

Let us introduce random variables $\mu_{\varepsilon,1,n} = \kappa_{\varepsilon,1,n}\mathrm{I}(\eta_{\varepsilon,1,n} = 1), n = 1, 2, \ldots$. Let now consider the sequence of random triplets $\langle \bar{\xi}_{\varepsilon,1,n} = \langle \xi_{\varepsilon,1,n}(t), t \geq 0\rangle, \kappa_{\varepsilon,1,n}, \mu_{\varepsilon,1,n}\rangle, n = 1, 2, \ldots$, the regenerative process $\xi_{\varepsilon,1}(t) = \xi_{\varepsilon,1,n}(t - \tau_{\varepsilon,1,n-1})$, for $t \in [\tau_{\varepsilon,1,n-1}, \tau_{\varepsilon,1,n})$, $n = 1, 2, \ldots$, with regeneration times $\tau_{\varepsilon,1,n} = \kappa_{\varepsilon,1,1} + \cdots + \kappa_{\varepsilon,1,n}, n = 1, 2, \ldots, \tau_{\varepsilon,1,0} = 0$, and the regenerative lifetime $\mu_{\varepsilon,1,+} = \tau_{\varepsilon,1,\nu_{\varepsilon,1}}$, where $\nu_{\varepsilon,1} = \min(n \geq 1 : \mu_{\varepsilon,1,n} < \kappa_{\varepsilon,1,n}) = \min(n \geq 1 : \eta_{\varepsilon,1,n} = 2)$.

Let us also denote $P_{\varepsilon,1,+}(t, A) = \mathsf{P}_1\{\xi_{\varepsilon,1}(t) \in A, \mu_{\varepsilon,1,+} > t\}$. In this case, the distribution function $F_{\varepsilon,1}(t) = \mathsf{P}\{\kappa_{\varepsilon,1,1} \leq t, \mu_{\varepsilon,1,1} \geq \kappa_{\varepsilon,1,1}\} = \mathsf{P}\{\kappa_{\varepsilon,1,1} \leq t, \eta_{\varepsilon,1,1} = 1\}, t \geq 0$, the stopping probability $f_{\varepsilon,1} = \mathsf{P}\{\mu_{\varepsilon,1,1} < \kappa_{\varepsilon,1,1}\} = \mathsf{P}\{\eta_{\varepsilon,1,1} = 2\} = p_{\varepsilon,12}$, and the expectation $e_{\varepsilon,1} = \mathsf{E}\kappa_{\varepsilon,1,1}\mathrm{I}(\mu_{\varepsilon,1,1} \geq \kappa_{\varepsilon,1,1}) = \mathsf{E}\kappa_{\varepsilon,1,1}\mathrm{I}(\eta_{\varepsilon,1,1} = 1) = e_{\varepsilon,11}$.

It is also readily seen that, for every $A \in \mathcal{B}_{\mathbb{X}}, t \geq 0$,

$$
\begin{aligned}
\check{q}_{\varepsilon,1}(t, A) &= \mathsf{P}_1\{\xi_\varepsilon(t) \in A, \eta_\varepsilon(t) = 1, \check{\tau}_{\varepsilon,1} > t\} \\
&= \mathsf{P}_1\{\xi_{\varepsilon,1}(t) \in A, \mu_{\varepsilon,1,+} > t\} = P_{\varepsilon,1,+}(t, A). \qquad (75)
\end{aligned}
$$

Conditions **F** – **J** and $\mathbf{N}_1$ imply that conditions **A** – **D** holds for the regenerative processes $\xi_{\varepsilon,1}(t), t \geq 0$ with regenerative times $\tau_{\varepsilon,1,n}, n = 1, 2, \ldots$ and regenerative lifetimes $\mu_{\varepsilon,1,+}$.

Let $s \in (0, \infty)$. We choose an arbitrary $0 \leq s_\varepsilon \to s$ as $\varepsilon \to 0$.

The above relation obviously implies that $s_\varepsilon v_\varepsilon \to \infty$. Conditions $\mathbf{N}_1$ and $\mathbf{K}_\beta$ obviously imply that $f_{\varepsilon,1}s_\varepsilon v_\varepsilon = p_{\varepsilon,12}s_\varepsilon v_\varepsilon = s_\varepsilon(1 + p_{\varepsilon,12}/p_{\varepsilon,21}) \to t_{s,\beta} = s(1 + \beta)$ as $\varepsilon \to 0$. Note that $t_{s,\beta} \in (0, \infty)$, for $\beta \in [0, \infty)$, while $t_{s,\infty} = \infty$.

Thus, all conditions of Theorem 3 holds for the regenerative processes $\xi_{\varepsilon,1}(t), t \geq 0$ with the regenerative times $\tau_{\varepsilon,1,n}, n = 1, 2, \ldots$ and the regenerative lifetimes $\mu_{\varepsilon,1,+}$. Therefore, the following relation holds, for any $A \in \Gamma$, and $s \in (0, \infty)$,

$$
\begin{aligned}
P_{\varepsilon,1,+}(s_\varepsilon v_\varepsilon, A) &= \check{q}_{\varepsilon,1}(s_\varepsilon v_\varepsilon, A) = \check{q}_{\varepsilon,v_\varepsilon,1}(s_\varepsilon, A) \\
&\to \check{q}_{0,v_0,1}(s, A) = e^{-t_{s,\beta}/e_{0,1}}\pi_{0,1}(A) \text{ as } \varepsilon \to 0. \qquad (76)
\end{aligned}
$$

If $\beta \in [0, \infty)$, then the limiting function $\check{q}_{0,v_0,1}(s, A) = e^{-s(1+\beta)/e_{0,1}}\pi_{0,1}(A)$, $s \in (0, \infty)$ is a non-trivial exponential function. However, if $\beta = \infty$, the limiting function $\check{q}_{0,v_0,1}(s, A) = 0, s \in (0, \infty)$.

In both cases, we can define $\check{q}_{0,v_0,1}(0, A) = \lim_{0<s\to 0} \check{q}_{0,v_0,1}(s, A)$.

Obviously, $\check{q}_{0,v_0,1}(0, A) = \pi_{0,1}(A)$, if $\beta \in [0, \infty)$, and $\check{q}_{0,v_0,1}(0, A) = 0$, if $\beta = \infty$.

Recall also the limiting distribution function $\check{Q}_{0,v_0,11}(t) = \mathsf{P}\{e_{0,1}\frac{1}{1+\beta}\zeta_1 + e_{0,2}\frac{1}{1+\beta^{-1}}\zeta_2 \leq t\}, t \geq 0$ given by relation (61). Here, ζ_1 and ζ_2 are two independent random variables, exponentially distributed, with parameter 1.

Now, we are prepared to define the corresponding limiting regenerative process $(\xi_{0,v_0}(t), \eta_{0,v_0}(t)), t \geq 0$ with regeneration times $\check{\tau}_{0,v_0,n}, n = 0, 1, \ldots$.

Let us $\xi_{i,n}, \kappa_{i,n}, i = 1, 2, n = 1, 2, \ldots$ be random variables, for which we assume that: (a) they are mutually independent; (b) their distributions do not depend on $n \geq 1$; (c) $\xi_{i,n}$ are random variables taking values in space $\mathbb{X}$ and such that, $\mathsf{P}\{\xi_{i,1} \in A\} = \pi_i(A), A \in \mathcal{B}_{\mathbb{X}}$, for $i = 1, 2$; (d) the random variables $\kappa_{i,n}$ are non-negative random variables and $\mathsf{P}\{\kappa_{1,1} \leq t\} = \mathsf{P}\{e_{0,1}\frac{1}{1+\beta}\zeta_1 \leq t\}, t \geq 0$ while $\mathsf{P}\{\kappa_{2,1} \leq t\} = \mathsf{P}\{e_{0,2}\frac{1}{1+\beta^{-1}}\zeta_2 \leq t\}, t \geq 0$.

Now, let us define the inter-regeneration times $\kappa_{0,v_0,n} = \kappa_{1,n} + \kappa_{2,n}, n = 1, 2, \ldots$, the regeneration times $\tau_{0,v_0,n} = \kappa_{0,v_0,1} + \cdots + \kappa_{0,v_0,n}, n = 1, 2, \ldots, \tau_{0,v_0,0} = 0$, and the regeneration process $\xi_{0,v_0}(t) = \xi_{1,n}$, $\eta_{0,v_0}(t) = 1$, for $t \in [\tau_{0,v_0,n-1}, \tau_{0,v_0,n-1} + \kappa_{1,n})$ and $\xi_{0,v_0}(t) = \xi_{2,n}$, $\eta_{0,v_0}(t) = 2$, for $t \in [\tau_{0,v_0,n-1} + \kappa_{1,n}, \tau_{0,v_0,n})$, for $n = 0, 1, \ldots$.

It is readily seen that $\mathsf{P}\{\xi_{0,v_0}(t) \in A, \tau_{0,v_0,1} > t\} = \check{q}_{0,v_0,1}(t, A), t \geq 0$ and $\mathsf{P}\{\tau_{0,v_0,1} \leq t\} = \check{Q}_{0,v_0,11}(t), t \geq 0$, where $\check{q}_{0,v_0,1}(t, A)$ and $\check{Q}_{0,v_0,11}(t)$ are given, respectively, by relations (76) and (61).

Therefore, the renewal equation (4) for probabilities $P_{0,v_0,11}(t, A) = \mathsf{P}\{\xi_{0,v_0}(t) \in A, \eta_{0,v_0}(t) = 1\}$ takes the following form,

$$P_{0,v_0,11}(t, A) = \check{q}_{0,v_0,1}(t, A) + \int_0^t P_{0,v_0,11}(t - s, A)\check{Q}_{0,v_0,11}(t), \ t \geq 0. \quad (77)$$

All conditions of Theorem 1 hold for the regenerative process $(\xi_{\varepsilon,v_\varepsilon}(t), \eta_{\varepsilon,v_\varepsilon}(t)), t \geq 0$ with regeneration times $\tau_{\varepsilon,v_\varepsilon,n}, n = 0, 1, \ldots$.

Indeed, Condition **F** implies that condition **A** holds for the above regenerative processes. Relation (61) and conditions **G**, **H**, **I** and $\mathbf{N}_1$ imply that condition **B** holds. Relation (64) and conditions and conditions **G**, **H**, **I** and $\mathbf{N}_1$ also imply that condition **C** holds.

Due to an arbitrary choice of $0 \leq s_\varepsilon \to s$ as $\varepsilon \to 0$, convergence in relation (76) is locally uniform in every point $s \in (0, \infty)$. Thus, by Lemma 1 given Subsection 4.3, the asymptotic relation in condition **D** holds for functions $\check{q}_{\varepsilon,v_\varepsilon,1}(s, A), s \in [0, \infty)$ for any $s \in (0, \infty)$. Convergence at point 0 is not guarantied. However, $m(\{0\}) = 0$. Thus, condition **D** holds for functions $\check{q}_{\varepsilon,v_\varepsilon,1}(s_\varepsilon, A), s \in [0,\infty)$, with the limiting function $\check{q}_{0,v_0,1}(s, A) = e^{-t_{s,\beta}/e_{0,11}}\pi_{0,1}(A), s \in [0,\infty)$.

By the above remarks, all conditions of Theorem 1 hold, and the ergodic relation given in this theorem takes place for probabilities $P_{\varepsilon,v_\varepsilon,11}(t'_\varepsilon, A) = P_{\varepsilon,11}(t'_\varepsilon v_\varepsilon, A)$ for any $0 \leq t'_\varepsilon \to \infty$ as $\varepsilon \to 0$,

$$P_{\varepsilon,v_\varepsilon,11}(t'_\varepsilon, A) = P_{\varepsilon,11}(t'_\varepsilon v_\varepsilon, A) \to \pi^{(\beta)}_{0,v_0,1}(A) = \frac{1}{\check{e}_{0,v_0,11}} \int_0^\infty e^{-t_{s,\beta}/e_{0,1}}\pi_{0,1}(A)m(ds) \text{ as } \varepsilon \to 0. \tag{78}$$

Relation (64) and formula $t_{s,\beta} = s(1+\beta)$ imply that probabilities $\pi^{(\beta)}_{0,v_0,1}(A)$ coincide with probabilities $\pi^{(\beta)}_{0,1}(A)$ given in relation (73). Indeed,

$$\pi^{(\beta)}_{0,v_0,1}(A) = \frac{1}{\check{e}_{0,v_0,11}} \int_0^\infty e^{-s(1+\beta)/e_{0,1}}\pi_{0,1}(A)m(ds) = \frac{e_{0,1}(1+\beta)^{-1}}{e_{0,1}(1+\beta)^{-1} + e_{0,2}(1+\beta^{-1})^{-1}}\pi_{0,1}(A) = \pi^{(\beta)}_{0,1}(A). \tag{79}$$

Thus, the following ergodic relation holds for any for $A \in \Gamma$ and $0 \leq t'_\varepsilon \to \infty$ as $\varepsilon \to 0$,

$$P_{\varepsilon,11}(t'_\varepsilon v_\varepsilon, A) \to \pi^{(\beta)}_{0,1}(A) \text{ as } \varepsilon \to 0. \tag{80}$$

Let us now consider the compressed version of the regenerative process $(\xi_\varepsilon(t), \eta_\varepsilon(t)), t \geq 0$ with the transition period $[0, \check{\tau}'_{\varepsilon,1})$ with regeneration times $\check{\tau}'_{\varepsilon,n}, n = 0, 1, \ldots$. It is the regenerative process $(\xi_{\varepsilon,v_\varepsilon}(t), \eta_{\varepsilon,v_\varepsilon}(t)), t \geq 0 = (\xi_\varepsilon(tv_\varepsilon), \eta_\varepsilon(tv_\varepsilon)), t \geq 0$ with regeneration times $\tau'_{\varepsilon,v_\varepsilon,n} = v_\varepsilon^{-1}\check{\tau}'_{\varepsilon,n}, n = 0, 1, \ldots$.

The shifted process $(\xi_{\varepsilon,v_\varepsilon}(\hat{\tau}'_{\varepsilon,v_\varepsilon,1} + t), \eta_{\varepsilon,v_\varepsilon}(\hat{\tau}'_{\varepsilon,v_\varepsilon,1} + t)), t \geq 0$ is a standard regenerative process. If $\eta_{\varepsilon,v_\varepsilon}(0) = 2$, then $\eta_{\varepsilon,v_\varepsilon}(\tilde{\tau}_{\varepsilon,v_\varepsilon,1}) = 1$. That is why, probabilities $P_{\varepsilon,v_\varepsilon,11}(t, A)$ play for this process the role of probabilities $P^{(1)}_\varepsilon(t, A)$ pointed out in Subsection 2.1.

Relation (60) and conditions **G**, **H**, **I** and $\mathbf{N}_1$ imply that condition **E** holds for the distribution functions $\tilde{Q}_{\varepsilon,v_\varepsilon,21}(t) = \mathsf{P}_2\{\tilde{\tau}_{\varepsilon,v_\varepsilon,1} \leq t\}, t \geq 0$.

Thus, all conditions of Theorem 2 hold, and the ergodic relation (80) for probabilities $P_{\varepsilon,v_\varepsilon,11}(t'_\varepsilon, A) = P_{\varepsilon,11}(t'_\varepsilon v_\varepsilon, A)$ also holds for probabilities $P_{\varepsilon,v_\varepsilon,21}(t'_\varepsilon, A) = P_{\varepsilon,21}(t'_\varepsilon v_\varepsilon, A)$.

Due to the symmetricity of conditions **G** – **J**, $\mathbf{K}_\beta$, and $\mathbf{N}_1$ with respect to the indices $i, j = 1, 2$, the ergodic relations, analogous to the mentioned above ergodic relations for probabilities $P_{\varepsilon,v_\varepsilon,11}(t'_\varepsilon, A) = P_{\varepsilon,11}(t'_\varepsilon v_\varepsilon, A)$ and $P_{\varepsilon,v_\varepsilon,21}(t'_\varepsilon, A) = P_{\varepsilon,21}(t'_\varepsilon v_\varepsilon, A)$, also take place for probabilities $P_{\varepsilon,v_\varepsilon,22}(t'_\varepsilon, A) = P_{\varepsilon,22}(t'_\varepsilon v_\varepsilon, A)$ and $P_{\varepsilon,v_\varepsilon,12}(t'_\varepsilon, A) = P_{\varepsilon,12}(t'_\varepsilon v_\varepsilon, A)$. They have the following forms, $P_{\varepsilon,v_\varepsilon,i2}(t'_\varepsilon, A) = P_{\varepsilon,i2}(t'_\varepsilon v_\varepsilon, A) \to \pi^{(\beta)}_{0,2}(A)$ as $\varepsilon \to 0$, for $i = 1, 2$.

The above analysis, in particular, relation (80), yields the description of asymptotic behaviour of probabilities $P_{\varepsilon,ij}(t_\varepsilon, A)$ for super-long times $0 \leq t_\varepsilon \to \infty$ as $\varepsilon \to 0$ satisfying the asymptotic relation $t_\varepsilon/v_\varepsilon \to \infty$ as $\varepsilon \to 0$. To see this, one should just represent such t_ε in the form, $t_\varepsilon = t'_\varepsilon v_\varepsilon$. Obviously, $t'_\varepsilon = t_\varepsilon/v_\varepsilon \to \infty$ as $\varepsilon \to 0$. □

4.5. Long time ergodic theorems for singularly perturbed alternating regenerative processes. In this subsection, we describe the asymptotic behaviour of probabilities $P_{\varepsilon,ij}(t_\varepsilon, A)$ for so-called "long" times $0 \leq t_\varepsilon \to \infty$ as $\varepsilon \to 0$, which satisfy asymptotic relation,

$$t_\varepsilon/v_\varepsilon \to t \in (0, \infty) \text{ as } \varepsilon \to 0. \tag{81}$$

Let $\beta \in (0, \infty)$, and $\eta^{(\beta)}(t), t \geq 0$ be a homogeneous continuous time Markov chain with the phase space $\mathbb{Y} = \{1, 2\}$, the transition probabilities of embedded Markov chain $p_{ij} = \mathrm{I}(i \neq j), i, j = 1, 2$, and the distribution functions of sojourn times in states 1 and 2, respectively, $F_1^{(\beta)}(t) = 1 - e^{-t(1+\beta)/e_{0,1}}, t \geq 0$ and $F_2^{(\beta)}(t) = 1 - e^{-t(1+\beta^{-1})/e_{0,2}}, t \geq 0$. We also assume that this Markov chain has continuous from the right trajectories.

Let us $p_{ij}^{(\beta)}(t) = \mathsf{P}_i\{\eta^{(\beta)}(t) = j\}, t \geq 0, i, j = 1, 2$ be transition probabilities for the Markov chain $\eta^{(\beta)}(t)$.

The explicit expression for the transition probabilities $p_{ij}^{(\beta)}(t)$ are well known, as the solutions of the corresponding forward Kolmogorov system of differential equations for these probabilities. Namely, the corresponding matrix $\|p_{ij}^{(\beta)}(t)\|$ has the following form, for $t \geq 0$,

$$\|p_{ij}^{(\beta)}(t)\| = \left\| \begin{array}{cc} \rho_1(\beta) + \rho_2(\beta)e^{-\lambda(\beta)t} & \rho_2(\beta) - \rho_2(\beta)e^{-\lambda(\beta)t} \\ \rho_1(\beta) - \rho_1(\beta)e^{-\lambda(\beta)t} & \rho_2(\beta) + \rho_1(\beta)e^{-\lambda(\beta)t} \end{array} \right\|, \tag{82}$$

where

$$\lambda_1(\beta) = \frac{1+\beta}{e_{0,1}}, \ \lambda_2(\beta) = \frac{1+\beta^{-1}}{e_{0,2}}, \ \lambda(\beta) = \lambda_1(\beta) + \lambda_2(\beta), \tag{83}$$

and

$$\rho_1(\beta) = \frac{\lambda_2(\beta)}{\lambda(\beta)} = \frac{e_{0,1}(1+\beta)^{-1}}{e(\beta)}, \ \rho_2(\beta) = \frac{\lambda_1(\beta)}{\lambda(\beta)} = \frac{e_{0,2}(1+\beta^{-1})^{-1}}{e(\beta)}. \tag{84}$$

Note that the Markov chain $\eta^{(\beta)}(t)$ is ergodic and $\rho_i(\beta), i = 1, 2$ are its stationary probabilities.

The corresponding limiting probabilities have in this case the following forms, for $A \in \mathcal{B}_{\mathbb{X}},\ i,j = 1,2,\ t \in (0,\infty)$,

$$\pi^{(\beta)}_{0,i1}(t,A) = \begin{cases} \pi_{0,1}(A) & \text{for } i = 1,2, \beta = 0, \\ p^{(\beta)}_{i1}(t)\pi_{0,1}(A) & \text{for } i = 1,2, \beta \in (0,\infty), \\ 0 & \text{for } i = 1,2, \beta = \infty, \end{cases} \tag{85}$$

and

$$\pi^{(\beta)}_{0,i2}(t,A) = \begin{cases} 0 & \text{for } i = 1,2, \beta = 0, \\ p^{(\beta)}_{i2}(t)\pi_{0,2}(A) & \text{for } i = 1,2, \beta \in (0,\infty), \\ \pi_{0,2}(A) & \text{for } i = 1,2, \beta = \infty. \end{cases} \tag{86}$$

The following theorem takes place.

Theorem 9. *Let conditions* $\mathbf{F} - \mathbf{J}$, $\mathbf{N}_1$ *hold and, also, condition* $\mathbf{K}_\beta$ *holds for some* $\beta \in [0,\infty]$. *Then, for every* $A \in \Gamma, i,j = 1,2,$ *and* $0 \leq t_\varepsilon \to \infty$ *as* $\varepsilon \to 0$ *such that* $t_\varepsilon / v_\varepsilon \to t \in (0,\infty)$ *as* $\varepsilon \to 0$,

$$P_{\varepsilon,ij}(t_\varepsilon, A) \to \pi^{(\beta)}_{0,ij}(t,A) \text{ as } \varepsilon \to 0. \tag{87}$$

Proof. Let us again us consider the renewal equation (77) for the compressed regenerative process $(\xi_{\varepsilon,v_\varepsilon}(t), \eta_{\varepsilon,v_\varepsilon}(t)), t \geq 0 = (\xi_\varepsilon(tv_\varepsilon), \eta_\varepsilon(tv_\varepsilon)), t \geq 0$ with regeneration times $\check{\tau}_{\varepsilon,v_\varepsilon,n} = v_\varepsilon^{-1}\check{\tau}_{\varepsilon,n}, n = 0,1,\ldots$.

As well known, the solution of this equation has the form,

$$P_{\varepsilon,v_\varepsilon,11}(t,A) = \int_0^t \check{q}_{\varepsilon,v_\varepsilon,1}(t-s,A)\check{U}_{\varepsilon,v_\varepsilon,11}(ds), t \geq 0, \tag{88}$$

where

$$\check{U}_{\varepsilon,v_\varepsilon,11}(t) = \sum_{n=0}^{\infty} \check{Q}^{*n}_{\varepsilon,v_\varepsilon,11}(t),\ t \geq 0, \tag{89}$$

is the corresponding renewal function.

Inequality $\check{Q}^{*n}_{\varepsilon,v_\varepsilon,11}(t) \leq \check{Q}^{n}_{\varepsilon,v_\varepsilon,11}(t)$ obviously holds any $t \geq 0$ and $n = 1,2,\ldots$. These inequalities and relation (61) imply that, $\overline{\lim}_{\varepsilon\to 0} \check{Q}^{*n}_{\varepsilon,v_\varepsilon,11}(t) \leq \overline{\lim}_{\varepsilon\to 0} \check{Q}^{n}_{\varepsilon,v_\varepsilon,11}(t) = \tilde{Q}^{n}_{0,v_0,11}(t) < 1$, since $\tilde{Q}_{0,v_0,11}(t) = \mathsf{P}\{e_{0,1}\frac{1}{1+\beta}\zeta_1 + e_{0,2}\frac{1}{1+\beta^{-1}}\zeta_2 \leq t\} < 1$. Thus, the series on the right hand side in (89) converge asymptotically uniformly, as $\varepsilon \to 0$.

Also, relation (61) implies that $\check{Q}^{*n}_{\varepsilon,v_\varepsilon,11}(\cdot) \Rightarrow \check{Q}^{*n}_{0,v_0,11}(\cdot)$ as $\varepsilon \to 0$.

The above remaks imply that, for $t > 0$,

$$\check{U}_{\varepsilon,v_\varepsilon,11}(t) \to \check{U}_{0,v_0,11}(t) \text{ as } \varepsilon \to 0. \tag{90}$$

The convergence relation in (90) holds for all $t > 0$, since $\tilde{Q}^{*n}_{0,v_0,11}(t), t \geq 0$ is a continuous distribution function and, in sequel, due to the above remarks, $\check{U}_{0,v_0,11}(t), t \geq 0$ is continuous function.

Relation (76) implies that, for every $t > 0$, functions $\check{q}_{\varepsilon,v_\varepsilon,1}(t-s, A) \xrightarrow{\mathrm{U}} \check{q}_{0,v_0,1}(t-s, A)$ as $\varepsilon \to 0$, for $s \in [0, t)$. At the same time, due to continuity function $\check{U}_{0,v_0,11}(t)$, for $t > 0$, measure $\check{U}_{0,v_0,11}(ds)$ has no atom at any point $t > 0$.

By the above remarks and relations (76), (90), Lemma 2 formulated in Subsection 4.3 imply, that the following relation holds, for $A \in \Gamma$ and $t > 0$,

$$\begin{aligned} P_{\varepsilon,v_\varepsilon,11}(t, A) &= \int_0^t \check{q}_{\varepsilon,v_\varepsilon,1}(t-s, A)\check{U}_{\varepsilon,v_\varepsilon,11}(ds) \\ &\to P_{0,v_0,11}(t, A) = \int_0^t \check{q}_{0,v_0,1}(t-s, A)\check{U}_{0,v_0,11}(ds) \\ &= \pi_1(A) \int_0^t e^{-(t-s)(1+\beta)/e_{0,1}} \check{U}_{0,v_0,11}(ds) \text{ as } \varepsilon \to 0. \end{aligned} \tag{91}$$

Next, we make an important remark that the scaling of time transformation with the compression factors v_ε and all following asymptotic relations presented above can be, in obvious way, repeated for any slightly modified compression factors $\dot{v}_\varepsilon = a_\varepsilon v_\varepsilon$, where $0 < a_\varepsilon \to 1$ as $\varepsilon \to 0$.

In particular, the modified asymptotic relation (91) takes the following form, for $A \in \Gamma$ and $t > 0$,

$$P_{\varepsilon,\dot{v}_\varepsilon,11}(t, A) = P_{\varepsilon,11}(ta_\varepsilon v_\varepsilon, A) \to P_{0,v_0,11}(t, A) \text{ as } \varepsilon \to 0. \tag{92}$$

Due to an arbitrary choice of $0 < a_\varepsilon \to 1$ as $\varepsilon \to 0$, relation (92) is, for every $t > 0$, equivalent to the following relation, which holds for any $A \in \Gamma$ and $0 \leq t''_\varepsilon \to t$ as $\varepsilon \to 0$,

$$P_{\varepsilon,11}(t''_\varepsilon v_\varepsilon, A) = P_{\varepsilon,v_\varepsilon,11}(t''_\varepsilon, A) \to P_{0,v_0,11}(t, A) \text{ as } \varepsilon \to 0. \tag{93}$$

As was pointed in Subsection 4.4, the shifted process $(\xi_{\varepsilon,v_\varepsilon}(\hat{\tau}'_{\varepsilon,v_\varepsilon,1} + t), \eta_{\varepsilon,v_\varepsilon}(\hat{\tau}'_{\varepsilon,v_\varepsilon,1} + t)), t \geq 0$ is a standard regenerative process. If $\eta_{\varepsilon,v_\varepsilon}(0) = 2$, then $\eta_{\varepsilon,v_\varepsilon}(\tilde{\tau}_{\varepsilon,v_\varepsilon,1}) = 1$. That is why, probabilities $P_{\varepsilon,v_\varepsilon,11}(t, A)$ play for this process the role of probabilities $P^{(1)}_\varepsilon(t, A)$ pointed out in Subsection 2.1. The distribution function for the duration of transition period is $\tilde{Q}_{\varepsilon,v_\varepsilon,21}(t) = \mathsf{P}_2\{\tilde{\tau}_{\varepsilon,v_\varepsilon,1} \leq t\}, t \geq 0$.

According relation (60), the distribution functions $\tilde{Q}_{\varepsilon,v_\varepsilon,21}(t)$ weakly converge as $\varepsilon \to 0$ to the distribution function $\tilde{Q}_{0,v_0,21}(t) = \mathsf{P}\{e_{0,2}\frac{1}{1+\beta^{-1}}\zeta \leq t\}$ which is continuous function for $t > 0$. If $\beta \in (0, \infty]$, then $\tilde{Q}_{0,v_0,21}(t) = 1 - e^{-t(1+\beta^{-1})/e_{0,2}}, t \geq 0$ is the exponential distribution function. If $\beta = 0$, then $\tilde{Q}_{0,v_0,21}(t) = \mathrm{I}(t \geq 0), t \geq 0$.

The renewal type transition relation (8) takes the following form,

$$P_{\varepsilon,v_\varepsilon,21}(t, A) = \int_0^t P_{\varepsilon,v_\varepsilon,11}(t-s, A)\tilde{Q}_{\varepsilon,v_\varepsilon,21}(ds), \ t \geq 0. \tag{94}$$

Relation (93) implies that, for every $t > 0$, functions $P_{\varepsilon,v_\varepsilon,11}(t-s, A) \stackrel{\mathrm{U}}{\longrightarrow} P_{0,v_0,11}(t-s, A)$ as $\varepsilon \to 0$, for $s \in [0, t)$. At the same time, due to continuity the distribution function $\tilde{Q}_{0,v_0,21}(t)$ for $t > 0$ measure $\tilde{Q}_{0,v_0,21}(ds)$ has no atom at any point $t > 0$. By these remarks and relations (60), (93), Lemma 2 formulated in Subsection 4.3 implies that the following relation holds, for $A \in \Gamma$ and $t > 0$,

$$\begin{aligned} P_{\varepsilon,v_\varepsilon,21}(t, A) &= \int_0^t P_{\varepsilon,v_\varepsilon,11}(t-s, A)\tilde{Q}_{\varepsilon,v_\varepsilon,21}(ds) \\ &\to \int_0^t P_{0,v_0,11}(t-s, A)\tilde{Q}_{0,v_0,21}(ds) = P_{0,v_0,21}(t, A). \end{aligned} \tag{95}$$

By arguments similar with those used for relations (91) – (93), one can, for every $t > 0$, improve relation (95) to the more advanced form of this relation, which holds for $A \in \Gamma$ and any $0 \leq t''_\varepsilon \to t$ as $\varepsilon \to 0$,

$$P_{\varepsilon,21}(t''_\varepsilon v_\varepsilon, A) = P_{\varepsilon,v_\varepsilon,21}(t''_\varepsilon, A) \to P_{0,v_0,21}(t, A) \text{ as } \varepsilon \to 0. \tag{96}$$

It is remains to give a more explicit expression for the limiting probabilities $P_{0,v_0,11}(t, A), t > 0$ and $P_{0,v_0,21}(t, A), t > 0$.

First, let us consider the case, where $\beta = 0$.

In this case, $\check{Q}_{0,v_0,11}(t) = \mathsf{P}\{e_{0,1}\zeta_1 \leq t\} = 1 - e^{-t/e_{0,1}}, t \geq 0, i = 1, 2$ is an exponential distribution function. Thus, the renewal function $\check{U}_{0,v_0,11}(t) = \mathrm{I}(t \geq 0) + \frac{1}{e_{0,1}}t, t \geq 0$. Also, $\tilde{Q}_{0,v_0,21}(t) = \mathrm{I}(t \geq 0), t \geq 0$. Finally, $t_{s,0} = s, s \geq 0$. That is why, for $A \in \mathcal{B}_{\mathbb{X}}$ and $t > 0$,

$$\begin{aligned} P_{0,v_0,11}(t, A) &= \pi_{0,1}(A)\int_0^t e^{-(t-s)/e_{0,1}}\check{U}_{0,v_0,11}(ds) \\ &= \pi_{0,1}(A)(e^{-t/e_{0,1}} + \int_0^t \frac{e^{-(t-s)/e_{0,1}}}{e_{0,1}}ds) \\ &= \pi_{0,1}(A)(e^{-t/e_{0,1}} + e^{-t/e_{0,1}}(e^{t/e_{0,1}} - 1)) = \pi_{0,1}(A). \end{aligned} \tag{97}$$

and

$$P_{0,v_0,21}(t, A) = \int_0^t P_{0,v_0,11}(t-s, A)\tilde{Q}_{0,v_0,21}(ds) = \pi_{0,1}(A). \tag{98}$$

Second, let us consider the case, where $\beta = \infty$.

In this case, $\tilde{Q}_{0,v_0,11}(t) = \mathsf{P}\{e_{0,2}\zeta_2 \leq t\} = 1 - e^{-t/e_{0,2}}, t \geq 0$ is an exponential distribution function. Thus, the renewal function $\check{U}_{0,v_0,11}(t) = \mathrm{I}(t \geq 0) + \frac{1}{e_{0,2}}t, t \geq 0$. Also, $\tilde{Q}_{0,v_0,21}(t) = \mathsf{P}\{e_{0,2}\zeta_2 \leq t\} = 1 - e^{-t/e_{0,2}}, t \geq 0$. Finally, $t_{s,\infty} = \infty, s \geq 0$. That is why, for $t > 0$,

$$P_{0,v_0,11}(t, A) = P_{0,v_0,21}(t, A) = 0. \tag{99}$$

Third, let us consider the main case, where $\beta \in (0, \infty)$.

Let us $\eta^{(\beta)}(t), t \geq 0$ be a continuous time homogeneous Markov chain introduced in the beginning of this subsection.

Let $\tau_n^{(\beta)} = \inf(t > \tau_{n-1}^{(\beta)}, \eta^{(\beta)}(t) \neq \eta^{(\beta)}(\tau_{n-1}^{(\beta)})), n = 1, 2, \ldots, \tau_0^{(\beta)} = 0$ be the sequential moments of jumps for the Markov chain $\eta^{(\beta)}(t)$.

The Markov chain $\eta^{(\beta)}(t)$ obviously is also an alternating regenerative process, with regeneration times $\tau_{2n}^{(\beta)}, n = 0, 1, \ldots$.

Let us assume that $\eta^{(\beta)}(0) = 1$. The transition probabilities $p_{11}^{(\beta)}(t), t \geq 0$ satisfy the following renewal equation,

$$p_{11}^{(\beta)}(t) = q_1^{(\beta)}(t) + \int_0^t p_{11}^{(\beta)}(t-s) F_{11}^{(\beta)}(ds), \ t \geq 0, \tag{100}$$

where $q_1^{(\beta)}(t) = \mathsf{P}_1\{\eta^{(\beta)}(t) = 1, \tau_2^{(\beta)} > t\}$ and $F_{11}^{(\beta)}(t) = \mathsf{P}_1\{\tau_2^{(\beta)} \leq t\}$, fior $t \geq 0$.

Let $U_{11}^{(\beta)}(t) = \sum_{n=0}^{\infty} F_{11}^{(\beta)*n}(t), t \geq 0$ be the corresponding renewal function generated by the distribution function $F_{11}^{(\beta)}(t)$. The transition probabilities $p_{11}^{(\beta)}(t)$ can be expressed as the solution of the renewal equation (100) in the following form,

$$p_{11}^{(\beta)}(t) = \int_0^t q_1^{(\beta)}(t) U_{11}^{(\beta)}(ds), \ t \geq 0. \tag{101}$$

Obviously,

$$q_1^{(\beta)}(t) = \mathsf{P}_1\{\tau_1^{(\beta)} > t\} = e^{-t(1+\beta)/e_{0,1}}, \ t \geq 0, \tag{102}$$

and

$$F_{11}^{(\beta)}(t) = F_1^{(\beta)}(t) * F_2^{(\beta)}(t) = \check{Q}_{0,v_0,11}(t), \ t \geq 0. \tag{103}$$

and, thus,

$$U_{11}^{(\beta)}(t) = \check{U}_{0,v_0,11}(t), \ t \geq 0. \tag{104}$$

Relations (101), (102), and (104) imply that,

$$p_{11}^{(\beta)}(t) = \int_0^t e^{-(t-s)(1+\beta)/e_{0,1}} \check{U}_{0,v_0,11}(ds), \ t \geq 0. \tag{105}$$

Finally, relations (91) and (105) imply that that the following equality takes place, for $t > 0$,

$$P_{0,v_0,11}(t, A) = p_{11}^{(\beta)}(t) \pi_{0,1}(A). \tag{106}$$

The distribution function $F_2^{(\beta)}(t) = \check{Q}_{0,v_0,21}(t) = 1 - e^{-t(1+\beta^{-1})/e_{0,2}}, t \geq 0$. Thus,

$$p_{21}^{(\beta)}(t) = \int_0^t p_{11}^{(\beta)}(t-s) \check{Q}_{0,v_0,21}(ds), t \geq 0, \tag{107}$$

and, therefore, relation (95) implies that following equality takes place, for $t > 0$,

$$P_{0,v_0,21}(t, A) = p_{21}^{(\beta)}(t)\pi_{0,1}(A). \quad (108)$$

Due to the symmetricity of conditions $\mathbf{F}$ – $\mathbf{J}$, $\mathbf{K}_\beta$, and $\mathbf{N}_1$ with respect to the indices $i, j = 1, 2$, the ergodic relations, analogous to the mentioned above ergodic relations for probabilities $P_{\varepsilon,v_\varepsilon,11}(t''_\varepsilon, A) = P_{\varepsilon,11}(t''_\varepsilon v_\varepsilon, A)$ and $P_{\varepsilon,v_\varepsilon,21}(t''_\varepsilon, A) = P_{\varepsilon,21}(t''_\varepsilon v_\varepsilon, A)$, also take place for probabilities $P_{\varepsilon,v_\varepsilon,22}(t''_\varepsilon, A) = P_{\varepsilon,22}(t''_\varepsilon v_\varepsilon, A)$ and $P_{\varepsilon,v_\varepsilon,12}(t''_\varepsilon, A) = P_{\varepsilon,12}(t''_\varepsilon v_\varepsilon, A)$.

The above analysis yields the description of asymptotic behaviour of probabilities $P_{\varepsilon,ij}(t_\varepsilon, A)$ for long times $0 \leq t_\varepsilon \to \infty$ as $\varepsilon \to 0$ satisfying the asymptotic relation $t_\varepsilon/v_\varepsilon \to t \in (0, \infty)$ as $\varepsilon \to 0$. To see this, one should just represent such t_ε in the form, $t_\varepsilon = t''_\varepsilon v_\varepsilon$. Obviously, $t''_\varepsilon = t_\varepsilon/v_\varepsilon \to t$ as $\varepsilon \to 0$. □

5. Short time ergodic theorems for singularly perturbed alternating regenerative processes

In this section, we present short time individual ergodic theorems for singularly perturbed alternating regenerative processes.

5.1. Short time ergodic theorems for singularly perturbed alternating regenerative processes, under the assumption that condition $\mathbf{K}_\beta$ holds for some $\beta \in (0, \infty)$. In this subsection, we describe the asymptotic behaviour of probabilities $P_{\varepsilon,ij}(t_\varepsilon, A)$ for so-called "short" times $0 \leq t_\varepsilon \to \infty$ as $\varepsilon \to 0$, which satisfy the following asymptotic relation,

$$t_\varepsilon/v_\varepsilon \to 0 \text{ as } \varepsilon \to 0. \quad (109)$$

We also assume that, additionally to conditions $\mathbf{N}_1$, condition $\mathbf{K}_\beta$ holds for some $\beta \in (0, \infty)$.

The corresponding limiting probabilities are, in this case, the same for any $\beta \in (0, \infty)$ and take the following form, for $A \in \mathcal{B}_{\mathbb{X}}$, $i, j = 1, 2$,

$$\pi_{0,ij}(A) = \mathrm{I}(j = i)\pi_{0,i}(A) = \begin{cases} \pi_{0,i}(A) & \text{for } j = i, \\ 0 & \text{for } j \neq i. \end{cases} \quad (110)$$

The following theorem takes place.

Theorem 10. *Let conditions* $\mathbf{F}$ – $\mathbf{J}$, $\mathbf{N}_1$ *hold and, also, condition* $\mathbf{K}_\beta$ *holds for some* $\beta \in (0, \infty)$. *Then, for every* $A \in \Gamma, i, j = 1, 2,$ *and* $0 \leq t_\varepsilon \to \infty$ *as* $\varepsilon \to 0$ *such that* $t_\varepsilon/v_\varepsilon \to 0$ *as* $\varepsilon \to 0$,

$$P_{\varepsilon,ij}(t_\varepsilon, A) \to \pi_{0,ij}(A) \text{ as } \varepsilon \to 0. \quad (111)$$

Proof. First, let us analyse the asymptotic behaviour of probabilities $P_{\varepsilon,11}(t_\varepsilon, A)$ and, thus, assume that $\eta_\varepsilon(0) = 1$.

We return back to the initial alternating regenerative process $(\xi_\varepsilon(t), \eta_\varepsilon(t))$, $t \geq 0$ with regeneration times $\tau_{\varepsilon,n}, n = 0, 1, \ldots$.

Recall the stopping time $\tilde{\tau}_{\varepsilon,1}$, which is the time of first hitting sate 2 by process $\eta_\varepsilon(t)$.

Let us again consider the regenerative process $\xi_{\varepsilon,1}(t), t \geq 0$ with regeneration times $\tau_{\varepsilon,1,n}, n = 0, 1, \ldots$, and the random lifetime $\mu_{\varepsilon,1,+}$ introduced in Subsection 4.4.

It is readily seen that, for every $t \geq 0$,

$$\tilde{Q}_{\varepsilon,12}(t) = \mathsf{P}_1\{\tilde{\tau}_{\varepsilon,1} \leq t\} = \mathsf{P}\{\mu_{\varepsilon,1,+} \leq t\} \tag{112}$$

and, for every $A \in \mathcal{B}_{\mathbb{X}}, t \geq 0$,

$$\mathsf{P}_1\{\xi_\varepsilon(t) \in A, \eta_\varepsilon(t) = 1, \tilde{\tau}_{\varepsilon,1} > t\} = \mathsf{P}\{\xi_{\varepsilon,1}(t) \in A, \mu_{\varepsilon,1,+} > t\}. \tag{113}$$

According relation (58), if $\eta_\varepsilon(0) = 1$, random variables,

$$v_\varepsilon^{-1}\tilde{\tau}_{\varepsilon,1} \stackrel{d}{\longrightarrow} e_{0,1}\frac{1}{1+\beta}\zeta \text{ as } \varepsilon \to 0, \tag{114}$$

where ζ is a random variable exponentially distributed, with parameter 1.

Since, we assumed that $t_\varepsilon/v_\varepsilon \to 0$ as $\varepsilon \to 0$, relations (112) and (114) imply that,

$$\begin{aligned}\mathsf{P}\{\mu_{\varepsilon,1,+} > t_\varepsilon\} &= \mathsf{P}_1\{\tilde{\tau}_{\varepsilon,1} > t_\varepsilon\} \\ &= \mathsf{P}_1\{v_\varepsilon^{-1}\tilde{\tau}_{\varepsilon,1} > t_\varepsilon v_\varepsilon^{-1}\} \to 1 \text{ as } \varepsilon \to 0.\end{aligned} \tag{115}$$

Relations (113) and (115) imply that

$$\begin{aligned}&\mathsf{P}_1\{\xi_\varepsilon(t_\varepsilon) \in A, \eta_\varepsilon(t_\varepsilon) = 1\} - \mathsf{P}_1\{\xi_\varepsilon(t_\varepsilon) \in A, \eta_\varepsilon(t_\varepsilon) = 1, \tilde{\tau}_{\varepsilon,1} > t_\varepsilon\} \\ &\quad \leq \mathsf{P}_1\{\tilde{\tau}_{\varepsilon,1} \leq t_\varepsilon\} \to 0 \text{ as } \varepsilon \to 0.\end{aligned} \tag{116}$$

and, analogously,

$$\begin{aligned}&\mathsf{P}\{\xi_{\varepsilon,1}(t_\varepsilon) \in A\} - \mathsf{P}\{\xi_{\varepsilon,1}(t_\varepsilon) \in A, \mu_{\varepsilon,1,+} > t_\varepsilon\} \\ &\quad \leq \mathsf{P}\{\mu_{\varepsilon,1,+} \leq t_\varepsilon\} \to 0 \text{ as } \varepsilon \to 0,\end{aligned} \tag{117}$$

These relations and Theorem 1, which can be applied to the regenerative processes $\xi_{\varepsilon,1}(t)$, imply that, for every $A \in \Gamma$,

$$\begin{aligned}\lim_{\varepsilon\to 0} \mathsf{P}_{11}(t_\varepsilon, A) &= \lim_{\varepsilon\to 0} \mathsf{P}_1\{\xi_\varepsilon(t_\varepsilon) \in A, \eta_\varepsilon(t_\varepsilon) = 1\} \\ &= \lim_{\varepsilon\to 0} \mathsf{P}_1\{\xi_\varepsilon(t_\varepsilon) \in A, \eta_\varepsilon(t_\varepsilon) = 1, \tilde{\tau}_{\varepsilon,1} > t_\varepsilon\} \\ &= \lim_{\varepsilon\to 0} \mathsf{P}\{\xi_{\varepsilon,1}(t_\varepsilon) \in A, \mu_{\varepsilon,1,+} > t_\varepsilon\} \\ &= \lim_{\varepsilon\to 0} \mathsf{P}\{\xi_{\varepsilon,1}(t_\varepsilon) \in A\} = \pi_{0,1}(A).\end{aligned} \tag{118}$$

Let us now analyse the asymptotic behaviour for probabilities $P_{\varepsilon,21}(t,A)$ and, thus, assume that $\eta_\varepsilon(0) = 2$.

In this case, relation (60) implies that random variables,

$$v_\varepsilon^{-1}\tilde{\tau}_{\varepsilon,1} \xrightarrow{d} e_{0,2}\frac{1}{1+\beta^{-1}}\zeta \text{ as } \varepsilon \to 0, \tag{119}$$

where ζ is a random variable exponentially distributed, with parameter 1.

Since, we assumed that $t_\varepsilon/v_\varepsilon \to 0$ as $\varepsilon \to 0$, the above convergence in distribution relation, obviously, implies that,

$$\mathsf{P}_2\{\tilde{\tau}_{\varepsilon,1} > t_\varepsilon\} = \mathsf{P}_2\{v_\varepsilon^{-1}\tilde{\tau}_{\varepsilon,1} > t_\varepsilon v_\varepsilon^{-1}\} \to 1 \text{ as } \varepsilon \to 0. \tag{120}$$

If $\eta_\varepsilon(0) = 2$, then, for every $t > 0$, event $\{\eta_\varepsilon(t) = 1\} \subseteq \{\tilde{\tau}_{\varepsilon,1} \leq t\}$. Thus, for every $A \in \Gamma$,

$$\begin{aligned} \mathsf{P}_{21}(t_\varepsilon, A) &= \mathsf{P}_2\{\xi_\varepsilon(t_\varepsilon) \in A, \eta_\varepsilon(t_\varepsilon) = 1\} \\ &\leq \mathsf{P}_2\{\tilde{\tau}_{\varepsilon,1} \leq t_\varepsilon\} \to 0 \text{ as } \varepsilon \to 0. \end{aligned} \tag{121}$$

Due to the symmetricity of conditions $\mathbf{F} - \mathbf{J}$ and $\mathbf{N}_1$ with respect to the indices $i, j = 1, 2$, the ergodic relations, analogous to the mentioned above ergodic relations for probabilities $P_{\varepsilon,11}(t_\varepsilon, A)$ and $P_{\varepsilon,21}(t_\varepsilon, A)$, also take place for probabilities $P_{\varepsilon,22}(t_\varepsilon, A)$ and $P_{\varepsilon,v_\varepsilon,12}(t_\varepsilon, A)$. □

5.2. Compression of time factors v_ε and w_ε. Let introduce function $w_\varepsilon = (p_{\varepsilon,12} + p_{\varepsilon,21})^{-1}$. This function possess useful asymptotic properties different of asymptotic properties of function $v_\varepsilon = p_{\varepsilon,12}^{-1} + p_{\varepsilon,21}^{-1}$.

The following lemma present some useful relations between functions v_ε and w_ε.

Lemma 3. *If conditions $\mathbf{M}_1$ holds and condition $\mathbf{K}_\beta$ holds, for some $\beta \in [0,\infty]$. Then, $0 < w_\varepsilon < v_\varepsilon < \infty, \varepsilon \in [01]$, and:*

(i) *If $\beta \in (0,\infty)$, then $v_\varepsilon \sim p_{\varepsilon,12}^{-1}(1+\beta) \sim p_{\varepsilon,21}^{-1}(1+\beta^{-1})$ as $\varepsilon \to 0$, while $w_\varepsilon \sim p_{\varepsilon,12}^{-1}(1+\beta^{-1})^{-1} \sim p_{\varepsilon,21}^{-1}(1+\beta)^{-1}$ as $\varepsilon \to 0$, and , thus, $w_\varepsilon \sim \frac{\beta}{(1+\beta)^2}v_\varepsilon$ as $\varepsilon \to 0$.*

(ii) *If $\beta = 0$, then $v_\varepsilon \sim p_{\varepsilon,12}^{-1}$ as $\varepsilon \to 0$, while $w_\varepsilon \sim p_{\varepsilon,21}^{-1} \prec v_\varepsilon$ as $\varepsilon \to 0$.*

(iii) *If $\beta = \infty$, then $v_\varepsilon \sim p_{\varepsilon,21}^{-1}$ as $\varepsilon \to 0$, while $w_\varepsilon \sim p_{\varepsilon,12}^{-1} \prec v_\varepsilon$ as $\varepsilon \to 0$.*

Here and henceforth, symbols $f'_\varepsilon \sim f''_\varepsilon$ as $\varepsilon \to 0$ and $f'_\varepsilon \prec f''_\varepsilon$ as $\varepsilon \to 0$ are used for two functions $0 < f'_\varepsilon, f''_\varepsilon \to \infty$ as $\varepsilon \to 0$ in the sense that, respectively, $f'_\varepsilon/f''_\varepsilon \to 1$ as $\varepsilon \to 0$ and $f'_\varepsilon/f''_\varepsilon \to 0$ as $\varepsilon \to 0$.

Proposition **(i)** of Lemma 3 implies that, in the case, where condition condition $\mathbf{K}_\beta$ holds, for some $\beta \in (0,\infty)$, relations $t_\varepsilon/v_\varepsilon \to t$ and $t_\varepsilon/w_\varepsilon \to t$ as $\varepsilon \to 0$ generate, for every $t \in [0,\infty]$, equivalent, in some sense, asymptotic time zones.

Propositions **(ii)** and **(iii)** of Lemma 3 imply that, in the case, where condition $\mathbf{K}_0$ or $\mathbf{K}_\infty$ holds, relations $t_\varepsilon / v_\varepsilon \to t$ and $t_\varepsilon / w_\varepsilon \to t$ as $\varepsilon \to 0$ generate, for every $t \in [0, \infty]$, essentially different asymptotic time zones.

We should assume in this case that "short" times $0 \leq t_\varepsilon \to \infty$ as $\varepsilon \to 0$ satisfy, additionally to the asymptotic relation (109), the following asymptotic relation,

$$t_\varepsilon / w_\varepsilon \to t \in [0, \infty] \text{ as } \varepsilon \to 0. \tag{122}$$

5.3. Short time ergodic theorems for singularly perturbed alternating regenerative processes, under the assumption that condition $\mathbf{K}_0$ or $\mathbf{K}_\infty$ hold and $w_\varepsilon \prec t_\varepsilon \prec v_\varepsilon$ as $\varepsilon \to 0$. In this subsection, we consider the case, where parameter $t = \infty$ in relation (122). In this case, relations (109) and (122) mean that,

$$w_\varepsilon \prec t_\varepsilon \prec v_\varepsilon \text{ as } \varepsilon \to 0. \tag{123}$$

The corresponding limiting probabilities take the following forms, for $A \in \mathcal{B}_{\mathbb{X}},\ i, j = 1, 2$,

$$\pi^{(0)}_{0,j}(A) = \begin{cases} \pi_{0,1}(A) & \text{for } j = 1, \\ 0 & \text{for } j = 2. \end{cases} \tag{124}$$

and

$$\pi^{(\infty)}_{0,j}(A) = \begin{cases} 0 & \text{for } j = 1, \\ \pi_{0,2}(A) & \text{for } j = 2. \end{cases} \tag{125}$$

It is useful to note that the above limiting probabilities $\pi^{(0)}_{0,j}(A)$ and $\pi^{(\infty)}_{0,j}(A)$ coincide with the corresponding limiting probabilities for semi-regularly perturbed alternating regenerative processes, respectively, with parameter $\beta = 0$, given in relation (36), and $\beta = \infty$, given in relation (37).

The following theorems take place.

Theorem 11. *Let conditions* $\mathbf{F}$ – $\mathbf{J}$, $\mathbf{N}_1$, *and* $\mathbf{K}_0$ *hold. Then, for every* $A \in \Gamma, i, j = 1, 2$, *and* $0 \leq t_\varepsilon \to \infty$ *as* $\varepsilon \to 0$ *such that* $t_\varepsilon / w_\varepsilon \to \infty$ *as* $\varepsilon \to 0$ *and* $t_\varepsilon / v_\varepsilon \to 0$ *as* $\varepsilon \to 0$,

$$P_{\varepsilon, ij}(t_\varepsilon, A) \to \pi^{(0)}_{0,j}(A) \text{ as } \varepsilon \to 0. \tag{126}$$

Theorem 12. *Let conditions* $\mathbf{F}$ – $\mathbf{J}$, $\mathbf{N}_1$, *and* $\mathbf{K}_\infty$ *holds. Then, for every* $A \in \Gamma, i, j = 1, 2$, *and* $0 \leq t_\varepsilon \to \infty$ *as* $\varepsilon \to 0$ *such that* $t_\varepsilon / w_\varepsilon \to \infty$ *as* $\varepsilon \to 0$ *and* $t_\varepsilon / v_\varepsilon \to 0$ *as* $\varepsilon \to 0$,

$$P_{\varepsilon, ij}(t_\varepsilon, A) \to \pi^{(\infty)}_{0,j}(A) \text{ as } \varepsilon \to 0. \tag{127}$$

Proof. First, let us prove Theorem 11.

It can be noted that the analysis of asymptotic behaviour for probabilities $P_{\varepsilon,11}(t_\varepsilon, A)$ can be performed in absolutely analogous way with those presented in relations (112) – (118), in the proof of Theorem 10. The only difference is that parameter $\beta = 0$, and, thus, the limiting random variable in the analogue of asymptotic relation (114) has the form, $e_{0,1}\zeta$, where ζ is a random variable exponentially distributed, with parameter 1. This analysis yields that the following asymptotic relation takes place, for every $A \in \Gamma$ and any $t_\varepsilon / v_\varepsilon \to 0$ as $\varepsilon \to 0$,

$$P_{\varepsilon,11}(t_\varepsilon, A) \to \pi_{0,1}(A) \text{ as } \varepsilon \to 0. \tag{128}$$

The asymptotic behaviour for probabilities $P_{\varepsilon,21}(t_\varepsilon, A)$ differs in this case of those presented in Theorem 10. As a matter of fact, the asymptotic relation analogous to (119) does not take place.

In this case, random variables $v_\varepsilon^{-1}\tilde{\tau}_{\varepsilon,1} \stackrel{d}{\longrightarrow} 0$ as $\varepsilon \to 0$. This asymptotic relation does not imply relation analogous to (115). The right normalising function for random variables $\tilde{\tau}_{\varepsilon,1}$ is, in this case, $w_\varepsilon \sim p_{\varepsilon,21}^{-1}$ as $\varepsilon \to 0$. According this relation and relation (59), if $\eta_\varepsilon(0) = 2$, then,

$$w_\varepsilon^{-1}\tilde{\tau}_{\varepsilon,1} \stackrel{d}{\longrightarrow} e_{0,2}\zeta \text{ as } \varepsilon \to 0, \tag{129}$$

where ζ is a random variable exponentially distributed, with parameter 1.

The following renewal type relation connects probabilities $P_{\varepsilon,11}(t_\varepsilon, A)$ and $P_{\varepsilon,21}(t_\varepsilon, A)$,

$$\begin{aligned} P_{\varepsilon,21}(t_\varepsilon, A) &= \int_0^{t_\varepsilon} P_{\varepsilon,11}(t_\varepsilon - s, A)\mathsf{P}_2\{\tilde{\tau}_{\varepsilon,1} \in ds\} \\ &= \int_0^{t_\varepsilon / w_\varepsilon} P_{\varepsilon,11}(t_\varepsilon - s w_\varepsilon, A)\mathsf{P}_2\{w_\varepsilon^{-1}\tilde{\tau}_{\varepsilon,1} \in ds\} \\ &= \int_0^{\infty} P_{\varepsilon,11}(t_\varepsilon - s w_\varepsilon, A)\mathsf{P}_2\{w_\varepsilon^{-1}\tilde{\tau}_{\varepsilon,1} \in ds\}, \end{aligned} \tag{130}$$

where function $P_{\varepsilon,11}(t_\varepsilon - s w_\varepsilon, A)$ is defined as 0 for $t_\varepsilon - s w_\varepsilon < 0$.

Let us take an arbitrary $s_\varepsilon \to s \in [0, \infty)$ as $\varepsilon \to 0$. Obviously, $(t_\varepsilon - s_\varepsilon w_\varepsilon)/w_\varepsilon = t_\varepsilon / w_\varepsilon - s_\varepsilon \to \infty$ and, thus, $(t_\varepsilon - s_\varepsilon w_\varepsilon) \to \infty$ as $\varepsilon \to 0$.

Also, $(t_\varepsilon - s_\varepsilon w_\varepsilon)/v_\varepsilon = t_\varepsilon / v_\varepsilon - s_\varepsilon w_\varepsilon / v_\varepsilon \to 0$ as $\varepsilon \to 0$.

That is why, according relation (128), the following asymptotic relation take place, for $A \in \Gamma$ and $s \in [0, \infty)$,

$$P_{\varepsilon,11}(t_\varepsilon - s_\varepsilon w_\varepsilon, A) \to \pi_{0,1}(A) \text{ as } \varepsilon \to 0. \tag{131}$$

Relations (129) and (131) imply, by Lemma 2 given in Subsection 4.3 that the following relation takes place, for $A \in \Gamma$,

$$P_{\varepsilon,21}(t_\varepsilon, A) \to \int_0^{\infty} \pi_{0,1}(A)\mathsf{P}\{e_{0,2}\zeta \in ds\} = \pi_{0,1}(A) \text{ as } \varepsilon \to 0. \tag{132}$$

As was pointed out in Subsection 2.5, the phase space $\mathbb{X} \in \Gamma$. Also, $\pi_{0,1}(\mathbb{X}) = 1$. Thus, relations (128) and (132) imply that the following relation holds, for $A \in \Gamma$ and $i = 1, 2$,

$$
\begin{aligned}
P_{\varepsilon,i2}(t_\varepsilon, A) \leq P_{\varepsilon,i2}(t_\varepsilon, \mathbb{X}) = 1 - P_{\varepsilon,i1}(t_\varepsilon, \mathbb{X}) \\
\to 1 - \pi_{0,1}(\mathbb{X}) = 0 \text{ as } \varepsilon \to 0.
\end{aligned} \tag{133}
$$

The proof of Theorem 11 is completed.

The proof of Theorem 12 is absolutely analogous to the proof of Theorem 11, due to simmetrisity conditions **F** – **J** and $\mathbf{N}_1$ with respect to indices $i, j = 1, 2$. The only formula (124) for the corresponding limiting probabilities should be replaced by formula (125). □

5.4. Short time ergodic theorems for singularly perturbed alternating regenerative processes, under the assumption that condition $\mathbf{K}_0$ or $\mathbf{K}_\infty$ hold and $t_\varepsilon / w_\varepsilon \to t \in (0, \infty)$ as $\varepsilon \to 0$. In this subsection, we consider the case, where parameter $t \in (0, \infty)$, in relation (122). In this case, relation (122) means that,

$$
t_\varepsilon \sim t w_\varepsilon \text{ as } \varepsilon \to 0, \ \text{ where } t \in (0, \infty). \tag{134}
$$

According propositions **(ii)** and **(iii)** of Lemma 3, if condition $\mathbf{K}_0$ or $\mathbf{K}_\infty$, then $w_\varepsilon \prec v_\varepsilon$ as $\varepsilon \to 0$, and, thus, relation (134) implies that a "short" time relation (109) holds, i.e., $t_\varepsilon / v_\varepsilon \to 0$ as $\varepsilon \to 0$.

The corresponding limiting probabilities take the following forms, for $A \in \Gamma, i, j = 1, 2$ and $t \in (0, \infty)$,

$$
\dot{\pi}^{(0)}_{0,ij}(t, A) = \begin{cases}
\pi_{0,1}(A) & \text{for } i = 1, j = 1, \\
0 & \text{for } i = 1, j = 2, \\
(1 - e^{-t/e_{0,2}})\pi_{0,1}(A) & \text{for } i = 2, j = 1, \\
e^{-t/e_{0,2}}\pi_{0,2}(A) & \text{for } i = 2, j = 2.
\end{cases} \tag{135}
$$

and

$$
\dot{\pi}^{(\infty)}_{0,ij}(t, A) = \begin{cases}
e^{-t/e_{0,1}}\pi_{0,1}(A) & \text{for } i = 1, j = 1, \\
(1 - e^{-t/e_{0,1}})\pi_{0,2}(A) & \text{for } i = 1, j = 2, \\
0 & \text{for } i = 2, j = 1, \\
\pi_{0,2}(A) & \text{for } i = 2, j = 2.
\end{cases} \tag{136}
$$

The following theorems take place.

Theorem 13. *Let conditions* **F** – **J**, $\mathbf{N}_1$ *and* $\mathbf{K}_0$ *hold. Then, for every* $A \in \Gamma, i, j = 1, 2$, *and any* $0 \leq t_\varepsilon \to \infty$ *as* $\varepsilon \to 0$ *such that* $t_\varepsilon / w_\varepsilon \to t \in (0, \infty)$ *as* $\varepsilon \to 0$,

$$
P_{\varepsilon,ij}(t_\varepsilon, A) \to \dot{\pi}^{(0)}_{ij}(t, A) \text{ as } \varepsilon \to 0. \tag{137}
$$

Theorem 14. *Let conditions* **F** – **J**, $\mathbf{N}_1$ *and* $\mathbf{K}_\infty$ *hold. Then, for every* $A \in \Gamma, i, j = 1, 2,$ *and* $0 \leq t_\varepsilon \to \infty$ *as* $\varepsilon \to 0$ *such that* $t_\varepsilon / w_\varepsilon \to t \in (0, \infty)$ *as* $\varepsilon \to 0$,

$$P_{\varepsilon,ij}(t_\varepsilon, A) \to \dot{\pi}^{(\infty)}_{0,ij}(t, A) \text{ as } \varepsilon \to 0. \tag{138}$$

Proof. First, let us prove Theorem 13.

It can be noted, as in the proof of Theorem 11, that the analysis of asymptotic behaviour for probabilities $P_{\varepsilon,11}(t_\varepsilon, A)$ can be performed in absolutely analogous way with those presented in relations (112) – (118), in the proof of Theorem 10. The only difference is that parameter $\beta = 0$, and, thus, the limiting random variable in the analogue of asymptotic relation (114) has the form, $e_{0,1}\zeta$, where ζ is a random variable exponentially distributed, with parameter 1.

This analysis yields that the asymptotic relation (128) takes place, i.e., $P_{\varepsilon,11}(t_\varepsilon, A) \to \pi_{0,1}(A)$ as $\varepsilon \to 0$, for $A \in \Gamma$ and any $t_\varepsilon / v_\varepsilon \to 0$ as $\varepsilon \to 0$.

Also, as in the proof of Theorem 11, the renewal type relation (130), connecting probabilities $P_{\varepsilon,11}(t_\varepsilon, A)$ and $P_{\varepsilon,21}(t_\varepsilon, A)$, takes place.

Let us take an arbitrary $s_\varepsilon \to s \in [0, \infty)$ as $\varepsilon \to 0$. Obviously, $(t_\varepsilon - s_\varepsilon w_\varepsilon)/w_\varepsilon = t_\varepsilon/w_\varepsilon - s_\varepsilon \to t - s$ as $\varepsilon \to 0$.

Thus, for $t > s$, the following relations holds, $(t_\varepsilon - s_\varepsilon w_\varepsilon) = (t_\varepsilon/w_\varepsilon - s_\varepsilon)w_\varepsilon \to \infty$ as $\varepsilon \to 0$ and $(t_\varepsilon - s_\varepsilon w_\varepsilon)/v_\varepsilon = (t_\varepsilon/w_\varepsilon - s_\varepsilon)w_\varepsilon/v_\varepsilon \to 0$ as $\varepsilon \to 0$.

Also, for $t < s$ function $(t_\varepsilon - s_\varepsilon w_\varepsilon) = (t_\varepsilon/w_\varepsilon - s_\varepsilon)w_\varepsilon \to -\infty$ for $\varepsilon \to 0$.

That is why, according relation (128) and the definition of $P_{\varepsilon,11}(t_\varepsilon - sw_\varepsilon, A) = 0$, for $t_\varepsilon - sw_\varepsilon < 0$, in relation (130), the following asymptotic relation holds, for $A \in \Gamma$ and $s \neq t$,

$$P_{\varepsilon,11}(t_\varepsilon - s_\varepsilon w_\varepsilon, A) \to \pi_{0,1}(A)\mathrm{I}(t > s) \text{ as } \varepsilon \to 0. \tag{139}$$

Note that convergence of $P_{\varepsilon,11}(t_\varepsilon - s_\varepsilon w_\varepsilon, A)$ as $\varepsilon \to 0$ is not guarantied for $s = t$. However, the distribution of limiting ransom variable in relation (129) is exponential and, thus, it has not an atom at any point $t > 0$.

Therefore, relations (129) and (139) imply, by Lemma 2 given in Subsection 4.3, that the following relation takes place, for $A \in \Gamma$ and any $0 \leq t_\varepsilon \to \infty$ as $\varepsilon \to 0$ such that $t_\varepsilon/w_\varepsilon \to t \in (0, \infty)$ as $\varepsilon \to 0$,

$$\begin{aligned} P_{\varepsilon,21}(t_\varepsilon, A) &= \int_0^\infty P_{\varepsilon,11}(t_\varepsilon - sw_\varepsilon, A)\mathsf{P}_2\{w_\varepsilon^{-1}\tilde{\tau}_{\varepsilon,1} \in ds\}, \\ &\to \int_0^\infty \pi_{0,1}(A)\mathrm{I}(t > s)e_{0,2}^{-1}e^{-s/e_{0,2}}ds \\ &= (1 - e^{-t/e_{0,2}})\pi_{0,1}(A) \text{ as } \varepsilon \to 0. \end{aligned} \tag{140}$$

It remains to give the asymptotic analysis of asymptotic behaviour for probabilities $P_{\varepsilon,12}(t_\varepsilon, A)$ and $P_{\varepsilon,22}(t_\varepsilon, A)$.

As was pointed out in Subsection 2.5, the phase space $\mathbb{X} \in \Gamma$. Also, $\pi_{0,1}(\mathbb{X}) = 1$. Thus, relation (128) implies that the following relation holds, for $A \in \Gamma$ and any $0 \leq t_\varepsilon \to \infty$ as $\varepsilon \to 0$ such that $t_\varepsilon/w_\varepsilon \to t \in (0, \infty)$ as $\varepsilon \to 0$,

$$\begin{aligned} P_{\varepsilon,12}(t_\varepsilon, A) &\leq P_{\varepsilon,12}(t_\varepsilon, \mathbb{X}) = 1 - P_{\varepsilon,11}(t_\varepsilon, \mathbb{X}) \\ &\to 1 - \pi_{0,1}(\mathbb{X}) = 0 \text{ as } \varepsilon \to 0. \end{aligned} \tag{141}$$

Let us introduce random variables $\mu_{\varepsilon,2,n} = \kappa_{\varepsilon,2,n}\mathrm{I}(\eta_{\varepsilon,2,n} = 2), n = 1, 2, \ldots$. Let now consider the random sequence of triplets $\langle \bar{\xi}_{\varepsilon,2,n} = \langle \xi_{\varepsilon,2,n}(t), t \geq 0\rangle, \kappa_{\varepsilon,2,n}, \mu_{\varepsilon,2,n}\rangle, n = 1, 2, \ldots$, the regenerative process $\xi_{\varepsilon,2}(t) = \xi_{\varepsilon,2,n}(t - \tau_{\varepsilon,2,n-1})$, for $t \in [\tau_{\varepsilon,2,n-1}, \tau_{\varepsilon,2,n})$, $n = 1, 2, \ldots$, with regeneration times $\tau_{\varepsilon,2,n} = \kappa_{\varepsilon,2,1} + \cdots + \kappa_{\varepsilon,2,n}, n = 1, 2, \ldots, \tau_{\varepsilon,2,0} = 0$, and the random lifetime $\mu_{\varepsilon,2,+} = \tau_{\varepsilon,2,\nu_{\varepsilon,2}}$, where $\nu_{\varepsilon,2} = \min(n \geq 1 : \mu_{\varepsilon,2,n} < \kappa_{\varepsilon,2,n}) = \min(n \geq 1 : \eta_{\varepsilon,2,n} = 1)$.

Let us also denote $P_{\varepsilon,2,+}(t, A) = \mathsf{P}_2\{\xi_{\varepsilon,2}(t) \in A, \mu_{\varepsilon,2,+} > t\}$. In this case, the distribution function $F_{\varepsilon,2}(t) = \mathsf{P}\{\kappa_{\varepsilon,2,1} \leq t, \mu_{\varepsilon,2,1} \geq \kappa_{\varepsilon,2,1}\} = \mathsf{P}\{\kappa_{\varepsilon,2,1} \leq t, \eta_{\varepsilon,2,1} = 2\}, t \geq 0$, the stopping probability $f_{\varepsilon,2} = \mathsf{P}\{\mu_{\varepsilon,2,1} < \kappa_{\varepsilon,2,1}\} = \mathsf{P}\{\eta_{\varepsilon,2,1} = 1\} = p_{\varepsilon,21}$, and the expectation $e_{\varepsilon,2} = \mathsf{E}\kappa_{\varepsilon,2,1}\mathrm{I}(\mu_{\varepsilon,2,1} \geq \kappa_{\varepsilon,2,1}) = \mathsf{E}\kappa_{\varepsilon,2,1}\mathrm{I}(\eta_{\varepsilon,2,1} = 2) = e_{\varepsilon,22}$.

The following relation obviously takes place, for $A \in \mathcal{B}_\mathbb{X}, t \geq 0$,

$$\begin{aligned} P_{\varepsilon,2,+}(t, A) &= \mathsf{P}\{\xi_{\varepsilon,2}(t) \in A, \mu_{\varepsilon,2,+} > t\} \\ &= \mathsf{P}_2\{\xi_\varepsilon(t) \in A, \tilde{\tau}_{\varepsilon,1} > t\}. \end{aligned} \tag{142}$$

Conditions $\mathbf{F} - \mathbf{J}$, $\mathbf{N}_1$ and $\mathbf{K}_0$ imply that conditions $\mathbf{A} - \mathbf{D}$ holds. Thus, conditions of Theorem 13 imply that all conditions of Theorem 3 hold for the regenerative processes $\xi_{\varepsilon,2}(t), t \geq 0$ with regenerative times $\tau_{\varepsilon,2,n}, n = 1, 2, \ldots$ and the regenerative lifetime $\mu_{\varepsilon,2,+}$.

Therefore, the following relation holds, for any $A \in \Gamma$, and any $0 \leq t_\varepsilon \to \infty$ as $\varepsilon \to 0$ such that $p_{\varepsilon,21}t_\varepsilon = t_\varepsilon/w_\varepsilon \to t \in (0, \infty)$ as $\varepsilon \to 0$,

$$\begin{aligned} \mathsf{P}_2\{\xi_\varepsilon(t_\varepsilon) \in A, \tilde{\tau}_{\varepsilon,1} > t_\varepsilon\} &= P_{\varepsilon,2,+}(t_\varepsilon, A) \\ &\to e^{-t/e_{0,2}}\pi_{0,2}(A) \text{ as } \varepsilon \to 0. \end{aligned} \tag{143}$$

The following renewal type equation connects probabilities $P_{\varepsilon,22}(t_\varepsilon, A)$ and $P_{\varepsilon,12}(t_\varepsilon, A)$,

$$\begin{aligned} P_{\varepsilon,22}(t_\varepsilon, A) &= \mathsf{P}_2\{\xi_\varepsilon(t_\varepsilon) \in A, \tilde{\tau}_{\varepsilon,1} > t_\varepsilon\} \\ &\quad + \int_0^{t_\varepsilon} P_{\varepsilon,12}(t_\varepsilon - s, A)\mathsf{P}_2\{\tilde{\tau}_{\varepsilon,1} \in ds\} \end{aligned} \tag{144}$$

The integral at the right hand side of the above relation can be represented in the following form,

$$\int_0^{t_\varepsilon} P_{\varepsilon,12}(t_\varepsilon - s, A)\mathsf{P}_2\{\tilde{\tau}_{\varepsilon,1} \in ds\} = \int_0^\infty P_{\varepsilon,12}(t_\varepsilon - sw_\varepsilon, A)\mathsf{P}_2\{w_\varepsilon^{-1}\tilde{\tau}_{\varepsilon,1} \in ds\}, \tag{145}$$

where function $P_{\varepsilon,12}(t_\varepsilon - sw_\varepsilon, A)$ is defined as 0 for $t_\varepsilon - sw_\varepsilon < 0$.

Analogously to relation (139), one can get using relation (141) and the definition of $P_{\varepsilon,12}(t_\varepsilon - sw_\varepsilon, A) = 0$, for $t_\varepsilon - sw_\varepsilon < 0$, in relation (145), the following asymptotic relation holds, for any $s_\varepsilon \to s \in [0, \infty)$ as $\varepsilon \to 0$, $A \in \Gamma$ and $s \neq t$,

$$P_{\varepsilon,12}(t_\varepsilon - s_\varepsilon w_\varepsilon, A) \to 0 \text{ as } \varepsilon \to 0. \tag{146}$$

Therefore, relations (129) and (146) imply, by Lemma 2 given in Subsection 4.3, that the following relation takes place, for $A \in \Gamma$ and any $0 \leq t_\varepsilon \to \infty$ as $\varepsilon \to 0$ such that $t_\varepsilon / w_\varepsilon \to t \in (0, \infty)$ as $\varepsilon \to 0$,

$$\int_0^\infty P_{\varepsilon,12}(t_\varepsilon - sw_\varepsilon, A)\mathsf{P}_2\{w_\varepsilon^{-1}\tilde{\tau}_{\varepsilon,1} \in ds\} \to \int_0^\infty 0 \cdot e_{0,2}^{-1} e^{-s/e_{0,2}} ds = 0 \text{ as } \varepsilon \to 0. \tag{147}$$

Relations (143) – (145) and (147) imply that the following relation holds for $A \in \Gamma$ and any $0 \leq t_\varepsilon \to \infty$ as $\varepsilon \to 0$ such that $t_\varepsilon / w_\varepsilon \to t \in (0, \infty)$ as $\varepsilon \to 0$,

$$P_{\varepsilon,22}(t_\varepsilon, A) \to e^{-t/e_{0,2}} \pi_{0,2}(A) \text{ as } \varepsilon \to 0. \tag{148}$$

The proof of Theorem 13 is completed.

The proof of Theorem 14 is absolutely analogous to the proof of Theorem 13, due to simmetrisity conditions **F** – **J** and $\mathbf{N}_1$ with respect to indices $i, j = 1, 2$. The only formula (135) for the corresponding limiting probabilities should be replaced by formula (136). □

5.5. Short time ergodic theorems for singularly perturbed alternating regenerative processes, under the assumption that condition $\mathbf{K}_0$ or $\mathbf{K}_\infty$ hold and $t_\varepsilon / w_\varepsilon \to 0$ as $\varepsilon \to 0$. In this subsection, we consider the case, where parameter $t = 0$ in relation (122). In this case, relation (122) means, for times $t_\varepsilon \to \infty$ as $\varepsilon \to 0$, that,

$$t_\varepsilon \prec w_\varepsilon \text{ as } \varepsilon \to 0. \tag{149}$$

The corresponding limiting probabilities are the same for both cases, where condition $\mathbf{K}_0$ or $\mathbf{K}_\infty$ holds, and take the following form, for $A \in$

$\Gamma, i,j = 1,2,$

$$\pi_{0,ij}(A) = \begin{cases} \pi_{0,i}(A) & \text{for } j = i, \\ 0 & \text{for } j \neq i. \end{cases} \tag{150}$$

The following theorem takes place.

Theorem 15. *Let conditions* $\mathbf{F}$ – $\mathbf{J}$, $\mathbf{N}_1$ *and* $\mathbf{K}_0$ *or* $\mathbf{K}_\infty$ *hold. Then, for every* $A \in \Gamma, i,j = 1,2,$ *and* $0 \leq t_\varepsilon \to \infty$ *as* $\varepsilon \to 0$ *such that* $t_\varepsilon/w_\varepsilon \to 0$ *as* $\varepsilon \to 0$,

$$P_{\varepsilon,ij}(t_\varepsilon, A) \to \pi_{0,ij}(A) \text{ as } \varepsilon \to 0. \tag{151}$$

Proof. Let us, first, assume that condition $\mathbf{K}_0$ holds.

Relation $t_\varepsilon/w_\varepsilon \to 0$ as $\varepsilon \to 0$ implies relation $t_\varepsilon/v_\varepsilon \to 0$ as $\varepsilon \to 0$. This makes it possible to repeat the part of proof of Theorem 10 given in relations (112) – (118) and to get the asymptotic relation,

$$\mathsf{P}_{11}(t_\varepsilon, A) \to \pi_{0,1}(A) \text{ as } \varepsilon \to 0. \tag{152}$$

In the case, where condition $\mathbf{K}_0$ holds, $w_\varepsilon \sim p_{\varepsilon,21}^{-1}$ as $\varepsilon \to 0$. According this relation and relation (59), if $\eta_\varepsilon(0) = 2$, then,

$$w_\varepsilon^{-1}\tilde{\tau}_{\varepsilon,1} \xrightarrow{d} e_{0,2}\zeta \text{ as } \varepsilon \to 0, \tag{153}$$

where ζ is a random variable exponentially distributed, with parameter 1.

Since, we assumed that $t_\varepsilon/w_\varepsilon \to 0$ as $\varepsilon \to 0$, the above convergence in distribution relation, obviously, implies that,

$$\mathsf{P}_2\{\tilde{\tau}_{\varepsilon,1} > t_\varepsilon\} = \mathsf{P}_2\{w_\varepsilon^{-1}\tilde{\tau}_{\varepsilon,1} > t_\varepsilon w_\varepsilon^{-1}\} \to 1 \text{ as } \varepsilon \to 0. \tag{154}$$

If $\eta_\varepsilon(0) = 2$, then, for every $t > 0$, event $\{\eta_\varepsilon(t) = 1\} \subseteq \{\tilde{\tau}_{\varepsilon,1} \leq t\}$. Thus, for every $A \in \Gamma$,

$$\begin{aligned} \mathsf{P}_{21}(t_\varepsilon, A) &= \mathsf{P}_2\{\xi_\varepsilon(t_\varepsilon) \in A, \eta_\varepsilon(t_\varepsilon) = 1\} \\ &\leq \mathsf{P}_2\{\tilde{\tau}_{\varepsilon,1} \leq t_\varepsilon\} \to 0 \text{ as } \varepsilon \to 0. \end{aligned} \tag{155}$$

As was pointed out in Subsection 2.5, the phase space $\mathbb{X} \in \Gamma$. Also, $\pi_{0,1}(\mathbb{X}) = 1$. Thus, relation (152) implies that the following relation holds, for $A \in \Gamma$ and any $0 \leq t_\varepsilon \to \infty$ as $\varepsilon \to 0$ such that $t_\varepsilon/w_\varepsilon \to 0$ as $\varepsilon \to 0$,

$$\begin{aligned} P_{\varepsilon,12}(t_\varepsilon, A) &\leq P_{\varepsilon,12}(t_\varepsilon, \mathbb{X}) = 1 - P_{\varepsilon,11}(t_\varepsilon, \mathbb{X}) \\ &\to 1 - \pi_{0,1}(\mathbb{X}) = 0 \text{ as } \varepsilon \to 0. \end{aligned} \tag{156}$$

Finally, let us analyse the asymptotic behaviour of probabilities $P_{\varepsilon,22}(t_\varepsilon, A)$ and, thus, assume that $\eta_\varepsilon(0) = 2$.

We return back to the initial alternating regenerative process $(\xi_\varepsilon(t), \eta_\varepsilon(t))$, $t \geq 0$ with regeneration times $\tau_{\varepsilon,n}, n = 0, 1, \ldots$.

Recall the stopping time $\tilde{\tau}_{\varepsilon,1}$, which is the time of first hitting sate 1 by process $\eta_\varepsilon(t)$.

Let us again consider the regenerative process $\xi_{\varepsilon,2}(t), t \geq 0$ with regeneration times $\tau_{\varepsilon,2,n}, n = 0, 1, \ldots$, and the random lifetime $\mu_{\varepsilon,2,+}$ introduced in Subsection 4.4.

It is readily seen that, for every $t \geq 0$,

$$\tilde{Q}_{\varepsilon,21}(t) = \mathsf{P}_2\{\tilde{\tau}_{\varepsilon,1} \leq t\} = \mathsf{P}\{\mu_{\varepsilon,2,+} \leq t\} \tag{157}$$

and, for every $A \in \mathcal{B}_{\mathbb{X}}, t \geq 0$,

$$\mathsf{P}_2\{\xi_\varepsilon(t) \in A, \eta_\varepsilon(t) = 2, \tilde{\tau}_{\varepsilon,1} > t\} = \mathsf{P}\{\xi_{\varepsilon,2}(t) \in A, \mu_{\varepsilon,2,+} > t\}. \tag{158}$$

Since, $t_\varepsilon / w_\varepsilon \to 0$ as $\varepsilon \to 0$, relations (153) and (157) imply that,

$$\begin{aligned} \mathsf{P}\{\mu_{\varepsilon,2,+} > t_\varepsilon\} &= \mathsf{P}_2\{\tilde{\tau}_{\varepsilon,1} > t_\varepsilon\} \\ &= \mathsf{P}_2\{w_\varepsilon^{-1}\tilde{\tau}_{\varepsilon,1} > t_\varepsilon w_\varepsilon^{-1}\} \to 1 \text{ as } \varepsilon \to 0. \end{aligned} \tag{159}$$

Relations (113) and (159) imply that

$$\begin{aligned} \mathsf{P}_2\{\xi_\varepsilon(t_\varepsilon) \in A, \eta_\varepsilon(t_\varepsilon) = 2\} &- \mathsf{P}_2\{\xi_\varepsilon(t_\varepsilon) \in A, \eta_\varepsilon(t_\varepsilon) = 2, \tilde{\tau}_{\varepsilon,1} > t_\varepsilon\} \\ &\leq \mathsf{P}_2\{\tilde{\tau}_{\varepsilon,1} \leq t_\varepsilon\} \to 0 \text{ as } \varepsilon \to 0. \end{aligned} \tag{160}$$

and, analogously,

$$\begin{aligned} \mathsf{P}\{\xi_{\varepsilon,2}(t_\varepsilon) \in A\} &- \mathsf{P}\{\xi_{\varepsilon,2}(t_\varepsilon) \in A, \mu_{\varepsilon,2,+} > t_\varepsilon\} \\ &\leq \mathsf{P}\{\mu_{\varepsilon,2,+} \leq t_\varepsilon\} \to 0 \text{ as } \varepsilon \to 0, \end{aligned} \tag{161}$$

These relations and Theorem 1, which can be applied to the regenerative processes $\xi_{\varepsilon,2}(t)$, imply that, for every $A \in \Gamma$,

$$\begin{aligned} \lim_{\varepsilon \to 0} \mathsf{P}_{22}(t_\varepsilon, A) &= \lim_{\varepsilon \to 0} \mathsf{P}_2\{\xi_\varepsilon(t_\varepsilon) \in A, \eta_\varepsilon(t_\varepsilon) = 2\} \\ &= \lim_{\varepsilon \to 0} \mathsf{P}_2\{\xi_\varepsilon(t_\varepsilon) \in A, \eta_\varepsilon(t_\varepsilon) = 2, \tilde{\tau}_{\varepsilon,1} > t_\varepsilon\} \\ &= \lim_{\varepsilon \to 0} \mathsf{P}\{\xi_{\varepsilon,2}(t_\varepsilon) \in A, \mu_{\varepsilon,2,+} > t_\varepsilon\} \\ &= \lim_{\varepsilon \to 0} \mathsf{P}\{\xi_{\varepsilon,2}(t_\varepsilon) \in A\} = \pi_{0,2}(A). \end{aligned} \tag{162}$$

In the case of holding condition $\mathbf{K}_\infty$, the proof is analogous. □

6. Ergodic theorems for super-singularly perturbed alternating regenerative processes

In this section, we present ergodic theorems for super-singularly perturbed alternating regenerative processes. As for singularly perturbed alternating regenerative processes, these theorems take different forms of super-long, long and short time ergodic theorems for different asymptotic time zones.

6.1. Super-singularly perturbed alternating regenerative processes. Let us consider the alternating regenerative processes with the super-singular modulation model, where, additionally to $\mathbf{F} - \mathbf{J}$, the following condition holds:

$\mathbf{N}_2$: **(a)** $p_{\varepsilon,12} = 0$, for $\varepsilon \in [0, 1]$, and $0 < p_{\varepsilon,21} \to p_{0,21} = 0$ as $\varepsilon \to 0$, or **(b)** $0 < p_{\varepsilon,12} \to p_{0,12} = 0$ as $\varepsilon \to 0$, and $p_{\varepsilon,21} = 0$, for $\varepsilon \in [0, 1]$.

In this case, $v_\varepsilon = \infty, \varepsilon \in (0, 1]$.

The role of time scaling factor is played function $w_\varepsilon, \varepsilon \in (0, 1]$. Note that $w_\varepsilon = p^{-1}_{\varepsilon,21}, \varepsilon \in (0, 1]$, if condition $\mathbf{N}_2$ **(a)** holds, while $w_\varepsilon = p^{-1}_{\varepsilon,12}, \varepsilon \in (0, 1]$, if condition $\mathbf{N}_2$ **(b)** holds.

We shall investigate asymptotic behaviour of probabilities $P_{\varepsilon,ij}(t_\varepsilon, A)$ under for $0 \leq t_\varepsilon \to \infty$ as $\varepsilon \to 0$ such that the following time scaling relation holds,

$$t_\varepsilon / w_\varepsilon \to t \in [0, \infty] \text{ as } \varepsilon \to 0. \tag{163}$$

It is readily seen that that conditions $\mathbf{N}_2$ **(a)** and $\mathbf{N}_2$ **(b)** are, in some sense, stronger forms, respectively, of conditions $\mathbf{K}_0$ and $\mathbf{K}_\infty$. That is why, it is expectable that the corresponding individual ergodic theorems for super-singularly perturbed alternating regenerative processes should take forms analogous to those presented for singularly perturbed alternating regenerative processes in short time ergodic Theorems 11 – 15, for models with asymptotic time zones generated by the asymptotic relation (163).

We also include in the class of super-singularly perturbed alternating regenerative processes the extremal case of absolutely singular perturbed alternating regenerative processes where, additionally to $\mathbf{F} - \mathbf{J}$, the following condition holds:

$\mathbf{N}_3$: $p_{\varepsilon,12}, p_{\varepsilon,21} = 0$, for $\varepsilon \in [0, 1]$.

6.2. Super-long time ergodic theorems for super-singularly perturbed alternating regenerative processes. In this subsection, we investigate asymptotic behaviour for probabilities $P_{\varepsilon,ij}(t_\varepsilon, A)$ for times $0 \leq t_\varepsilon \to$

∞ as $\varepsilon \to 0$ satisfying the following relation,

$$t_\varepsilon / w_\varepsilon \to \infty \text{ as } \varepsilon \to 0. \tag{164}$$

The corresponding limiting probabilities take the following form, for $A \in \Gamma, i, j = 1, 2$,

$$\pi^{(0)}_{0,j}(A) = \begin{cases} \pi_{0,1}(A) & \text{for } j = 1, \\ 0 & \text{for } j = 2. \end{cases} \tag{165}$$

and

$$\pi^{(\infty)}_{0,j}(A) = \begin{cases} 0 & \text{for } j = 1, \\ \pi_{0,2}(A) & \text{for } j = 2. \end{cases} \tag{166}$$

The following theorems takes place.

Theorem 16. *Let conditions* **F** – **J** *and* $\mathbf{N}_2$ **(a)** *hold. Then, for every* $A \in \Gamma, i, j = 1, 2,$ *and* $0 \leq t_\varepsilon \to \infty$ *as* $\varepsilon \to 0$ *such that* $t_\varepsilon / w_\varepsilon \to \infty$ *as* $\varepsilon \to 0$,

$$P_{\varepsilon,ij}(t_\varepsilon, A) \to \pi^{(0)}_{0,j}(A) \text{ as } \varepsilon \to 0. \tag{167}$$

Theorem 17. *Let conditions* **F** – **J** *and* $\mathbf{N}_2$ **(b)** *hold. Then, for every* $A \in \Gamma, i, j = 1, 2,$ *and* $0 \leq t_\varepsilon \to \infty$ *as* $\varepsilon \to 0$ *such that* $t_\varepsilon / w_\varepsilon \to \infty$ *as* $\varepsilon \to 0$,

$$P_{\varepsilon,ij}(t_\varepsilon, A) \to \pi^{(\infty)}_{0,j}(A) \text{ as } \varepsilon \to 0. \tag{168}$$

Proof. The asymptotic behaviour for probabilities $P_{\varepsilon,11}(t_\varepsilon, A)$ is obviously given by Theorems 1. Indeed, if $\eta_\varepsilon(0) = 1$, then condition $\mathbf{N}_2$ **(a)** implies that the process $\xi_\varepsilon(t), t \geq 0$ coincides with the process $\xi_{\varepsilon,1}(t), t \geq 0$, while the process $\eta_\varepsilon(t) = 1, t \geq 0$. Thus, the following relation takes place, for any $A \in \Gamma$, and any $0 \leq t_\varepsilon \to \infty$ as $\varepsilon \to 0$,

$$P_{\varepsilon,11}(t_\varepsilon, A) \to \pi_{0,1}(A) \text{ as } \varepsilon \to 0. \tag{169}$$

Also, for any $A \in \Gamma$ and $t \geq 0$,

$$P_{\varepsilon,12}(t, A) = 0. \tag{170}$$

According relation (59), if $\eta_\varepsilon(0) = 2$, then,

$$w_\varepsilon^{-1} \tilde{\tau}_{\varepsilon,1} \xrightarrow{d} e_{0,2}\zeta \text{ as } \varepsilon \to 0, \tag{171}$$

where ζ is a random variable exponentially distributed, with parameter 1.

The following renewal type relation connects probabilities $P_{\varepsilon,11}(t_\varepsilon, A)$ and $P_{\varepsilon,21}(t_\varepsilon, A)$,

$$\begin{aligned} P_{\varepsilon,21}(t_\varepsilon, A) &= \int_0^{t_\varepsilon} P_{\varepsilon,11}(t_\varepsilon - s, A) \mathsf{P}_2\{\tilde{\tau}_{\varepsilon,1} \in ds\} \\ &= \int_0^{\infty} P_{\varepsilon,11}(t_\varepsilon - s w_\varepsilon, A) \mathsf{P}_2\{w_\varepsilon^{-1} \tilde{\tau}_{\varepsilon,1} \in ds\}, \end{aligned} \tag{172}$$

where function $P_{\varepsilon,11}(t_\varepsilon - sw_\varepsilon, A)$ is defined as 0 for $t_\varepsilon - sw_\varepsilon < 0$.

Let us take an arbitrary $s_\varepsilon \to s \in [0, \infty)$ as $\varepsilon \to 0$. Obviously, $(t_\varepsilon - s_\varepsilon w_\varepsilon)/w_\varepsilon = t_\varepsilon/w_\varepsilon - s_\varepsilon \to \infty$ as $\varepsilon \to 0$. That is why, according relation (169), the following asymptotic relation take place, for $A \in \Gamma$ and $s \in [0, \infty)$,

$$P_{\varepsilon,11}(t_\varepsilon - s_\varepsilon w_\varepsilon, A) \to \pi_{0,1}(A) \text{ as } \varepsilon \to 0. \tag{173}$$

Relations (171) and (173) imply, by Lemma 2 given in Subsection 4.3 that the following relation takes place, for $A \in \Gamma$,

$$P_{\varepsilon,21}(t_\varepsilon, A) \to \int_0^\infty \pi_{0,1}(A) \mathsf{P}\{e_{0,2}\zeta \in ds\} = \pi_{0,1}(A) \text{ as } \varepsilon \to 0. \tag{174}$$

As was pointed out in Section 3, the phase space $\mathbb{X} \in \Gamma$. Also, $\pi_{0,1}(\mathbb{X}) = 1$. Thus, relations (174) implies that the following relation holds, for $A \in \Gamma$,

$$\begin{aligned} P_{\varepsilon,22}(t_\varepsilon, A) \leq P_{\varepsilon,22}(t_\varepsilon, \mathbb{X}) = 1 - P_{\varepsilon,21}(t_\varepsilon, \mathbb{X}) \\ \to 1 - \pi_{0,1}(\mathbb{X}) = 0 \text{ as } \varepsilon \to 0. \end{aligned} \tag{175}$$

The proof of Theorem 16 is completed.

The proof of Theorem 17 is absolutely analogous. □

6.3. Long time ergodic theorems for super-singularly perturbed alternating regenerative processes. In this subsection, we investigate asymptotic behaviour for probabilities $P_{\varepsilon,ij}(t_\varepsilon, A)$ for times $0 \leq t_\varepsilon \to \infty$ as $\varepsilon \to 0$ satisfying the following relation,

$$t_\varepsilon/w_\varepsilon \to t \in (0, \infty) \text{ as } \varepsilon \to 0. \tag{176}$$

The corresponding limiting probabilities take that following form for $A \in \Gamma, i, j = 1, 2$ and $t \in (0, \infty)$,

$$\dot{\pi}_{0,ij}^{(0)}(t, A) = \begin{cases} \pi_{0,1}(A) & \text{for } i = 1, j = 1, \\ 0 & \text{for } i = 1, j = 2, \\ (1 - e^{-t/e_{0,2}})\pi_{0,1}(A) & \text{for } i = 2, j = 1, \\ e^{-t/e_{0,2}}\pi_{0,2}(A) & \text{for } i = 2, j = 2. \end{cases} \tag{177}$$

and

$$\dot{\pi}_{0,ij}^{(\infty)}(t, A) = \begin{cases} e^{-t/e_{0,1}}\pi_{0,1}(A) & \text{for } i = 1, j = 1, \\ (1 - e^{-t/e_{0,1}})\pi_{0,2}(A) & \text{for } i = 1, j = 2, \\ 0 & \text{for } i = 2, j = 1, \\ \pi_{0,2}(A) & \text{for } i = 2, j = 2. \end{cases} \tag{178}$$

The following theorems take place.

Theorem 18. *Let conditions* **F** – **J** *and* $\mathbf{N}_2$ **(a)** *hold. Then, for every* $A \in \Gamma, i, j = 1, 2,$ *and* $0 \leq t_\varepsilon \to \infty$ *as* $\varepsilon \to 0$ *such that* $t_\varepsilon / w_\varepsilon \to t \in (0, \infty)$ *as* $\varepsilon \to 0,$

$$P_{\varepsilon,ij}(t_\varepsilon, A) \to \dot{\pi}^{(0)}_{ij}(t, A) \text{ as } \varepsilon \to 0. \tag{179}$$

Theorem 19. *Let conditions* **F** – **J** *and* $\mathbf{N}_2$ **(b)** *hold. Then, for every* $A \in \Gamma, i, j = 1, 2,$ *and* $0 \leq t_\varepsilon \to \infty$ *as* $\varepsilon \to 0$ *such that* $t_\varepsilon / w_\varepsilon \to t \in (0, \infty)$ *as* $\varepsilon \to 0,$

$$P_{\varepsilon,ij}(t_\varepsilon, A) \to \dot{\pi}^{(\infty)}_{0,ij}(t, A) \text{ as } \varepsilon \to 0. \tag{180}$$

Proof. The asymptotic behaviour of probabilities $P_{\varepsilon,1j}(t_\varepsilon, A), j = 1, 2$ is given, under the assumption that condition $\mathbf{N}_2$ **(a)** holds, is given by relations (169) and (170), in the proof of Theorem 16.

Recall again relation (59). If $\eta_\varepsilon(0) = 2$, then, for $u \geq 0$,

$$\mathsf{P}_2\{w_\varepsilon^{-1}\tilde{\tau}_{\varepsilon,1} \leq u\} \to 1 - e^{-u/e_{0,2}} \text{ as } \varepsilon \to 0. \tag{181}$$

Also recall the renewal type relation connecting probabilities $P_{\varepsilon,11}(t_\varepsilon, A)$ and $P_{\varepsilon,21}(t_\varepsilon, A)$,

$$\begin{aligned} P_{\varepsilon,21}(t_\varepsilon, A) &= \int_0^{t_\varepsilon} P_{\varepsilon,11}(t_\varepsilon - s, A)\mathsf{P}_2\{\tilde{\tau}_{\varepsilon,1} \in ds\} \\ &= \int_0^{\infty} P_{\varepsilon,11}(t_\varepsilon - sw_\varepsilon, , A)\mathsf{P}_2\{w_\varepsilon^{-1}\tilde{\tau}_{\varepsilon,1} \in ds\}, \end{aligned} \tag{182}$$

where function $P_{\varepsilon,11}(t_\varepsilon - sw_\varepsilon, A)$ is defined as 0 for $t_\varepsilon - sw_\varepsilon < 0$.

Let us take an arbitrary $s_\varepsilon \to s \in [0, \infty)$ as $\varepsilon \to 0$. Obviously, $(t_\varepsilon - s_\varepsilon w_\varepsilon)/w_\varepsilon = t_\varepsilon/w_\varepsilon - s_\varepsilon \to t - s$ as $\varepsilon \to 0$. That is why, according relation (169) and the above definition of $P_{\varepsilon,11}(t_\varepsilon - sw_\varepsilon, A) = 0$, for $t_\varepsilon - sw_\varepsilon < 0$. the following asymptotic relation holds, for $A \in \Gamma$ and $s \neq t$,

$$P_{\varepsilon,11}(t_\varepsilon - s_\varepsilon w_\varepsilon, A) \to \pi_{0,1}(A)\mathrm{I}(t > s) \text{ as } \varepsilon \to 0. \tag{183}$$

Note that convergence of $P_{\varepsilon,11}(t_\varepsilon - s_\varepsilon w_\varepsilon, A)$ as $\varepsilon \to 0$ is not guarantied for $s = t$. However the limiting distribution in relation (181) is exponential and, thus, it has not an atom at any point $t > 0$.

Therefore, relations (181) and (183) imply, by Lemma 2 given Subsection 4.3, that the following relation takes place, for $A \in \Gamma$ and $t \in (0, \infty)$,

$$\begin{aligned} P_{\varepsilon,21}(t_\varepsilon, A) &\to \int_0^{\infty} \pi_{0,1}(A)\mathrm{I}(t > s)e_{0,2}^{-1}e^{-s/e_{0,2}}ds \\ &= (1 - e^{-t/e_{0,2}})\pi_{0,1}(A) \text{ as } \varepsilon \to 0. \end{aligned} \tag{184}$$

It remains to give the asymptotic analysis of asymptotic behaviour for probabilities $P_{\varepsilon,22}(t_\varepsilon, A)$.

Let us introduce random variables $\mu_{\varepsilon,2,n} = \kappa_{\varepsilon,2,n}\mathrm{I}(\eta_{\varepsilon,2,n} = 2), n = 1, 2, \ldots$. Let now consider the random sequence of triplets $\langle \bar{\xi}_{\varepsilon,2,n} = \langle \xi_{\varepsilon,2,n}(t), t \geq 0\rangle, \kappa_{\varepsilon,2,n}, \mu_{\varepsilon,2,n}\rangle, n = 1, 2, \ldots$, the regenerative process $\xi_{\varepsilon,2}(t) = \xi_{\varepsilon,2,n}(t - \tau_{\varepsilon,2,n-1})$, for $t \in [\tau_{\varepsilon,2,n-1}, \tau_{\varepsilon,2,n})$, $n = 1, 2, \ldots$, with regeneration times $\tau_{\varepsilon,2,n} = \kappa_{\varepsilon,2,1} + \cdots + \kappa_{\varepsilon,2,n}, n = 1, 2, \ldots, \tau_{\varepsilon,2,0} = 0$, and the random lifetime $\mu_{\varepsilon,2,+} = \tau_{\varepsilon,2,\nu_{\varepsilon,2}}$, where $\nu_{\varepsilon,2} = \min(n \geq 1 : \mu_{\varepsilon,2,n} < \kappa_{\varepsilon,2,n}) = \min(n \geq 1 : \eta_{\varepsilon,2,n} = 1)$.

Let us also denote $P_{\varepsilon,2,+}(t, A) = \mathsf{P}_2\{\xi_{\varepsilon,2}(t) \in A, \mu_{\varepsilon,2,+} > t\}$. In this case, the distribution function $F_{\varepsilon,2}(t) = \mathsf{P}\{\kappa_{\varepsilon,2,1} \leq t, \mu_{\varepsilon,2,1} \geq \kappa_{\varepsilon,2,1}\} = \mathsf{P}\{\kappa_{\varepsilon,2,1} \leq t, \eta_{\varepsilon,2,1} = 2\}, t \geq 0$, the stopping probability $f_{\varepsilon,2} = \mathsf{P}\{\mu_{\varepsilon,2,1} < \kappa_{\varepsilon,2,1}\} = \mathsf{P}\{\eta_{\varepsilon,2,1} = 1\} = p_{\varepsilon,21}$, and expectation $e_{\varepsilon,2} = \mathsf{E}\kappa_{\varepsilon,2,1}\mathrm{I}(\mu_{\varepsilon,2,1} \geq \kappa_{\varepsilon,2,1}) = \mathsf{E}\kappa_{\varepsilon,2,1}\mathrm{I}(\eta_{\varepsilon,2,1} = 2) = e_{\varepsilon,22}$.

Condition $\mathbf{N}_2$ **(a)** implies that, for every $A \in \mathcal{B}_{\mathbb{X}}, t \geq 0$,

$$\begin{aligned} P_{\varepsilon,22}(t, A) &= \mathsf{P}_2\{\xi_\varepsilon(t) \in A, \tilde{\tau}_{\varepsilon,1} > t\} \\ &= \mathsf{P}\{\xi_{\varepsilon,2}(t) \in A, \mu_{\varepsilon,2,+} > t\} = P_{\varepsilon,2,+}(t, A). \end{aligned} \tag{185}$$

Conditions **F** – **J** and $\mathbf{N}_2$ **(a)** and imply that conditions **A** – **D** holds. Thus, conditions of Theorem 18 imply that all conditions of Theorem 3 hold for the regenerative processes $\xi_{\varepsilon,2}(t), t \geq 0$ with regenerative times $\tau_{\varepsilon,2,n}, n = 1, 2, \ldots$ and random lifetimes $\mu_{\varepsilon,2,+}$. Therefore, the following relation holds, for any $A \in \Gamma$, and $t_\varepsilon \to t \in (0, \infty)$ as $\varepsilon \to 0$,

$$P_{\varepsilon,22}(t_\varepsilon, A) = P_{\varepsilon,2,+}(t_\varepsilon, A) \to e^{-t/e_{0,2}}\pi_{0,2}(A) \text{ as } \varepsilon \to 0. \tag{186}$$

The proof of Theorem 18 is completed.

The proof of Theorem 19 is absolutely analogous. □

6.4. Short time ergodic theorems for super-singularly perturbed alternating regenerative processes. In this subsection, we investigate asymptotic behaviour for probabilities $P_{\varepsilon,ij}(t_\varepsilon, A)$ for times $0 \leq t_\varepsilon \to \infty$ as $\varepsilon \to 0$ satisfying the following relation,

$$t_\varepsilon / w_\varepsilon \to 0 \text{ as } \varepsilon \to 0. \tag{187}$$

The corresponding limiting probabilities are the same for both case, where condition $\mathbf{N}_2$ **(a)** or $\mathbf{N}_2$ **(b)** holds. They take the following form, for $A \in \Gamma, i, j = 1, 2$,

$$\pi_{0,ij}(A) = \begin{cases} \pi_{0,i}(A) & \text{for } j = i, \\ 0 & \text{for } j \neq i. \end{cases} \tag{188}$$

The following theorem takes place.

Theorem 20. *Let conditions* **F** – **J** *and* $\mathbf{N}_2$ *hold. Then, for every* $A \in \Gamma, i, j = 1, 2,$ *and* $0 \leq t_\varepsilon \to \infty$ *as* $\varepsilon \to 0$ *such that* $t_\varepsilon / w_\varepsilon \to 0$ *as* $\varepsilon \to 0$,

$$P_{\varepsilon,ij}(t_\varepsilon, A) \to \pi_{0,ij}(A) \text{ as } \varepsilon \to 0. \tag{189}$$

Proof. Let us, first, assume that condition $\mathbf{N}_2$ **(a)** holds.

The asymptotic behaviour of probabilities $P_{\varepsilon,1j}(t_\varepsilon, A), j = 1, 2$ is given, under the assumption that condition $\mathbf{N}_2$ **(a)** holds, by relations (169) and (170), in the proof of Theorem 17.

It is readily seen that, for every $t \geq 0$,

$$\tilde{Q}_{\varepsilon,21}(t) = \mathsf{P}_2\{\tilde{\tau}_{\varepsilon,1} \leq t\} = \mathsf{P}\{\mu_{\varepsilon,2,+} \leq t\} \tag{190}$$

and, for every $A \in \mathcal{B}_{\mathbb{X}}, t \geq 0$,

$$\mathsf{P}_2\{\xi_\varepsilon(t) \in A, \eta_\varepsilon(t) = 2, \tilde{\tau}_{\varepsilon,1} > t\} = \mathsf{P}\{\xi_{\varepsilon,2}(t) \in A, \mu_{\varepsilon,2,+} > t\}. \tag{191}$$

According relation (59), if $\eta_\varepsilon(0) = 2$, random variables, $w_\varepsilon^{-1}\tilde{\tau}_{\varepsilon,1} \stackrel{d}{\longrightarrow} e_{0,2}\zeta$ as $\varepsilon \to 0$, where ζ is a random variable exponentially distributed with parameter 1. Since, we assumed that $t_\varepsilon / w_\varepsilon \to 0$ as $\varepsilon \to 0$, the above convergence in distribution relation and relation (190) imply that,

$$\begin{aligned} \mathsf{P}\{\mu_{\varepsilon,2,+} > t_\varepsilon\} &= \mathsf{P}_2\{\tilde{\tau}_{\varepsilon,1} > t_\varepsilon\} \\ &= \mathsf{P}_2\{w_\varepsilon^{-1}\tilde{\tau}_{\varepsilon,1} > t_\varepsilon w_\varepsilon^{-1}\} \to 1 \text{ as } \varepsilon \to 0. \end{aligned} \tag{192}$$

Relations (191) and (192) imply that

$$\begin{aligned} \mathsf{P}_2\{\xi_\varepsilon(t_\varepsilon) \in A, \eta_\varepsilon(t_\varepsilon) = 2\} - \mathsf{P}_2\{\xi_\varepsilon(t_\varepsilon) \in A, \eta_\varepsilon(t_\varepsilon) = 2, \tilde{\tau}_{\varepsilon,1} > t_\varepsilon\} \\ \leq \mathsf{P}_2\{\tilde{\tau}_{\varepsilon,1} \leq t_\varepsilon\} \to 0 \text{ as } \varepsilon \to 0, \end{aligned} \tag{193}$$

and, analogously,

$$\begin{aligned} \mathsf{P}\{\xi_{\varepsilon,2}(t_\varepsilon) \in A\} - \mathsf{P}\{\xi_{\varepsilon,2}(t_\varepsilon) \in A, \mu_{\varepsilon,2,+} > t_\varepsilon\} \\ \leq \mathsf{P}\{\mu_{\varepsilon,2,+} \leq t_\varepsilon\} \to 0 \text{ as } \varepsilon \to 0, \end{aligned} \tag{194}$$

These relations and Theorem 1, which can be applied to the regenerative processes $\xi_{\varepsilon,2}(t)$, imply that, for every $A \in \Gamma$,

$$\begin{aligned} \lim_{\varepsilon \to 0} \mathsf{P}_{22}(t_\varepsilon, A) &= \lim_{\varepsilon \to 0} \mathsf{P}_2\{\xi_\varepsilon(t_\varepsilon) \in A, \eta_\varepsilon(t_\varepsilon) = 2\} \\ &= \lim_{\varepsilon \to 0} \mathsf{P}_2\{\xi_\varepsilon(t_\varepsilon) \in A, \eta_\varepsilon(t_\varepsilon) = 2, \tilde{\tau}_{\varepsilon,1} > t_\varepsilon\} \\ &= \lim_{\varepsilon \to 0} \mathsf{P}\{\xi_{\varepsilon,2}(t_\varepsilon) \in A, \mu_{\varepsilon,2,+} > t_\varepsilon\} \\ &= \lim_{\varepsilon \to 0} \mathsf{P}\{\xi_{\varepsilon,2}(t_\varepsilon) \in A\} = \pi_{0,2}(A). \end{aligned} \tag{195}$$

If $\eta_\varepsilon(0) = 2$, then, for every $t > 0$, event $\{\eta_\varepsilon(t) = 1\} \subseteq \{\tilde{\tau}_{\varepsilon,1} \leq t\}$. Thus, for every $A \in \Gamma$,

$$\begin{aligned} \mathsf{P}_{21}(t_\varepsilon, A) &= \mathsf{P}_2\{\xi_\varepsilon(t_\varepsilon) \in A, \eta_\varepsilon(t_\varepsilon) = 1\} \\ &\leq \mathsf{P}_2\{\tilde{\tau}_{\varepsilon,1} \leq t_\varepsilon\} \to 0 \text{ as } \varepsilon \to 0. \end{aligned} \tag{196}$$

The proof for the case, where condition $\mathbf{N}_2$ **(b)** holds, is absolutely analogous to the above proof, due to the simmetrisity conditions $\mathbf{F}$ – $\mathbf{J}$ and $\mathbf{N}_2$ **(a)** and **(b)** with respect to indices $i, j = 1, 2$. $\square$

6.5. Ergodic theorems for absolutely singular perturbed alternating regenerative processes. This is the extremal and trivial case, where condition $\mathbf{N}_3$ holds.

In this case, the process $\xi_\varepsilon(t), t \geq 0$ coincides with the process $\xi_{\varepsilon,i}(t), t \geq 0$ and the process $\eta_\varepsilon(t) = i, t \geq 0$, if $\eta_\varepsilon(0) = i$, for $i = 1, 2$.

Thus, the asymptotic behaviour for probabilities $P_{\varepsilon,ii}(t_\varepsilon, A)$ is given by Theorem 1.

Also, probabilities $P_{\varepsilon,12}(t, A), P_{\varepsilon,21}(t, A) = 0$, for $t \geq 0$.

The above remarks can be summarised in following theorem.

Theorem 21. *Let conditions* $\mathbf{F}$ – $\mathbf{J}$ *and* $\mathbf{N}_3$ *hold. Then, for every* $A \in \Gamma, i, j = 1, 2$, *and any* $0 \leq t_\varepsilon \to \infty$ *as* $\varepsilon \to 0$,

$$P_{\varepsilon,ij}(t_\varepsilon, A) \to \pi_{0,ij}(A) \text{ as } \varepsilon \to 0. \tag{197}$$

6.6. One- and multi-dimensional distributions for perturbed alternating regenerative processes. Individual ergodic theorems presented in this paper give ergodic relations for one-dimensional distributions $P_{\varepsilon,ij}(t, A) = \mathsf{P}_i\{\xi_\varepsilon(t) \in A, \eta_\varepsilon(t) = j\}$ for alternating regenerative processes with semi-Markov modulation $(\xi_\varepsilon(t), \eta_\varepsilon(t))$.

This makes it possible to weaken the model assumption **(j)** formulated in Subsection 2.4. This assumption concerns multi-dimensional joint distributions of random variables $\xi_{\varepsilon,i,n}(t_k), k = 1, \ldots, r$ and $\kappa_{\varepsilon,i,n}, \eta_{\varepsilon,i,n}$. This assumption can be replaced by the weaker assumption that the joint distributions of random variables $\xi_{\varepsilon,i,n}(t)$ and $\kappa_{\varepsilon,i,n}, \eta_{\varepsilon,i,n}$ do not depend on $n \geq 1$, for every $t \geq 0$ and $i = 1, 2$.

Process $(\xi_\varepsilon(t), \eta_\varepsilon(t), t \geq 0$ still will process a weaken, say, one-dimensional regenerative property, which, in fact, means that one-dimensional distributions $P_{\varepsilon,ij}(t, A) = \mathsf{P}_i\{\xi_\varepsilon(t) \in A, \eta_\varepsilon(t) = j\}, t \geq 0, i = 1, 2$ satisfy the system of renewal type equations (13). Respectively, formulations of conditions, propositions and proofs of Theorems 4 – 21 still remain to be valid.

6.7. Alternating regenerative processes with transition periods. Ergodic theorems for perturbed alternating regenerative processes can be generalised to such processes with transition periods. In this case, the model assumption **(j)** formulated in Subsection 2.4 is assumed to hold only for $n \geq 2$. The alternating regenerative process $(\xi_\varepsilon(t), \eta_\varepsilon(t)), t \geq 0$ has the transition period $[0, \tau_{\varepsilon,1})$, while the shifted process $(\xi^{(1)}_\varepsilon(t), \eta^{(1)}_\varepsilon(t)) = (\xi_\varepsilon(\tau_{\varepsilon,1}+ t), \eta_\varepsilon(\tau_{\varepsilon,1} + t)) \geq 0$ is a usual alternating regenerative process.

All quantities appearing in conditions **G** – **J** the renewal type equations (13) and relations (12) and (15) should be, in this case, defined using shifted sequence of triplets $\langle \bar{\xi}_{\varepsilon,i,2} = \langle \xi_{\varepsilon,i,2}(t), t \geq 0\rangle, \kappa_{\varepsilon,i,2}, \eta_{\varepsilon,i,2}\rangle, i = 1, 2$. It is also natural to index the above mentioned quantities by the upper index $^{(1)}$, for example, to use notation $P^{(1)}_{\varepsilon,i,j}(t, A) = \mathsf{P}_i\{\xi^{(1)}_\varepsilon(t) \in A, \eta^{(1)}_\varepsilon(t) = j\}$, etc. Probabilities $P^{(1)}_{\varepsilon,ij}(t, A)$ satisfy the system of renewal type equations (13). Theorems 4 – 21 present, in this case, the corresponding ergodic relations for these probabilities.

Instead of condition **E**, condition **G** should be assumed to hold for probabilities $\tilde{p}_{\varepsilon,ij} = \mathsf{P}\{\eta_{\varepsilon,i,1} = j\}, i, j = 1, 2$ and condition **H** (with omitted the non-arithmetic assumption) for transition probabilities $\tilde{Q}_{\varepsilon,ij}(t) = \mathsf{P}\{\kappa_{\varepsilon,i,1} \leq t, \eta_{\varepsilon,i,1} = j\}, t \geq 0, i, j = 1, 2$. The corresponding ergodic relations for probabilities $P_{\varepsilon,ij}(t_\varepsilon, A) = \mathsf{P}_i\{\xi_\varepsilon(t) \in A, \eta_\varepsilon(t) = j\}$ take the form similar with the asymptotic relation (16). If, for example, $P^{(1)}_{\varepsilon,ij}(t_\varepsilon, A) \to \pi^{(\beta)}_{0,ij}(t, A)$ as $\varepsilon \to 0$, for $i = 1, 2$, then, $P_{\varepsilon,ij}(t_\varepsilon, A) \to \tilde{p}_{0,i1}\pi^{(\beta)}_{0,1j}(t, A) + \tilde{p}_{0,i2}\pi^{(\beta)}_{0,2j}(t, A)$ as $\varepsilon \to 0$.

7. Summary of results

In this section, a summary of results obtained in the paper and a list of some open directions for further extension of its results are given.

7.1. Summary of results. As it was pointed in the introduction, the paper presents results of complete analysis and classification of ergodic theorems for perturbed alternating regenerative processes modulated by two states semi-Markov processes.

It is shown that the forms of the corresponding ergodic relations and limiting probabilities appearing in these relations are essentially determined by two parameters.

The first one is parameter $\beta \in [0, \infty]$, which asymptotically balance switching probabilities $p_{\varepsilon,12}$ and $p_{\varepsilon,21}$ between two alternative variants of regenerative processes, in the form of asymptotic relation, $p_{\varepsilon,12}/p_{\varepsilon,21} \to \beta$ as $\varepsilon \to 0$.

The second one is a time scaling parameter $t \in [0, \infty]$, which determines the asymptotic time zones for time $t_\varepsilon \to \infty$ as $\varepsilon \to 0$, in the form of one of two asymptotic relations, $t_\varepsilon/v_\varepsilon \to t$ or $t_\varepsilon/w_\varepsilon \to t$ as $\varepsilon \to 0$, with time scaling factors, respectively, $v_\varepsilon = p^{-1}_{\varepsilon,12} + p^{-1}_{\varepsilon,21}$ or $w_\varepsilon = (p_{\varepsilon,12} + p_{\varepsilon,21})^{-1}$.

The variants of ergodic relations are presented in Theorems 4 – 21, which we split in groups as ergodic theorems for regularly perturbed alternating regenerative processes, and short, long, and super-long time ergodic theorems for singularly and super-singularly perturbed alternating regenerative processes.

The classification of the corresponding individual ergodic theorems is summarised in the following Table 1 (where numbers of theorems, their conditions, the corresponding asymptotic time zones, and the limiting probabilities are given, respectively, in columns 1, 2, 3 and 4).

T	Conditions	Asymptotic time zones	Limiting probabilities
	Regular perturbations		
T	Conditions	Asymptotic time zones	Limiting probabilities
4	$\mathbf{F}-\mathbf{J}$, $\mathbf{M}_1$, $\beta=1$	$t_\varepsilon \to \infty$	$\pi^{(1)}_{0,j}(A)$
5	$\mathbf{F}-\mathbf{J}$, $\mathbf{M}_2$, $\beta \in (0,\infty)$	$t_\varepsilon \to \infty$	$\pi^{(\beta)}_{0,j}(A)$
6	$\mathbf{F}-\mathbf{J}$, $\mathbf{M}_3$, $\beta=0$	$t_\varepsilon \to \infty$	$\pi^{(0)}_{0,j}(A)$
7	$\mathbf{F}-\mathbf{J}$, $\mathbf{M}_3$, $\beta=\infty$	$t_\varepsilon \to \infty$	$\pi^{(\infty)}_{0,j}(A)$
	Singular perturbations		
T	Conditions	Asymptotic time zones	Limiting probabilities
8	$\mathbf{F}-\mathbf{J}$, $\mathbf{N}_1$, $\mathbf{K}_\beta, \beta \in [0,\infty]$	$v_\varepsilon \prec t_\varepsilon$	$\pi^{(\beta)}_{0,j}(A)$
9	$\mathbf{F}-\mathbf{J}$, $\mathbf{N}_1$, $\mathbf{K}_\beta$, $\beta \in [0,\infty]$	$t_\varepsilon \sim tv_\varepsilon, t \in (0,\infty)$	$\pi^{(\beta)}_{0,ij}(t,A)$
10	$\mathbf{F}-\mathbf{J}$, $\mathbf{N}_1$, $\mathbf{K}_\beta$, $\beta \in (0,\infty)$	$t_\varepsilon \prec v_\varepsilon, t_\varepsilon \to \infty$	$\pi_{0,ij}(A)$
11	$\mathbf{F}-\mathbf{J}$, $\mathbf{N}_1$, $\mathbf{K}_0$	$w_\varepsilon \prec t_\varepsilon \prec v_\varepsilon$	$\pi^{(0)}_{0,j}(A)$
12	$\mathbf{F}-\mathbf{J}$, $\mathbf{N}_1$, $\mathbf{K}_\infty$	$w_\varepsilon \prec t_\varepsilon \prec v_\varepsilon$	$\pi^{(\infty)}_{0,j}(A)$
13	$\mathbf{F}-\mathbf{J}$, $\mathbf{N}_1$, $\mathbf{K}_0$	$t_\varepsilon \sim tw_\varepsilon, t \in (0,\infty)$	$\dot{\pi}^{(0)}_{0,ij}(t,A)$
14	$\mathbf{F}-\mathbf{J}$, $\mathbf{N}_1$, $\mathbf{K}_\infty$	$t_\varepsilon \sim tw_\varepsilon, t \in (0,\infty)$	$\dot{\pi}^{(\infty)}_{0,ij}(t,A)$
15	$\mathbf{F}-\mathbf{J}$, $\mathbf{N}_1$, $\mathbf{K}_0$ or $\mathbf{K}_\infty$	$t_\varepsilon \prec w_\varepsilon, t_\varepsilon \to \infty$	$\pi_{0,ij}(A)$
	Super-singular perturbations		
T	Conditions	Asymptotic time zones	Limiting probabilities
16	$\mathbf{F}-\mathbf{J}$, $\mathbf{N}_2$ **(a)**	$w_\varepsilon \prec t_\varepsilon$	$\pi^{(0)}_{0,j}(A)$
17	$\mathbf{F}-\mathbf{J}$, $\mathbf{N}_2$ **(b)**	$w_\varepsilon \prec t_\varepsilon$	$\pi^{(\infty)}_{0,j}(A)$
18	$\mathbf{F}-\mathbf{J}$, $\mathbf{N}_2$ **(a)**	$t_\varepsilon \sim tw_\varepsilon, t \in (0,\infty)$	$\dot{\pi}^{(0)}_{0,ij}(t,A)$
19	$\mathbf{F}-\mathbf{J}$, $\mathbf{N}_2$ **(b)**	$t_\varepsilon \sim tw_\varepsilon, t \in (0,\infty)$	$\dot{\pi}^{(\infty)}_{0,ij}(t,A)$
20	$\mathbf{F}-\mathbf{J}$, $\mathbf{N}_2$	$t_\varepsilon \prec w_\varepsilon, t_\varepsilon \to \infty$	$\pi_{0,ij}(A)$
21	$\mathbf{F}-\mathbf{J}$, $\mathbf{N}_3$	$t_\varepsilon \to \infty$	$\pi_{0,ij}(A)$

Table 1: Classification of ergodic theorems

It should be noted that the limiting probabilities appearing in Theorems 4 – 21 have the forms $\pi^{(\beta)}_{0,j}(A) = \rho_j(\beta)\pi_{0,j}(A)$, $\pi^{(\beta)}_{0,ij}(t,A) = p^{(\beta)}_{ij}(t)\pi_{0,j}(A)$ and $\dot{\pi}^{(0)}_{0,ij}(t,A) = \dot{p}^{(0)}_{ij}(t)\pi_{0,j}(A)$, $\dot{\pi}^{(\infty)}_{0,ij}(t,A) = \dot{p}^{(\infty)}_{ij}(t)\pi_{0,j}(A)$. Coefficients $\rho_j(\beta)$

and $p_{ij}^{(\beta)}(t)$, $\dot{p}_{ij}^{(0)}(t)$, $\dot{p}_{ij}^{(\infty)}(t)$ can be interpreted as, respectively, stationary probabilities and transition probabilities for some semi-Markov processes or Markov chains controlling switching of regimes for the limiting alternating regenerative processes, while $\pi_{0,j}(A)$ are the stationary probabilities for these limiting regenerative processes corresponding to different regimes.

Also, it is worth noting that limiting probabilities $\pi_{0,j}^{(\beta)}(A)$ and $\pi_{0,ij}^{(\beta)}(t, A)$, $\dot{\pi}_{0,ij}^{(0)}(t, A), \dot{\pi}_{0,ij}^{(\infty)}(t, A)$ possess some natural continuity properties as functions of parameters $\beta \in [0, \infty]$ and $t \in [0, \infty]$.

In particular, the limiting probabilities $\pi_{0,j}^{(\beta)}(A)$, which appear, for regularly perturbed alternating regenerative processes, in Theorems 4 – 7, and, for singularly and super-singularly perturbed alternating regenerative processes, in Theorems 8, 9, 11, 12, 16, and 17, are continuous functions of parameter $\beta \in [0, \infty]$.

Analogously, the limiting probabilities $\pi_{0,ij}^{(\beta)}(t, A)$, which appear, for singularly perturbed alternating regenerative processes, in Theorem 9, are continuous functions of parameter $(\beta, t) \in [0, \infty] \times [0, \infty]$, except points $(0, 0)$ and $(\infty, 0)$. Also, the limiting probabilities $\dot{\pi}_{0,ij}^{(0)}(t, A)$ and $\dot{\pi}_{0,ij}^{(\infty)}(t, A)$, which appear, for singularly and super-singularly perturbed alternating regenerative processes, in Theorems 13, 14, 18, and 19, are continuous functions of parameter $t \in [0, \infty]$.

Moreover, the corresponding limits, $\pi_{0,ij}^{(\beta)}(0, A) = \lim_{t\to 0} \pi_{0,ij}^{(\beta)}(t, A) = \pi_{0,ij}(A)$, for $\beta \in (0, \infty)$, while $\pi_{0,ij}^{(0)}(0, A) = \lim_{t\to 0} \pi_{0,ij}^{(0)}(t, A) = \pi_{0,j}^{(0)}(A)$ and $\pi_{0,ij}^{(\infty)}(0, A) = \lim_{t\to 0} \pi_{0,ij}^{(\infty)}(t, A) = \pi_{0,j}^{(\infty)}(A)$. Also, the limit, $\pi_{0,ij}^{(\beta)}(\infty, A) = \lim_{t\to\infty} \pi_{0,ij}^{(\beta)}(t, A) = \pi_{0,j}^{(\beta)}(A)$, for $\beta \in [0, \infty]$. Here, $\pi_{0,ij}(A)$ are the limiting probabilities appearing in Theorems 10, 15, 20 and 21.

The latter asymptotic relations have a natural explanation. As a matter of fact, there exists some kind of "competition" between the velocities with which the switching probabilities $p_{\varepsilon,12}, p_{\varepsilon,21}$ tends to zero and time t_ε tends to infinity, for singularly and super-singularly perturbed alternating regenerative processes. Probabilities $p_{\varepsilon,12}, p_{\varepsilon,21}$ determine the "grade of singularity" for perturbed alternating regenerative processes. These processes become more singular if parameter $\beta_\varepsilon = p_{\varepsilon,12}/p_{\varepsilon,21}$ takes values close to 0 or ∞. The time parameter t controls the "grade of ergodicity" for perturbed alternating regenerative processes. Values of β_ε closer to 0 or ∞ and smaller values of parameter t promote convergence of probabilities $P_{\varepsilon,ij}(t_\varepsilon, A)$ to limiting probabilities $\pi_{0,ij}(A) = \mathrm{I}(j = i)\pi_{0,i}(A)$, characteristic for absolutely singular alternating regenerative processes (for which switching of regimes is impossible). Larger values of switching probabilities and parameter t promote manifestation of ergodic phenomena and convergence of probabilities $P_{\varepsilon,ij}(t_\varepsilon, A)$

to limiting probabilities $\pi_{0,j}^{(\beta)}(A) = \rho_j(\beta)\pi_{0,j}(A)$, which are characteristic for regular alternating regenerative processes.

6.2. Directions for future research. Let us list some directions for further continuation of research studies, which results are presented in the paper.

It is clear that analogous individual ergodic theorems can be obtained for perturbed alternating regenerative processes with discrete time.

Individual ergodic theorems presented in this paper relate to one-dimensional distributions of alternating regenerative processes. It would be useful to get also analogous ergodic theorems for multi-dimensional distributions.

A very interesting and prospective direction for future studies is individual ergodic theorems for singularly and super-singularly perturbed multi-alternating regenerative processes. These are models analogous to those studied in the present paper, but with alternative regenerative processes choosing from some parametric finite or more general sets, which serve as the phase space for the corresponding switching (modulating) semi-Markov processes.

An important is model of alternating regenerative processes with terminating regeneration times, where the regenerative processes $\xi_{\varepsilon,i,n}(t), t \geq 0$ and random vectors $(\kappa_{\varepsilon,i,n}, \eta_{\varepsilon,i,n})$ are independent.

Another important model is where the processes $\xi_{\varepsilon,i,n}(t), t \geq 0$ are of Markov processes, random variables $\kappa_{\varepsilon,i,n}$ are some Markov moments for these processes, and the switching random variables $\eta_{\varepsilon,i,n}$ are determined by some events for random trajectories $\xi_{\varepsilon,i,n}(t), t \in [0, \kappa_{\varepsilon,i,n})$.

An unbounded area of applications constitute queuing, reliability, control and other types of stochastic systems with alternating regimes of function.

Results in the listed above directions shall be presented in future publications.

References

[1] Alimov, D., Shurenkov, V.M. (1990). Markov renewal theorems in a scheme of series. Ukr. Mat. Zh., **21**, no. 11, 1443–1448 (English translation in Ukr. Math. J., **42**, no. 11, 1283–1288).

[2] Anisimov, V.V. (1988). Random Processes with Discrete Components. Vysshaya Shkola and Izdatel'stvo Kievskogo Universiteta, Kiev, 183 pp.

[3] Anisimov, V.V. (2008). Switching Processes in Queueing Models. Applied Stochastic Methods Series. ISTE, London and Wiley, Hoboken, NJ, 345 pp.

[4] Asmussen, S. (2003). Applied Probability and Queues. Second edition. Applications of Mathematics, Stochastic Modelling and Applied Probability, **51**, Springer, New York, xii+438 pp.

[5] Avrachenkov, K.E., Filar, J.A., Howlett, P.G. (2013). Analytic Perturbation Theory and Its Applications. SIAM, Philadelphia, PA, xii+372 pp.

[6] Bini, D.A., Latouche, G., Meini, B. (2005). Numerical Methods for Structured Markov Chains. Numerical Mathematics and Scientific Computation, Oxford Science Publications, Oxford University Press, New York, xii+327 pp.

[7] Cohen, J.W. (1976). On Regenerative Processes in Queueing Theory. Lecture Notes in Economics and Mathematical Systems, **121**, Springer, Berlin, ix+93 pp.

[8] Courtois, P.J. (1977). Decomposability. Queueing and Computer System Applications. ACM Monograph Series, Academic Press, New York, xiii+201 pp.

[9] Cox, D.R. (1962). Renewal Theory. Methuen, London and Wiley, New York, ix+142 pp.

[10] Englund, E. (2001). Nonlinearly Perturbed Renewal Equations with Applications. Doctoral dissertation, Umeå University.

[11] Englund, E., Silvestrov, D.S. (1997). Mixed large deviation and ergodic theorems for regenerative processes with discrete time. In: Jagers, P., Kulldorff, G., Portenko, N., Silvestrov, D. (Eds.) Proceedings of the Second Scandinavian–Ukrainian Conference in Mathematical Statistics, Vol. I, Umeå, 1997. Theory Stoch. Process., **3(19)**, no. 1-2, 164–176.

[12] Feller, W. (1971). An introduction to probability theory and its applications. Vol. II. Second edition, Wiley, New York, xxiv+669 pp.

[13] Gyllenberg, M., Silvestrov, D.S. (2000). Nonlinearly perturbed regenerative processes and pseudo-stationary phenomena for stochastic systems. Stoch. Process. Appl., **86**, 1–27.

[14] Gyllenberg, M., Silvestrov, D.S. (2008). Quasi-Stationary Phenomena in Nonlinearly Perturbed Stochastic Systems. De Gruyter Expositions in Mathematics, **44**, Walter de Gruyter, Berlin, ix+579 pp.

[15] Iglehart, D.L., Shedler, G.S. (1980). Regenerative Simulation of Response Times in Networks of Queues. Lecture Notes in Control and Information Sciences, **26**, Springer, Berlin, xii+204 pp.

[16] Kartashov, N.V. (1996). Strong Stable Markov Chains. VSP, Utrecht and TBiMC, Kiev, 138 pp.

[17] Kingman, J.F.C. (1972). Regenerative Phenomena. Wiley Series in Probability and Mathematical Statistics. Wiley, London, xii+190 pp.

[18] Koroliuk, V.S., Limnios, N. (2005). Stochastic Systems in Merging Phase Space. World Scientific, Singapore, xv+331 pp.

[19] Korolyuk, V.S., Korolyuk, V.V. (1999). Stochastic Models of Systems. Mathematics and its Applications, **469**, Kluwer, Dordrecht, xii+185 pp.

[20] Korolyuk, V., Swishchuk, A. (1995). Semi-Markov Random Evolutions. Mathematics and its Applications, **308**, Kluwer, Dordrecht, x+310 pp. (English revised edition of: Semi-Markov Random Evolutions. Naukova Dumka, Kiev, 1992, 254 pp.).

[21] Korolyuk, V. S., Turbin, A.F. (1976). Semi-Markov Processes and its Applications. Naukova Dumka, Kiev, 184 pp.

[22] Korolyuk, V.S., Turbin, A.F. (1993). Mathematical Foundations of the State Lumping of Large Systems. Mathematics and its Applications, **264**, Kluwer, Dordrecht, x+278 pp. (English edition of: Mathematical Foundations of the State Lumping of Large Systems. Naukova Dumka, Kiev, 1978, 218 pp.).

[23] Kovalenko, I.N., Kuznetsov, N.Yu., Shurenkov, V.M. (1996). Models of Random Processes. A Handbook for Mathematicians and Engineers. CRC Press, Boca Raton, FL, 446 pp. (A revised edition of: Stochastic Processes. A Handbook. Naukova Dumka, Kiev, 1983, 366 pp.).

[24] Lindvall, T. (2002). Lectures on the Coupling Method. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics, Wiley, New York, xiv+257 pp. (A revised reprint of the 1992 original).

[25] Ni, Y. (2011). Nonlinearly Perturbed Renewal Equations: Asymptotic Results and Applications. Doctoral dissertation **106**, Mälardalen University, Västerås.

[26] Ni, Y., Silvestrov D., Malyarenko, A. (2008). Exponential asymptotics for nonlinearly perturbed renewal equation with non-polynomial perturbations. J. Numer. Appl. Math., **1**(96), 173–197.

[27] Petersson, M. (2016). Perturbed Discrete Time Stochastic Models. Doctoral dissertation, Stockholm University.

[28] Rolski, T., Schmidli, H., Schmidt, V., Teugels, J. (1999). Stochastic Processes for Insurance and Finance. Wiley Series in Probability and Statistics, Wiley, New York, xviii+654 pp.

[29] Ross, S.M. (2014). Introduction to Probability Models. Eleventh edition. Elsevier/Academic Press, Amsterdam, xvi+767 pp.

[30] Serfozo, R. (2009). Basics of Applied Stochastic Processes. Probability and Its Applications. Springer, Berlin, xiv+443 pp.

[31] Shedler, G.S. (1987). Regeneration and Networks of Queues. Applied Probability. A Series of the Applied Probability Trust. Springer, New York, viii+224 pp.

[32] Shedler, G.S. (1993). Regenerative Stochastic Simulation. Statistical Modeling and Decision Science. Academic Press, Boston, MA, x+400 pp.

[33] Shurenkov, V.M. (1980). Transition phenomena of the renewal theory in asymptotical problems of theory of random processes 1. Mat. Sbornik, **112**, 115–132 (English translation in Math. USSR: Sbornik, **40**, no. 1, 107–123).

[34] Shurenkov, V.M. (1980). Transition phenomena of the renewal theory in asymptotical problems of theory of random processes 2. Mat. Sbornik, **112**, 226–241 (English translation in Math. USSR: Sbornik, **40**, no. 2, 211–225).

[35] Shurenkov, V.M. (1998). Ergodic Theorems and Related Problems. VSP, Utrecht, viii+96 pp. (English edition of: Ergodic Theorems and Related Problems of Theory of Random Processes. Naukova Dumka, Kiev, 1981, 119 pp.).

[36] Shurenkov, V.M. (1989). Ergodic Markov Processes. Probability Theory and Mathematical Statistics, Nauka, Moscow, 332 pp.

[37] Silvestrov, D.S. (1976). A generalization of the renewal theorem. Dokl. Akad. Nauk Ukr. SSR, Ser. A, no.11, 978–982.

[38] Silvestrov, D.S. (1978). The renewal theorem in a series scheme. 1. Teor. Veroyatn. Mat. Stat. **18**, 144–161 (English translation in Theory Probab. Math. Statist., **18**, 155–172).

[39] Silvestrov, D.S. (1979). The renewal theorem in a series scheme 2. Teor. Veroyatn. Mat. Stat., **20**, 97–116 (English translation in Theory Probab. Math. Statist., **20**, 113–130).

[40] Silvestrov, D.S. (1980). Semi-Markov Processes with a Discrete State Space. Library for an Engineer in Reliability, Sovetskoe Radio, Moscow, 272 pp.

[41] Silvestrov, D.S. (1980). Synchronized regenerative processes and explicit estimates for the rate of convergence in ergodic theorems, Dokl. Acad. Nauk Ukr. SSR, Ser. A, no. 11, 22–25.

[42] Silvestrov, D.S. (1983). Method of a single probability space in ergodic theorems for regenerative processes 1. Math. Operat. Statist., Ser. Optim., **14**, 285–299.

[43] Silvestrov, D.S. (1984). Method of a single probability space in ergodic theorems for regenerative processes 2. Math. Operat. Statist., Ser. Optim., **15**, 601–612.

[44] Silvestrov, D.S. (1984). Method of a single probability space in ergodic theorems for regenerative processes 3. Math. Operat. Statist., Ser. Optim., **15**, 613–622.

[45] Silvestrov, D. (1994). Coupling for Markov renewal processes and the rate of convergence in ergodic theorems for processes with semi-Markov switchings. Acta Applic. Math., **34**, 109–124.

[46] Silvestrov, D.S. (1995). Exponential asymptotic for perturbed renewal equations. Teor. Ĭmovirn. Mat. Stat., **52**, 143–153 (English translation in Theory Probab. Math. Statist., **52**, 153–162).

[47] Silvestrov D.S. (2004). Limit Theorems for Randomly Stopped Stochastic Processes. Probability and Its Applications, Springer, London, xiv+398 pp.

[48] Silvestrov D.S. (2010). Nonlinearly Perturbed Stochastic Processes and Systems, In: Rykov, V., Balakrishnan, N., Nikulin, M. (Eds.) Mathematical and Statistical Models and Methods in Reliability, Birkhäuser, Chapter 2, 19–38.

[49] Silvestrov D. (2014). Improved asymptotics for ruin probabilities. In: Silvestrov, D., Martin-Löf, A. (Eds). Modern Problems in Insurance Mathematics, EAA series, Springer, Cham, Chapter 5, 37–68.

[50] Silvestrov, D.S., Petersson, M. (2014). Exponential expansions for perturbed discrete time renewal equations. In: Karagrigoriou, A., Lisnianski, A., Kleyner, A., Frenkel, I. (Eds.) Applied Reliability Engineering and Risk Analysis. Probabilistic Models and Statistical Inference, Wiley, New York, Chapter 23, 349–362.

[51] Silvestrov, D., Silvestrov, S. (2016). Asymptotic expansions for stationary distributions of perturbed semi-Markov processes. In: Silvestrov, S., Rančić, M. (Eds). Engineering Mathematics II. Algebraic, Stochastic and Analysis Structures for Networks, Data Classification and Optimization, Springer Proceedings in Mathematics & Statistics **179**, Springer, Cham, Chapter 10, 151–222.

[52] Silvestrov, D., Silvestrov, S. (2017). Nonlinearly Perturbed Semi-Markov Processes. Springer Briefs in Probability and Mathematical Statistics, Springer, Cham, xiv+143 pp.

[53] Smith, W.L. (1955). Regenerative Stochastic Processes. Proceedings of the Royal Society, Ser. A: Mathematical, Physical and Engineering Sciences. **232**, 6–31.

[54] Stewart, G.W. (1998). Matrix Algorithms. Vol. I. Basic Decompositions. SIAM, Philadelphia, PA, xx+458 pp.

[55] Stewart, G.W. (2001). Matrix Algorithms. Vol. II. Eigensystems. SIAM, Philadelphia, PA, xx+469 pp.

[56] Stewart, G.W., Sun, J.G. (1990). Matrix Perturbation Theory. Computer Science and Scientific Computing. Academic Press, Boston, xvi+365 pp.

[57] Thorisson, H. (2000). Coupling, Stationarity and Regeneration. Probability and its Applications, Springer, New York, xiv+517 pp.

[58] Yeleiko, Ya.I., Shurenkov, V.M. (1995).Transient phenomena in a class of matrix-valued stochastic evolutions. Theor. Ĭmovirn. Mat. Stat., **52**, 72–76 (English translation in Theory Probab. Math. Statist., **52**, 75–79).

[59] Yin, G.G., Zhang, Q. (2005). Discrete-Time Markov Chains. Two-Time-Scale Methods and Applications. Stochastic Modelling and Applied Probability, **55**, Springer, New York, xix+348 pp.

[60] Yin, G.G., Zhang, Q. (2013). Continuous-Time Markov Chains and Applications. A Two-Time-Scale Approach. Second edition, Stochastic Modelling and Applied Probability, **37**, Springer, New York, xxii+427 pp. (An extended variant of the first 1998 edition).