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Abstract

In this paper, using the shrinkage-based approach for portfolio weights and modern results from random matrix theory we construct an effective procedure for testing the efficiency of the expected utility (EU) portfolio and discuss the asymptotic behavior of the proposed test statistic under the high-dimensional asymptotic regime, namely when the number of assets p increases at the same rate as the sample size n such that their ratio p/n approaches a positive constant $c \in (0,1)$ as $n \to \infty$. We provide an extensive simulation study where the power function and receiver operating characteristic curves of the test are analyzed. In the empirical study, the methodology is applied to the returns of S&P 500 constituents.

Keywords: Finance; Portfolio analysis; Mean-variance optimal portfolio; Statistical test; Shrinkage estimator; Random matrix theory.

1   Introduction

Following the mean-variance approach of Markowitz (1952), which is considered to be one of the most popular portfolio choice strategies, the weights of an optimal portfolio are obtained by minimizing the portfolio variance for a predefined level of the portfolio expected return. This set of optimal portfolios determines the efficient frontier in the mean-variance space. The Markowitz approach formalizes the advantages of portfolio diversification and has become a benchmark for both researchers and practitioners in portfolio management.

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Markowitz optimal portfolios, also known as mean-variance optimal portfolios, can also be obtained as solutions of other optimization problems (e.g., Bodnar et al. (2013)), like by maximizing the expected quadratic utility (EU) function (see, Ingersoll (1987)) expressed as

$$\mathbf{w}'\boldsymbol{\mu} - \frac{\gamma}{2}\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} \to max \quad \text{subject to} \quad \mathbf{w}'\mathbf{1}_p = 1, \tag{1}$$

where $\mathbf{w} = (w_1, \ldots, w_p)'$ is the vector of portfolio weights, $\mathbf{1}_p$ is the p-dimensional vectors of ones, $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are the mean vector and the covariance matrix of the random vector of asset returns $\mathbf{x} = (x_1, \ldots, x_p)'$. The quantity $\gamma > 0$ measures the investors attitude towards risk. If $\gamma = \infty$, then the investor is fully risk averse and determines the investment strategy by minimizing the portfolio variance without paying attention to the expected portfolio return, i.e., constructs the so-called global minimum variance (GMV) portfolio. Under the assumption that the asset returns are normally distributed, the problem of maximization the mean-variance objective function (1) is equivalent to the maximization of the expected exponential utility, which implies constant absolute risk aversion (CARA). In this case, γ is equal to the investors absolute risk aversion coefficient (see, e.g., Ingersoll (1987)).

We denote the solution of (1) by $\mathbf{w}_{EU}$ and it is given by

$$\mathbf{w}_{EU} = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}_p}{\mathbf{1}_p'\boldsymbol{\Sigma}^{-1}\mathbf{1}_p} + \gamma^{-1}\mathbf{Q}\boldsymbol{\mu}, \tag{2}$$

where

$$\mathbf{Q} = \boldsymbol{\Sigma}^{-1} - \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}_p\mathbf{1}_p'\boldsymbol{\Sigma}^{-1}}{\mathbf{1}_p'\boldsymbol{\Sigma}^{-1}\mathbf{1}_p}. \tag{3}$$

The case of fully risk averse investor, i.e., $\gamma = \infty$, leads to the weights of the GMV portfolio expressed as

$$\mathbf{w}_{GMV} = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}_p}{\mathbf{1}_p'\boldsymbol{\Sigma}^{-1}\mathbf{1}_p}. \tag{4}$$

The derived formulas of optimal portfolio weights (2) and (4) cannot directly be used in practice, since they both depend on unknown parameters of the data generating process. The mean vector $\boldsymbol{\mu}$ and the covariance matrix $\boldsymbol{\Sigma}$ are not observable in practice and have to be estimated by using historical data for asset returns. This, however, introduces further sources of risk into the investment process, namely the estimation risk which has been ignored for a long time in finance.

The most commonly used approach to estimate the weights of optimal portfolios is based on simple replacing the unknown first two moments of the asset returns by their sample counterparts. As a result, we obtain a "plug-in" estimator for the optimal portfolio weights also known as its sample estimator, which is a traditional way to construct a portfolio in practice. Assuming that the asset returns are independent and normally distributed Okhrin and Schmid (2006) obtain the asymptotic distribution of the sample estimator of the EU portfolio weights, while the corresponding exact distributional results can be found in Bodnar and Schmid (2011). Further theoretical and practically relevant findings related to the characterization of the distribution of the sample estimator of the optimal portfolio weights and their characteristics can be found in Yang et al. (2015), Woodgate and Siegel (2015), Simaan et al. (2018), Zhao et al. (2019), among

others.

The use of the "plug-in" estimators in practice has been widely criticized in statistical and financial literature. One of the main drawbacks of the sample estimators is the investors overoptimism about the optimality of the constructed portfolio. Several studies (see, e.g., Siegel and Woodgate (2007), Kan and Smith (2008), Bodnar and Bodnar (2010)) show with theoretical and empirical arguments that the plug-in estimator of the efficient frontier overestimates the location of the true efficient frontier in the mean-variance space. This leads to too optimistic trading strategies which perform in practice typically much worse than expected.

In recent years, other types of estimators for the optimal portfolio weights have been introduced in the literature. Some estimators attempt to improve the estimators for the parameters of the asset returns. Relying on the idea of Stein (1956) we can use a shrinkage estimator for the mean vector and for the covariance matrix or its inverse (see, e.g., Bodnar et al. (2014) and Bodnar et al. (2016)). Alternatively, one can apply the shrinkage method directly to portfolio weights as suggested by Golosnoy and Okhrin (2007), Okhrin and Schmid (2008), Frahm and Memmel (2010), etc. The goal of the approach is to reduce the estimation uncertainty and to decrease the variance in the estimated portfolio weights.

The problem of assessing the estimation risk, when an optimal portfolio is constructed, becomes very challenging from the high-dimensional perspectives, i.e., when both the number of included assets p and the sample size n tend to infinity simultaneously such that p/n tends to the concentration ratio $c > 0$ as $n \to \infty$ (see, Bai, Liu and Wong (2009), Bai and Shi (2011)). Under the classical asymptotic regime, when the number of assets p is fixed and substantially smaller then the sample size n, the traditional "plug-in" estimator of optimal portfolio weights is consistent (see, Okhrin and Schmid (2006), Bodnar and Schmid (2011)). On the other hand, the sample estimators of the mean vector and of the covariance matrix are not longer feasible under the high-dimensional asymptotics (Bai and Silverstein (2010), Bai and Shi (2011), Bodnar, Okhrin and Parolya (2019)), which has a negative impact on the performance of the asset allocation strategy. Moreover, the inverse covariance matrix does not exist anymore for $c > 1$ and the optimal portfolios cannot be constructed in a traditional way.

Nowadays, the technological advances and the availability of financial information make the whole universe of assets easily accessible for private and institutional investors (see, Hautsch and Voigt (2019)). This leads to portfolios consisting of hundreds of assets and to a high demand for new results on constructing optimal portfolios in a high-dimensional setting. Similarly as in the low-dimensional case, the first line of the research deals with deriving improved estimators for the mean vector and the covariance matrix of asset returns. These are used to obtain improved plug-in estimators of the optimal portfolio weights (see, Ledoit and Wolf (2017), Holgersson et al. (2020)). The second possibility is to improve the estimators of the optimal portfolio weights directly. This can be achieved by taking their functional dependence on the mean vector and of covariance matrix. Following this approach Bodnar et al. (2018) suggest the optimal shrinkage estimator for the GMV portfolio weights, while Bodnar et al. (2020) propose the optimal shrinkage estimator for the EU portfolio weights. Both estimators are derived by using recent results in random matrix theory and appear to be feasible even in the case of $c > 1$. Other optimal portfolio choice strategies under the high-dimensional regime were established by

Rubio et al. (2012), Benidis et al. (2018), Zhao et al. (2019).

It is important to note that the statistical methods developed for estimating optimal portfolio weights can be linked to the classical methods used in statistical signal processing. For example, the Capon or minimum variance spatial filter is equivalent to the GMV portfolio in signal processing literature (see Verdú (1998) and Van Trees (2002)). The estimation risk of the high-dimensional minimum variance beamformer is studied in Rubio et al. (2012) and Yang et al. (2018), while its constrained versions are discussed in Li et al. (2004). Moreover, Mestre and Lagunas (2006) discuss the finite-sample size effect on minimum variance filter and Zhang et al. (2013) present an improved calibration of the precision matrix. Further literature on the applications of random matrix theory to signal processing and portfolio optimization can be found in Feng and Palomar (2016) and references therein.

We contribute to the recent literature in portfolio theory and signal processing theory by developing new statistical tests on the weights of the EU portfolio in a high-dimensional setting. From practical point of view an investor will have an opportunity to test if the current large portfolio coincides with a prespecified benchmark portfolio or there are significant deviations. From the theoretical perspective we contribute by derivation of confidence intervals and test theory for expressions including functions of both the mean vector and the covariance matrix. This directly extends the existent results on testing the structure of the covariance matrix in high-dimensional settings (see, e.g., Bai, Jiang, Yao and Zheng (2009), Yao et al. (2015), Bodnar, Dette and Parolya (2019)). The new approach is based on the shrinkage estimator of the EU portfolio weights and extends the one derived for the weights of the GMV portfolio in Bodnar, Dmytriv, Parolya and Schmid (2019) by taking the uncertainty about the estimated mean vector into account when the high-dimensional asymptotic distribution of the test statistic is derived. One of the main advantages of the approach is that the whole high-dimensional vector of portfolio weights can be tested in a single step. Moreover, the investor can make a decision about the efficiency of the holding portfolio based on the result of the testing procedure.

The rest of paper is organized as follows. In Section 2, we describe the existent approaches in testing the finite number of the EU portfolio weights in both low and high dimensions. New test based on the shrinkage approach is suggested in Section 3. Here, the asymptotic distribution of the test statistic is derived under both the null and the alternative hypotheses under high-dimensional settings. In Section 4.1, we compare the new test with the existent approaches in terms of size and power properties, while an empirical illustration is provided in Section 4.2. Concluding remarks are presented in Section .

2 Sample estimator of the EU portfolio and test theory

We consider a financial market consisting of p risky assets. Let $\mathbf{x}_t$ denote the p-dimensional vector of the returns on risky assets at time t. Suppose that $E(\mathbf{x}_t) = \boldsymbol{\mu}$ and $Cov(\mathbf{x}_t) = \boldsymbol{\Sigma}$ where $\boldsymbol{\Sigma}$ is assumed to be positive definite. Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be a sample of asset return vectors consisting of their n independent realizations and let $\mathbf{X}_n = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ stand for the $p \times n$ data matrix. Throughout of the paper we assume that the asset returns are independent and identically normally distributed, i.e. $\mathbf{x}_i \sim \mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, $i = 1, \ldots, n$.

The sample estimators of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are given by

$$\bar{\mathbf{x}}_n = \frac{1}{n} \sum_{j=1}^{n} \mathbf{x}_j \text{ and } \hat{\boldsymbol{\Sigma}}_n = \frac{1}{n-1} \sum_{j=1}^{n} \left(\mathbf{x}_j - \bar{\mathbf{x}}_n\right) \left(\mathbf{x}_j - \bar{\mathbf{x}}_n\right)'. \tag{5}$$

Replacing $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ in (2) by their sample estimators from (5), we obtain the sample estimator of the EU portfolio weights expressed as

$$\hat{\mathbf{w}}_{EU} = \frac{\hat{\boldsymbol{\Sigma}}_n^{-1} \mathbf{1}_p}{\mathbf{1}_p' \hat{\boldsymbol{\Sigma}}_n^{-1} \mathbf{1}_p} + \gamma^{-1} \hat{\mathbf{Q}}_n \hat{\boldsymbol{\mu}}_n,$$

where

$$\hat{\mathbf{Q}}_n = \hat{\boldsymbol{\Sigma}}_n^{-1} - \frac{\hat{\boldsymbol{\Sigma}}_n^{-1} \mathbf{1}_p \mathbf{1}_p' \hat{\boldsymbol{\Sigma}}_n^{-1}}{\mathbf{1}_p' \hat{\boldsymbol{\Sigma}}_n^{-1} \mathbf{1}_p}. \tag{6}$$

Okhrin and Schmid (2006) derive the analytical expression for the expectation and the co-variance matrix of $\hat{\mathbf{w}}_{EU}$ and obtain its asymptotic distribution assuming that the portfolio size is considerably smaller than the sample size. These results are extended in Bodnar and Schmid (2011) who derive the finite-sample distribution of the estimated EU portfolio weights and use these results in the derivation of an asymptotic tests on the weights which we present in the next subsection.

2.1 Tests based on Mahalanobis distance

At each time point an investor has to decide whether the holding portfolio is efficient or it has to be adjusted (see, Bodnar and Schmid (2008), Bodnar and Schmid (2011)). This problem can be presented as a special case of the general linear hypotheses formulated for the portfolio weights. Let $\mathbf{L}$ denote the $k \times p$ dimensional matrix of constants with $k < p - 1$ and let $\mathbf{r}$ be the k-dimensional vector of constants. Bodnar and Schmid (2011) consider the following hypotheses for linear combinations of the EU portfolio weights

$$H_0 : \mathbf{L}\mathbf{w}_{EU} = \mathbf{r} \qquad \text{against} \qquad H_1 : \mathbf{L}\mathbf{w}_{EU} \neq \mathbf{r}, \tag{7}$$

If one sets $\mathbf{L} = [\mathbf{I}_k \ \mathbf{O}_{k,p-k}]$ in (7) where $\mathbf{I}_k$ is the k-dimensional identity matrix and $\mathbf{O}_{k,p-k}$ is the $k \times (p-k)$ matrix with zeros, then the null hypothesis states that the first k weights in $\mathbf{w}_{EU}$ are equal to the corresponding components defined by $\mathbf{r}$. It also has to be noted that whole structure of the EU portfolio cannot be tested by using (7) because of the restriction imposed on the number of linear combinations which should be smaller than $p-1$. Thus, the test on the whole vector of the EU portfolio weights should be performed by testing at least two null hypotheses of the form (7) by selecting matrices $\mathbf{L}$ in each of the null hypotheses such that all elements in $\mathbf{w}_{EU}$ are tested. This leads to a multiple testing problem also discussed below.

In order to test (7) for a given matrix $\mathbf{L}$ and a vector $\mathbf{r}$, Bodnar and Schmid (2011) suggest

the following test statistic:

$$
\begin{aligned}
T_{\mathbf{L}} &= (n-p+1)\left(\hat{\mathbf{w}}_{\mathbf{L}} - \mathbf{r}\right)' \left(\frac{\mathbf{L}\hat{\mathbf{Q}}_n\mathbf{L}'}{\mathbf{1}_p'\hat{\boldsymbol{\Sigma}}_n^{-1}\mathbf{1}_p} + \gamma^{-1}\frac{\mathbf{L}\hat{\mathbf{Q}}_n\mathbf{L}'}{\bar{\mathbf{x}}_n'\hat{\mathbf{Q}}_n\bar{\mathbf{x}}_n} + \gamma^{-2}(\mathbf{L}\hat{\mathbf{Q}}_n\mathbf{L}'\bar{\mathbf{x}}_n'\hat{\mathbf{Q}}_n\bar{\mathbf{x}}_n - \mathbf{L}\hat{\mathbf{Q}}_n\bar{\mathbf{x}}_n\bar{\mathbf{x}}_n'\hat{\mathbf{Q}}_n\mathbf{L}') \right)^{-1} \\
&\times \left(\hat{\mathbf{w}}_{\mathbf{L}} - \mathbf{r}\right),
\end{aligned}
\tag{8}
$$

where

$$
\hat{\mathbf{w}}_{\mathbf{L}} = \mathbf{L}\hat{\mathbf{w}}_{EU} = \frac{\mathbf{L}\hat{\boldsymbol{\Sigma}}_n^{-1}\mathbf{1}_p}{\mathbf{1}_p'\hat{\boldsymbol{\Sigma}}_n^{-1}\mathbf{1}_p} + \gamma^{-1}\mathbf{L}\hat{\mathbf{Q}}_n\bar{\mathbf{x}}_n.
\tag{9}
$$

Bodnar and Schmid (2011) show that the test statistic $T_{\mathbf{L}}$ can be asymptotically well approximated by a non-central χ^2-distribution with k degrees of freedom and the non-centrality parameter

$$
\lambda = n\left(\mathbf{w}_{\mathbf{L}} - \mathbf{r}\right)' \left(\frac{\mathbf{L}\mathbf{Q}\mathbf{L}'}{\mathbf{1}_p'\boldsymbol{\Sigma}^{-1}\mathbf{1}_p} + \gamma^{-1}\frac{\mathbf{L}\mathbf{Q}_n\mathbf{L}'}{\boldsymbol{\mu}'\mathbf{Q}\boldsymbol{\mu}} + \gamma^{-2}(\mathbf{L}\mathbf{Q}\mathbf{L}'\boldsymbol{\mu}'\mathbf{Q}\boldsymbol{\mu} - \mathbf{L}\mathbf{Q}\boldsymbol{\mu}\boldsymbol{\mu}'\mathbf{Q}\mathbf{L}') \right)^{-1} \left(\mathbf{w}_{\mathbf{L}} - \mathbf{r}\right) \tag{10}
$$

with

$$
\mathbf{w}_L = \mathbf{L}\mathbf{w}_{EU} = \frac{\mathbf{L}\boldsymbol{\Sigma}^{-1}\mathbf{1}_p}{\mathbf{1}_p'\boldsymbol{\Sigma}^{-1}\mathbf{1}_p} + \gamma^{-1}\mathbf{L}\mathbf{Q}\boldsymbol{\mu},
\tag{11}
$$

when both p and k are relatively small with respect to the sample size n. As a special case, we obtain the asymptotic distribution of $T_{\mathbf{L}}$ under the null hypothesis. This appears to be a χ^2-distribution, i.e. $T_{\mathbf{L}} \sim \chi_k^2$ under the null hypothesis in (7).

Since the asymptotic distribution of the test statistic $T_{\mathbf{L}}$ is obtained under classical asymptotic regime, this test, in general, is not applicable when the portfolio size is comparable to the sample size. We illustrate this point in Figure 1. Here we plot the kernel density estimator (KDE) of the distribution of the test statistic $T_{\mathbf{L}}$ under the null hypothesis together with the asymptotic χ^2-distribution (green and red curves, respectively). For this purpose we generate samples from a multivariate normal distribution with mean vector and covariance matrix as specified in the numerical study of Section 4.1. The vector $\mathbf{r}$ consists of the first k components of the true EU portfolio weights and we set $\mathbf{L} = [\mathbf{I}_k \; \mathbf{O}_{k,p-k}]$. For each sample we compute the value of the test statistic $T_{\mathbf{L}}$ and then plot the KDE. To robustify the conclusions we set $\gamma = 5$, $p = 300$, $c_n = p/n \in \{0.3, 0.8\}$ and $k \in \{10, 30, 100\}$. We observe that already for $k = 10$ the difference between the KDE and the asymptotic distribution is very large and this evidence becomes stronger if k increases. For $k = 100$ the KDE shifts strongly to the right and is not shown to retain the same scaling on the x-axis. Table 1 gives the realized sizes (type I errors) of the considered test based on the 5000 independent replications and with the nominal level $\alpha = 0.05$. For different values of $k \in \{10, 30, 100\}$, it can be seen that $T_{\mathbf{L}}$ is highly inconsistent and has a much higher size than the nominal value α. We conclude that the test is highly unreliable if we wish to test many or all weights simultaneously.

6

2.2 Improvement of the test based on Mahalanobis distance for large-dimensional portfolios

Bodnar, Dette, Parolya and Thorsén (2019) show that the sample estimator of the EU portfolio weights is not consistent under the high-dimensional asymptotic regime, i.e., when $p/n \to c \in [0,1)$ as $n \to \infty$. Moreover, they derive a consistent estimator for the elements of $\mathbf{w}_{EU}$ and use these findings to construct a high-dimensional asymptotic test on the finite number of linear combinations of the EU portfolio weights.

Let $\mathbf{L}$ be a $k \times p$ matrix of constant as defined in Section 2.1 and let

$$\hat{\mathbf{w}}_{GMV;\mathbf{L}} = \mathbf{L}\hat{\mathbf{w}}_{GMV} = \frac{\mathbf{L}\hat{\boldsymbol{\Sigma}}_n^{-1}\mathbf{1}_p}{\mathbf{1}_p'\hat{\boldsymbol{\Sigma}}_n^{-1}\mathbf{1}_p}, \quad \hat{s} = \bar{\mathbf{x}}_n'\hat{\mathbf{Q}}_n\bar{\mathbf{x}}_n$$

$$\text{and} \quad \hat{\boldsymbol{\eta}}_{\mathbf{L}} = \frac{\mathbf{L}\hat{\mathbf{Q}}_n\bar{\mathbf{x}}_n}{\bar{\mathbf{x}}_n'\hat{\mathbf{Q}}_n\bar{\mathbf{x}}_n}. \tag{12}$$

Assuming that k is finite, i.e. considerably smaller than both p and n, Bodnar, Dette, Parolya and Thorsén (2019) prove that

$$\hat{\mathbf{w}}_{GMV;\mathbf{L}} \overset{a.s.}{\to} \mathbf{L}\mathbf{w}_{GMV}, \quad \hat{s}_c = (1 - c_n)\hat{s} - c_n \overset{a.s.}{\to} s \tag{13}$$

$$\text{and} \quad \hat{\boldsymbol{\eta}}_{\mathbf{L};c} = \frac{\hat{s}_c + c_n}{\hat{s}_c}\hat{\boldsymbol{\eta}}_{\mathbf{L}} \overset{a.s.}{\to} \boldsymbol{\eta}_{\mathbf{L}} \tag{14}$$

for $c_n = p/n \to c \in [0,1)$ as $n \to \infty$ with

$$s = \boldsymbol{\mu}'\mathbf{Q}\boldsymbol{\mu} \quad \text{and} \quad \boldsymbol{\eta}_{\mathbf{L}} = \frac{\mathbf{L}\mathbf{Q}\boldsymbol{\mu}}{\boldsymbol{\mu}'\mathbf{Q}\boldsymbol{\mu}}. \tag{15}$$

The symbol $\overset{a.s.}{\to}$ denotes the almost surely convergence.

Using (13), Bodnar, Dette, Parolya and Thorsén (2019) propose a high-dimensional asymptotic test on the hypotheses (7) with the test statistic given by

$$T_{\mathbf{L};c} = (n - p)\left(\hat{\mathbf{w}}_{\mathbf{L};c} - \mathbf{r}\right)'\hat{\boldsymbol{\Omega}}_{\mathbf{L};c}^{-1}\left(\hat{\mathbf{w}}_{\mathbf{L};c} - \mathbf{r}\right), \tag{16}$$

where

$$\hat{\mathbf{w}}_{\mathbf{L};c} = \hat{\mathbf{w}}_{GMV;\mathbf{L}} + \gamma^{-1}\hat{s}_c\hat{\boldsymbol{\eta}}_{\mathbf{L};c} \tag{17}$$

and

$$\hat{\boldsymbol{\Omega}}_{\mathbf{L};c} = \left(\left(\left(\frac{1 - c_n}{\hat{s}_c + c_n} + (\hat{s}_c + c_n)\gamma^{-1}\right)\gamma^{-1} + \hat{V}_c\right)(1 - c_n)\mathbf{L}\hat{\mathbf{Q}}_n\mathbf{L}^\top \right.$$
$$\left. + \gamma^{-2}\left\{\frac{2(1 - c_n)c_n^3}{(\hat{s}_c + c_n)^2} + 4(1 - c_n)c_n\frac{\hat{s}_c(\hat{s}_c + 2c_n)}{(\hat{s}_c + c_n)^2} + \frac{2(1 - c_n)c_n^2(\hat{s}_c + c_n)^2}{\hat{s}_c^2} - \hat{s}_c^2\right\}\hat{\boldsymbol{\eta}}_{\mathbf{L};c}\hat{\boldsymbol{\eta}}_{\mathbf{L};c}', \tag{18}$$

where

$$\hat{V}_c = \frac{\hat{V}_{GMV}}{1 - c_n} \quad \text{with} \quad \hat{V}_{GMV} = \frac{1}{\mathbf{1}_p'\hat{\boldsymbol{\Sigma}}_n^{-1}\mathbf{1}_p} \tag{19}$$

are the consistent and the sample estimators of the variance of the GMV portfolio (4), that is (see, e.g., Bodnar et al. (2018, p.387))

$$\hat{V}_c \overset{a.s.}{\to} V_{GMV} = \frac{1}{\mathbf{1}_p' \mathbf{\Sigma}^{-1} \mathbf{1}_p}$$

for $c_n = p/n \to c \in [0, 1)$ as $n \to \infty$.

The application of the results of Theorem 4.4 in Bodnar, Dette, Parolya and Thorsén (2019) leads to the high-dimensional asymptotic distribution of $T_{\mathbf{L};c}$ under both hypotheses in (7). Namely, it holds that the asymptotic distribution of $T_{\mathbf{L};c}$ under H_1 is well approximated by a non-central χ^2-distribution with k degrees of freedom and non-centrality parameter given by

$$\lambda_c = (n - p)(\mathbf{w_L} - \mathbf{r})' \mathbf{\Omega}_{\mathbf{L};c}^{-1} (\mathbf{w_L} - \mathbf{r}), \tag{20}$$

where

$$
\begin{aligned}
\mathbf{\Omega}_{\mathbf{L};c} &= \left(\left(\frac{1 - c}{s + c} + (s + c)\gamma^{-1} \right) \gamma^{-1} + V_{GMV} \right) (1 - c) \mathbf{L} \mathbf{Q} \mathbf{L}^\top \\
&+ \gamma^{-2} \left\{ \frac{2(1 - c)c^3}{(s + c)^2} + 4(1 - c)c\frac{s(s + 2c)}{(s + c)^2} + \frac{2(1 - c)c^2(s + c)^2}{s^2} - s^2 \right\} \boldsymbol{\eta_L} \boldsymbol{\eta_L'}.
\end{aligned}
$$

Moreover, $T_{\mathbf{L};c} \overset{d}{\to} \chi_k^2$ under H_0, where the symbol $\overset{d}{\to}$ denotes the convergence in distribution.

In Figure 1 we present the KDE of the distribution of $T_{\mathbf{L};c}$ (blue curve) and compare it to its high-dimensional asymptotic distribution (red curve). The kernel density estimator as well as the sizes of the test are obtained under the same simulation setup as one used at the end of Section 2.1. The approximation works well and much better than in the case of $T_{\mathbf{L}}$ for smaller values of k, but discrepancy becomes large if k increases. The same conclusion can be drawn from Table 1. Here the method proposed by Bodnar, Dette, Parolya and Thorsén (2019) has a much better realized size which still increases dramatically with growing k.

c = 0.3			
	$k = 10$	$k = 30$	$k = 100$
$T_{\mathbf{L}}$	0.528	0.891	1
$T_{\mathbf{L};c}$	0.061	0.071	0.181
c = 0.8			
	$k = 10$	$k = 30$	$k = 100$
$T_{\mathbf{L}}$	0.226	0.765	1
$T_{\mathbf{L};c}$	0.069	0.105	0.221

Table 1: Empirical sizes of the tests based on $T_{\mathbf{L}}$ and $T_{\mathbf{L};c}$ using $5 \cdot 10^3$ independent replications.
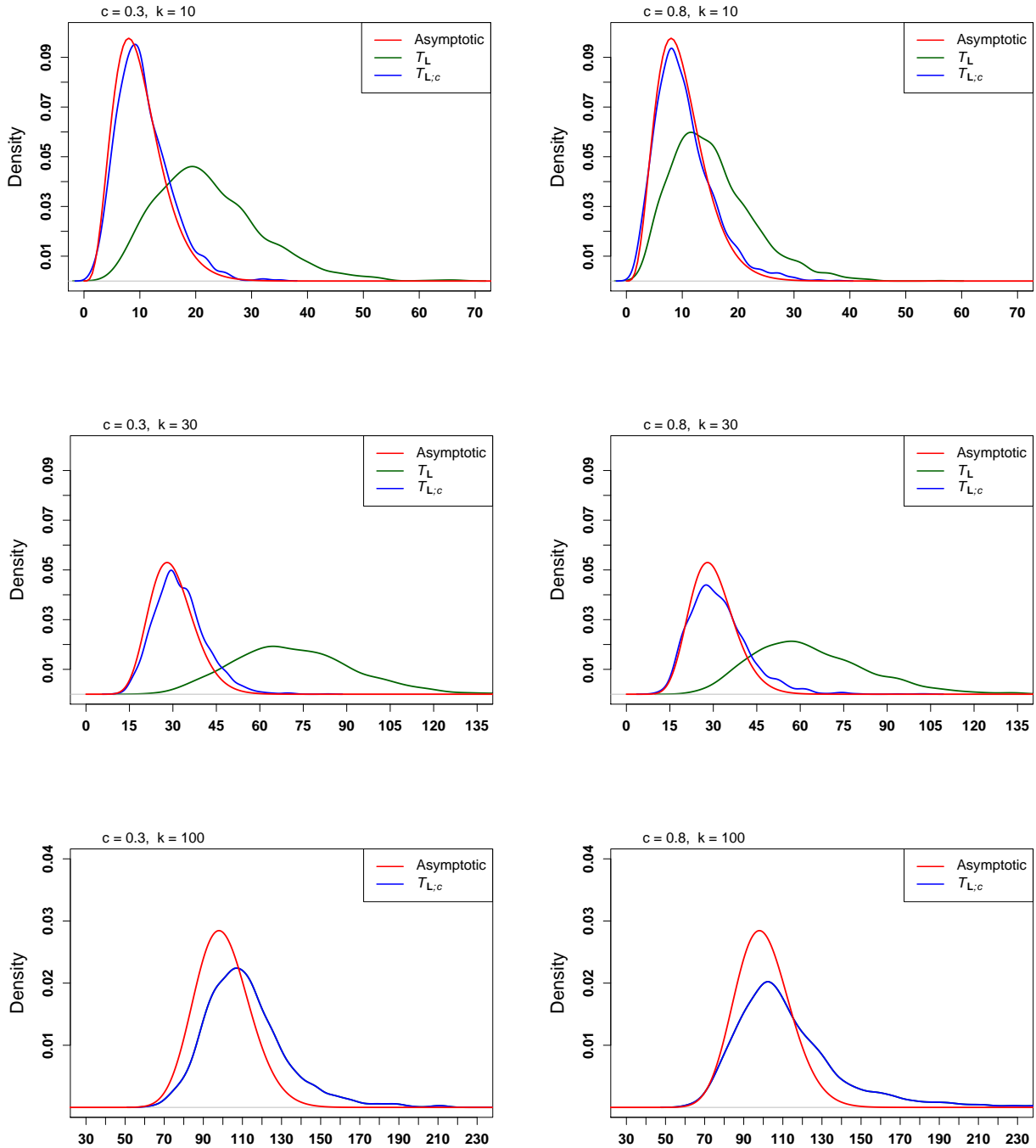
Figure 1: The high-dimensional asymptotic χ^2 approximation of the densities of $T_{\mathbf{L}}$ and $T_{\mathbf{L};c}$ together with their kernel density estimators for $\gamma = 5$, $p = 300$, $c_n = p/n \in \{0.3, 0.8\}$ and $k \in \{10, 30, 100\}$.

3 Test based on the shrinkage approach

Both tests based on the Mahalanobis distance are designed to test a finite number of linear restrictions imposed on the EU portfolio weights. Although the high-dimensional test shows a considerable improvement in terms of the size (see, Figures 1 and Table 1), this test, similarly to the test based on the statistic $T_{\mathbf{L}}$, cannot be applied to test the structure of the whole EU portfolio. In practice, one has to fix the number k of the EU portfolio weights (or their linear

restrictions) and apply the test $T_{\mathbf{L};c}$ several times in order to cover the whole vector $\mathbf{w}_{EU}$. This approach is a single-step multiple test (see, Dickhaus (2014)) with the number of marginal hypotheses to be tested equal to $[p/k]+1$. Since the dependence structure between the marginal tests is very complicated, one has to monitor the overall type I error rate by using the so-called Bonferroni correction (see, Dickhaus (2014)). This would worse the power properties of each individual test, especially when the number of tests is relatively large.

As a solution to this challenging problem, we suggest a new approach for testing the structure of the EU portfolio by a single test. The new procedure is based on the shrinkage estimator of the EU portfolio weights as suggested by Bodnar et al. (2020) and extend our previous results obtained for the GMV portfolio in Bodnar, Dmytriv, Parolya and Schmid (2019), which is a very special case of the EU portfolio. In contrast to the EU portfolio, the weights of the GMV portfolio do not depend on the mean vector. As a result, the derivation of the test for the EU portfolio becomes a very challenging task and completely new results in random matrix theory have to be derived to handle it.

3.1   Optimal shrinkage estimator of the EU portfolio weights

The shrinkage estimator for the EU portfolio weights is a convex combination of the sample estimator and a fixed well behaved target portfolio $\mathbf{b} \in \mathbb{R}^p$ with bounded expected return and variance, i.e., $R_b = \mathbf{b}'\boldsymbol{\mu} < \infty$ and $V_b = \mathbf{b}'\boldsymbol{\Sigma}^{-1}\mathbf{b} < \infty$ uniformly in p. Thus, the shrinkage estimator is expressed as

$$\hat{\mathbf{w}}_{GSE} = \alpha_n \hat{\mathbf{w}}_{EU} + (1 - \alpha_n)\mathbf{b} \quad \text{with} \quad \mathbf{b}'\mathbf{1}_p = 1, \tag{21}$$

where α_n is the shrinkage intensity. One of the main ideas behind the shrinkage estimator (21) is to reduce the large variability present in the sample estimator $\hat{\mathbf{w}}_{EU}$ by shrinking it to a vector of constants. This approach might introduce a bias in the estimator, but on the other side it reduces the variability of the sample estimator considerably.

Bodnar et al. (2020) determine the optimal shrinkage intensity α_n^* as the solution of the maximization problem based on the mean-variance objective function. It is given by

$$\alpha_n^* = \frac{(\hat{\mathbf{w}}_{EU} - \mathbf{b})'(\boldsymbol{\mu} - \gamma\boldsymbol{\Sigma}\mathbf{b})}{(\hat{\mathbf{w}}_{EU} - \mathbf{b})'\boldsymbol{\Sigma}(\hat{\mathbf{w}}_{EU} - \mathbf{b})} \tag{22}$$

Since the expression of α_n^* depends on both the population mean vector and covariance matrix and on their sample counterparts, it cannot be directly applied in practice. As such, Bodnar et al. (2020) propose a two-stage procedure. First, the deterministic quantity α^* which is asymptotically equivalent to α_n^* is found. Second, it is consistently estimated under the high-dimensional asymptotic regime.

It holds that (see, Bodnar et al. (2020, Theorem 2.1))

$$\alpha^* = \gamma^{-1}\frac{(R_{GMV} - R_b)\left(1 + \frac{1}{1-c}\right) + \gamma(V_b - V_{GMV}) + \frac{\gamma^{-1}}{1-c}s}{\frac{1}{1-c}V_{GMV} - 2\left(V_{GMV} + \frac{\gamma^{-1}}{1-c}(R_b - R_{GMV})\right) + \gamma^{-2}\left(\frac{s}{(1-c)^3} + \frac{c}{(1-c)^3}\right) + V_b}, \tag{23}$$

where $R_{GMV} = \frac{\mathbf{1}'_p \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}}{\mathbf{1}'_p \boldsymbol{\Sigma}^{-1} \mathbf{1}_p}$ is the expected return of the GMV portfolio. Following Bodnar et al. (2020) we assume throughout the paper that uniformly in p the quadratic form $\mathbf{1}' \boldsymbol{\Sigma}^{-1} \mathbf{1}_p$ is bounded away from zero and $\boldsymbol{\mu}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}$ is bounded from above by some positive constant. These conditions guarantee among others the boundedness of R_{GMV}, V_{GMV} and s as $p \to \infty$, thus, keeping the limiting expressions coming further well defined asymptotically. Consistent estimators for the variance of the GMV portfolio V_{GMV} and for the slope parameter of the efficient frontier s are given in (19) and (13), respectively. Bodnar et al. (2020) show that the sample estimators of R_{GMV}, R_b, and V_b are consistent, that is

$$
\begin{aligned}
\hat{R}_{GMV} &= \frac{\mathbf{1}'_p \hat{\boldsymbol{\Sigma}}_n^{-1} \bar{\mathbf{x}}_n}{\mathbf{1}'_p \hat{\boldsymbol{\Sigma}}_n^{-1} \mathbf{1}_p} \overset{a.s}{\to} R_{GMV}, \\
\hat{R}_b &= \mathbf{b}' \bar{\mathbf{x}}_n \overset{a.s}{\to} R_b, \\
\hat{V}_b &= \mathbf{b}' \hat{\boldsymbol{\Sigma}}_n \mathbf{b} \overset{a.s}{\to} V_b,
\end{aligned}
\tag{24}
$$

for $p/n \to c \in [0, 1)$ as $n \to \infty$.

Hence, a consistent estimator for α^* is constructed as

$$
\hat{\alpha}_c^* = \gamma^{-1} \frac{(\hat{R}_{GMV} - \hat{R}_b)\left(1 + \frac{1}{1-c_n}\right) + \gamma(\hat{V}_b - \hat{V}_c) + \frac{\gamma^{-1}}{1-c_n}\hat{s}_c}{\frac{1}{1-c_n}\hat{V}_c - 2\left(\hat{V}_c + \frac{\gamma^{-1}}{1-c_n}(\hat{R}_b - \hat{R}_{GMV})\right) + \gamma^{-2}\left(\frac{\hat{s}_c}{(1-c_n)^3} + \frac{c_n}{(1-c_n)^3}\right) + \hat{V}_b},
\tag{25}
$$

while the bona fide shrinkage estimator for the weights of the EU portfolio are expressed as

$$
\hat{\mathbf{w}}_{BFGSE} = \hat{\alpha}_c^* \hat{\mathbf{w}}_{EU} + (1 - \hat{\alpha}_c^*)\mathbf{b}.
\tag{26}
$$

Next, we prove that $\hat{\alpha}_c^*$ is asymptotically normally distributed. This result will then be used to derive a test for the structure of the EU portfolio in Section 3.2. Let $\alpha^* = \frac{A}{B}$ and $\hat{\alpha}_c^* = \frac{\hat{A}_n}{\hat{B}_n}$. Then, we get

$$
\begin{aligned}
\sqrt{n}(\hat{\alpha}_c^* - \alpha^*) &= \sqrt{n}\left(\frac{\hat{A}_n - A}{\hat{B}_n} - \frac{A(\hat{B}_n - B)}{B\hat{B}_n}\right) \\
&= \frac{1}{\hat{B}_n}\left(\sqrt{n}(\hat{A}_n - A) - \frac{A}{B}\sqrt{n}(\hat{B}_n - B)\right) \\
&= \frac{\mathbf{d}'}{\hat{B}_n}\sqrt{n}\mathbf{t} + o_P(1)
\end{aligned}
\tag{27}
$$

for $p/n \to c + o(n^{-1/2})$ as $n \to \infty$ with

$$
\mathbf{t} = \begin{pmatrix} \hat{R}_{GMV} - R_{GMV} \\ \hat{V}_c - V_{GMV} \\ \hat{s}_c - s \\ \hat{R}_b - R_b \\ \hat{V}_b - V_b \end{pmatrix} \quad \text{and} \quad \mathbf{d} = \begin{pmatrix} 1 + \frac{1}{1-c_n}\left(1 - 2\frac{A}{B}\right) \\ -\gamma\left(1 + \frac{A}{B}\left(\frac{1}{1-c_n} - 2\right)\right) \\ \frac{\gamma^{-1}}{1-c_n}\left(1 - \frac{1}{(1-c_n)^2}\frac{A}{B}\right) \\ -1 - \frac{1}{1-c_n}\left(1 - 2\frac{A}{B}\right) \\ \gamma\left(1 - \frac{A}{B}\right) \end{pmatrix},
\tag{28}
$$

where the symbol $o_P(1)$ denotes a sequence which tends almost surely to zero. In Theorem 1 we derive the asymptotic distribution of $\mathbf{t}$.

Theorem 1 *Let* $\mathbf{x}_1, \ldots, \mathbf{x}_n$ *be independent and identically distributed with* $\mathbf{x}_i \sim \mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ *for* $i = 1, \ldots, n$ *with* $\boldsymbol{\Sigma}$ *positive definite. Then it holds that*

$$\sqrt{n}\mathbf{t} \xrightarrow{d} \mathcal{N}_5(\mathbf{0}, \boldsymbol{\Omega}_\alpha) \tag{29}$$

for $p/n \to c \in [0, 1)$ *as* $n \to \infty$ *where*

$$\boldsymbol{\Omega}_\alpha = \begin{pmatrix} \frac{V_{GMV}(s+1)}{1-c} & 0 & 0 & V_{GMV} & -2V_{GMV}(R_b - R_{GMV}) \\ 0 & 2\frac{V_{GMV}^2}{1-c} & 0 & 0 & 2V_{GMV}^2 \\ 0 & 0 & 2\frac{((s+1)^2+c-1)}{1-c} & 2(R_b - R_{GMV}) & -2(R_b - R_{GMV})^2 \\ V_{GMV} & 0 & 2(R_b - R_{GMV}) & V_b & 0 \\ -2V_{GMV}(R_b - R_{GMV}) & 2V_{GMV}^2 & -2(R_b - R_{GMV})^2 & 0 & 2V_b^2 \end{pmatrix}. \tag{30}$$

Since

$$\hat{B}_n \xrightarrow{a.s} B \quad \text{for} \quad \frac{p}{n} \to c \in [0, 1) \quad \text{as} \quad n \to \infty,$$

the application of Slutsky's lemma (c.f., DasGupta (2008, Theorem 1.5)) leads to the asymptotic distribution of $\hat{\alpha}_c^*$ as given in Theorem 2.

Theorem 2 *Under the assumptions of Theorem 1, it holds that*

$$\sqrt{n}(\hat{\alpha}_c^* - \alpha^*) \xrightarrow{d} \mathcal{N}(0, C_\alpha), \tag{31}$$

for $p/n \to c \in [0, 1)$ *as* $n \to \infty$ *where*

$$C_\alpha = \frac{1}{B^2}\mathbf{d}'\boldsymbol{\Omega}_\alpha\mathbf{d}. \tag{32}$$

Finally, using (13), (19), and (24) a consistent estimator for C_α is given by

$$\hat{C}_\alpha = \frac{1}{\hat{B}_n^2}\mathbf{d}'\hat{\boldsymbol{\Omega}}_{\alpha;c}\mathbf{d}, \tag{33}$$

where $\hat{\boldsymbol{\Omega}}_{\alpha;c}$ is a consistent estimator for $\boldsymbol{\Omega}_\alpha$ expressed as

$$\hat{\boldsymbol{\Omega}}_{\alpha;c} = \begin{pmatrix} \frac{\hat{V}_c(\hat{s}_c+1)}{1-c} & 0 & 0 & \hat{V}_c & -2\hat{V}_c(\hat{R}_b - \hat{R}_{GMV}) \\ 0 & 2\frac{\hat{V}_c^2}{1-c} & 0 & 0 & 2\hat{V}_c^2 \\ 0 & 0 & 2\frac{((\hat{s}_c+1)^2+c-1)}{1-c} & 2(\hat{R}_b - \hat{R}_{GMV}) & -2(\hat{R}_b - \hat{R}_{GMV})^2 \\ \hat{V}_c & 0 & 2(\hat{R}_b - \hat{R}_{GMV}) & \hat{V}_b & 0 \\ -2\hat{V}_c(\hat{R}_b - \hat{R}_{GMV}) & 2\hat{V}_c^2 & -2(\hat{R}_b - \hat{R}_{GMV})^2 & 0 & 2\hat{V}_b^2 \end{pmatrix}. \tag{34}$$

Remark 1 *In the case of the investor who invests into the GMV portfolio* ($\gamma = \infty$), *the formulas* (23) *and* (25) *simplify to*

$$\alpha^* = \frac{(1-c)(V_b - V_{GMV})}{c + (1-c)(V_b - V_{GMV})} \quad \text{and} \quad \hat{\alpha}_c^* = \frac{(1-c)(\hat{V}_b - \hat{V}_c)}{c + (1-c)(\hat{V}_b - \hat{V}_c)}.$$

Moreover, the application of Theorem 1 leads to

$$\sqrt{n}(\hat{\alpha}_c^* - \alpha^*) \to \mathcal{N}\left(0, \frac{2(1-c)c^2(L_b+1)}{((1-c)R_\mathbf{b}+c)^4}((2-c)L_b+c)\right) \tag{35}$$

for $p/n \to c \in (0,1)$ *as* $n \to \infty$ *with* $L_b = V_b/V_{GMV} - 1$, *which coincides with the results obtained in Theorem 2 of Bodnar, Dmytriv, Parolya and Schmid (2019).*

3.2 Test based on a shrinkage estimator

We use the properties of the shrinkage intensity α^* and of its consistent estimator $\hat{\alpha}_c^*$ to derive an asymptotic test on the structure of the EU portfolio. The testing hypotheses are given by

$$H_0 : \mathbf{w}_{EU} = \mathbf{w}_0 \qquad \text{against} \qquad H_1 : \mathbf{w}_{EU} \neq \mathbf{w}_0, \tag{36}$$

which, in contrast to the hypotheses considered in Section 2, allow to test the structure of the whole vector of the EU portfolio weights by using a single test avoiding the problem of multiplicity.

Following Bodnar, Dmytriv, Parolya and Schmid (2019), the idea behind a statistical test based on the shrinkage approach is the usage $\mathbf{w}_0$ as a fixed target portfolio, i.e., to set $\mathbf{b} = \mathbf{w}_0$ in (21). Since $\mathbf{w}_0$ is the EU optimal portfolio under the null hypothesis in (36), its expected return and variance should satisfy

$$R_{\mathbf{w}_0} = R_{GMV} + \gamma^{-1}s \quad \text{and} \quad V_{\mathbf{w}_0} = V_{GMV} + \gamma^{-2}s. \tag{37}$$

As a result, the numerator in (23) becomes

$$\begin{aligned} A(\mathbf{w}_0) &= (R_{GMV} - R_b)\left(1 + \frac{1}{1-c}\right) + \gamma(V_b - V_{GMV}) + \frac{\gamma^{-1}}{1-c}s \\ &= -\gamma^{-1}s\left(1 + \frac{1}{1-c}\right) + \gamma^{-1}s + \frac{\gamma^{-1}}{1-c}s = 0, \end{aligned}$$

proving that

$$\alpha^* = 0 \quad \text{under} \quad H_0. \tag{38}$$

Hence, for testing (36), one can derive a test on the hypotheses

$$H_0 : \alpha^*(\mathbf{w}_0) = 0 \qquad \text{against} \qquad H_1 : \alpha^*(\mathbf{w}_0) \neq 0, \tag{39}$$

where the notation $\alpha^*(\mathbf{w}_0)$ denotes the optimal shrinkage intensity as in (23) computed with target portfolio $\mathbf{w}_0$. It has to be noted that the hypotheses (36) and (39) are not equivalent. Nevertheless, the rejection of the null hypothesis in (39) ensures the rejection of the null hypothesis in (36) meaning that $\mathbf{w}_0$ is not the EU optimal portfolio.

Let $\hat{\alpha}_c^*(\mathbf{w}_0)$ be the consistent estimator of $\alpha^*(\mathbf{w}_0)$ as constructed in (25) when the shrinkage

13

target is $\mathbf{b} = \mathbf{w}_0$. Then the application of Theorem 2 shows that

$$\hat{\alpha}_c^*(\mathbf{w}_0) \overset{a.s.}{\to} 0 \quad \text{for} \quad \frac{p}{n} \to c \in [0,1) \quad \text{as} \quad n \to \infty,$$

when the null hypothesis in (36) is true.

Moreover, since the numerator in the expression of $\alpha^*(\mathbf{w}_0)$ in (23) under the null hypothesis in (39) is equal to zero, i.e. $A = 0$ where A is defined before (27), we get the following stochastic representation of $\sqrt{n}\hat{\alpha}_c^*(\mathbf{w}_0)$ expressed as

$$\sqrt{n}\hat{\alpha}_c^*(\mathbf{w}_0) = \frac{1}{\hat{B}_n}\mathbf{d}_0'\sqrt{n}\mathbf{t} \quad \text{with} \quad \mathbf{d}_0 = \begin{pmatrix} 1 + \frac{1}{1-c_n} \\ -\gamma \\ \frac{\gamma^{-1}}{1-c_n} \\ -1 - \frac{1}{1-c_n} \\ \gamma \end{pmatrix} \tag{40}$$

and $\mathbf{t}$ is defined in (28). The application of Theorem 1 then leads to the following result

Theorem 3 *Assume that the conditions of Theorem 1 are fulfilled. Then, under the null hypothesis in* (39), *it holds that*

$$\sqrt{n}\hat{\alpha}_c^*(\mathbf{w}_0) \overset{d}{\to} \mathcal{N}(0, C_{\alpha;0}), \tag{41}$$

for $p/n \to c \in [0,1)$ *as* $n \to \infty$ *with* $C_{\alpha;0} = \frac{1}{B^2}\mathbf{d}_0'\mathbf{\Omega}_\alpha\mathbf{d}_0$ *where* $\mathbf{\Omega}_\alpha$ *is given in* (30) *and* B *is defined before* (27).

Replacing B and $\mathbf{\Omega}_\alpha$ by their consistent estimators $\hat{B}_n^2$ and $\hat{\mathbf{\Omega}}_{\alpha;c}$, we get a consistent estimator for $C_{\alpha;0}$ expressed as

$$\hat{C}_{\alpha;0} = \frac{1}{\hat{B}_n^2}\mathbf{d}_0'\hat{\mathbf{\Omega}}_{\alpha;c}\mathbf{d}_0. \tag{42}$$

Then for testing hypotheses (39), we obtain the following test statistic

$$T_\alpha = \sqrt{n}\frac{\hat{\alpha}_c^*(\mathbf{w}_0)}{\sqrt{\hat{C}_{\alpha;0}}} = \sqrt{n}\frac{\hat{\alpha}_c^*(\mathbf{w}_0)\hat{B}_n}{\sqrt{\mathbf{d}_0'\hat{\mathbf{\Omega}}_{\alpha;c}\mathbf{d}_0}}, \tag{43}$$

where $\hat{\alpha}_c^*(\mathbf{w}_0)$ with $\mathbf{b} = \mathbf{w}_0$ and $\hat{\mathbf{\Omega}}_{\alpha;c}$ are given in (25) and (34), respectively. Under the null hypothesis in (39) we get that

$$T_\alpha \overset{d}{\to} \mathcal{N}(0,1)$$

for $p/n \to c \in [0,1)$ as $n \to \infty$ and, hence, the hypothesis that $\mathbf{w}_0$ are the weights of the EU portfolio is rejected as soon as $|T_\alpha| > z_{1-\beta/2}$ where $z_{1-\beta/2}$ is the $(1-\beta/2)$ quantile of the standard normal distribution. Under the alternative hypothesis in (39), the distribution of $\sqrt{n}\hat{\alpha}_c^*(\mathbf{w}_0)$ can still be well approximated by the normal distribution under the high-dimensional asymptotic regime and $\mathbf{d}_0'\hat{\mathbf{\Omega}}_{\alpha;c}\mathbf{d}_0$ provides a consistent estimator of its asymptotic variance. On the other side, it does not hold that $\hat{\alpha}_c^*(\mathbf{w}_0) \overset{a.s.}{\to} 0$ and consequently, the test based on T_α can detect the deviation in the null hypotheses of both (36) and (39).

Remark 2 *Using that* $s = \gamma(R_{\mathbf{w}_0} - R_{GMV})$ *(see* (37)*) and* $\hat{R}_{\mathbf{w}_0}$ *and* $\hat{R}_{GMV}$ *are consistent*

14

estimators of $R_{\mathbf{w}_0}$ and R_{GMV}, respectively (see (24)), another consistent estimator of $\mathbf{\Omega}_\alpha$ under H_0 in (39) is given by

$$
\widetilde{\mathbf{\Omega}}_{\alpha;c} = \begin{pmatrix}
\frac{\hat{V}_c(\gamma(\hat{R}_{\mathbf{w}_0}-\hat{R}_{GMV})+1)}{1-c} & 0 & 0 & \hat{V}_c & -2\hat{V}_c(\hat{R}_{\mathbf{w}_0}-\hat{R}_{GMV}) \\
0 & 2\frac{\hat{V}_c^2}{1-c} & 0 & 0 & 2\hat{V}_c^2 \\
0 & 0 & 2\frac{((\gamma(\hat{R}_{\mathbf{w}_0}-\hat{R}_{GMV})+1)^2+c-1)}{1-c} & 2(\hat{R}_{\mathbf{w}_0}-\hat{R}_{GMV}) & -2(\hat{R}_{\mathbf{w}_0}-\hat{R}_{GMV})^2 \\
\hat{V}_c & 0 & 2(\hat{R}_{\mathbf{w}_0}-\hat{R}_{GMV}) & \hat{V}_{\mathbf{w}_0} & 0 \\
-2\hat{V}_c(\hat{R}_{\mathbf{w}_0}-\hat{R}_{GMV}) & 2\hat{V}_c^2 & -2(\hat{R}_{\mathbf{w}_0}-\hat{R}_{GMV})^2 & 0 & 2\hat{V}_{\mathbf{w}_0}^2
\end{pmatrix}.
\tag{44}
$$

Then, the hypotheses in (39) can also be tested by using the following test statistic

$$
\widetilde{T}_\alpha = \sqrt{n}\frac{\hat{\alpha}_c^*(\mathbf{w}_0)\hat{B}_n}{\sqrt{\mathbf{d}_0'\widetilde{\mathbf{\Omega}}_{\alpha;c}\mathbf{d}_0}}
\tag{45}
$$

which is asymptotically standard normally distributed under H_0 in (39).

Remark 3 *Using the duality between the test theory and confidence interval (see, Aitchison (1964)), the null hypothesis in (39) and consequently in (36) are rejected at significance level β as soon as the $(1-\beta)$ confidence interval constructed for $\alpha^*(\mathbf{w}_0)$ does not include zero. This confidence interval in the case of the test T_α has the boundaries*

$$
\hat{\alpha}_c^*(\mathbf{w}_0) \pm \frac{z_{1-\beta/2}}{\sqrt{n}}\frac{\sqrt{\mathbf{d}_0'\hat{\mathbf{\Omega}}_{\alpha;c}\mathbf{d}_0}}{\hat{B}_n},
\tag{46}
$$

while for the test based on $\widetilde{T}_\alpha$ we get

$$
\hat{\alpha}_c^*(\mathbf{w}_0) \pm \frac{z_{1-\beta/2}}{\sqrt{n}}\frac{\sqrt{\mathbf{d}_0'\widetilde{\mathbf{\Omega}}_{\alpha;c}\mathbf{d}_0}}{\hat{B}_n}.
\tag{47}
$$

To assess the precision of the asymptotic distribution we use a similar setting as in the last section. In Figure 2 we show the KDEs of the distribution of the test statistics T_α and $\widetilde{T}_\alpha$ under the null hypothesis together with their high-dimensional asymptotic distribution. The latter approximates the simulated exact distributions very precisely, although the the fit appears to be slightly better for T_α. The empirical size on both cases is close to the nominal size of 5% as it is shown in Table 2. Summarizing, we conclude that the high-dimensional asymptotic distribution provide a good approximation for proposed test statistics for different values of c.

	$\mathbf{c = 0.3}$	$\mathbf{c = 0.8}$
T_α	0.048	0.054
$\widetilde{T}_\alpha$	0.052	0.053

Table 2: Empirical sizes of the two tests based on T_α and $\widetilde{T}_\alpha$ using $5 \cdot 10^3$ replications.
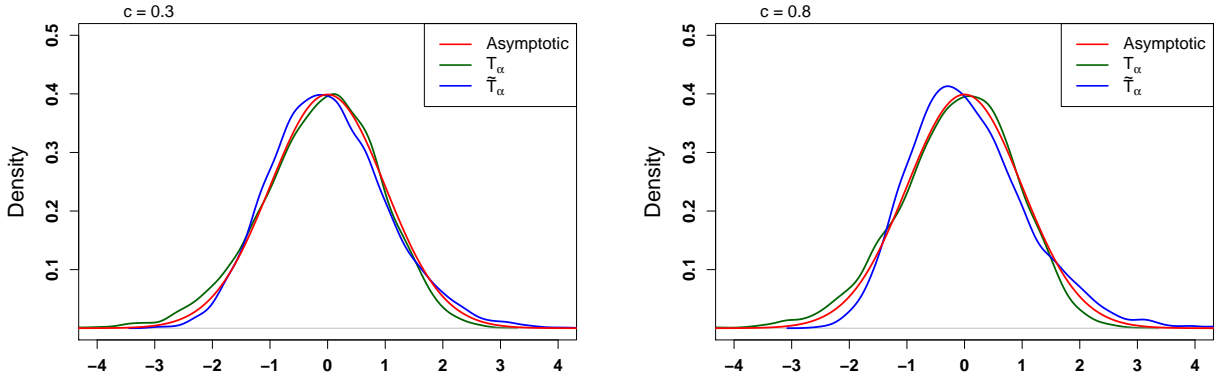
Figure 2: The high-dimensional asymptotic normal approximation of the densities of T_α and $\widetilde{T}_\alpha$ together with their kernel density estimators for $\gamma = 5$, $p = 300$ and $c_n = p/n \in \{0.3, 0.8\}$.

4 Simulation and empirical study

The performance of the derived test is investigated throughout an extensive simulation study. In particular, we explore the behavior of the test with respect to its power characteristics and receiver operative characteristic curves. Additionally, we apply the derived inference procedure to the real data in this section.

4.1 Simulation study

The sample of asset returns $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x_n}$ are generated independently from $\mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. To mimic the bahavior of real data we generate the eigenvalues of population covariance matrix $\boldsymbol{\Sigma}$ according to the law $\lambda_i = 0.1 e^{\delta c(i-1)/p}$, $i = 1, \ldots p$ (see, Bodnar et al. (2020)) and take its eigenvectors from the spectral decomposition of the standard Wishart random matrix. Then, the covariance matrix is given as follows

$$\boldsymbol{\Sigma} = \Theta \boldsymbol{\Lambda} \Theta', \tag{48}$$

where $\boldsymbol{\Lambda}$ is a diagonal matrix of the predefined eigenvalues and Θ is a $p \times p$ matrix of eigenvectors. By changing the value of δ, we can control the conditional index of the covariance matrix for different values of c. We set condition index equals to 450. This setting reflects the parametrisation we observed in the empirical study in the next section. The mean vector is randomly generated from $U(-0.2, 0.2)$, which also corresponds to the natural behavior of daily asset returns.

We assume that the portfolio weights and thus the shrinkage intensity change due to a change in the mean of asset returns. Under the alternative hypothesis, there is an additive shift to the mean vector of the asset returns defined as

$$\boldsymbol{\mu}_1 = \boldsymbol{\mu} + \epsilon, \tag{49}$$

16

where
$$\epsilon = -a \cdot (\underbrace{1, \ldots, 1}_{m}, \underbrace{0, \ldots, 0}_{m}),$$

where $a = 0.01\kappa$, $\kappa \in \{0, 1, 2, \ldots, 35\}$, $m = 0.5p$. Thus we assume that the expected return on the assets with high variance decreases.

We conduct the test at the significance level $\alpha = 0.05$. We put $p = 300$ and $c \in \{0.3, 0.8\}$. The number of repetitions is 10^5 and $\gamma = 5$. For the ROC curves we fix a at 0.08. The results are illustrated in Figure 3. It can be seen that both tests display an overall consistency and a good performance in terms of power functions and ROC curves. The behavior is better for smaller values of c and not substantially worse in case of $c = 0.8$. The test based on the test statistic given in (45) outperforms the test given in (43) and demonstrates a satisfactory power.
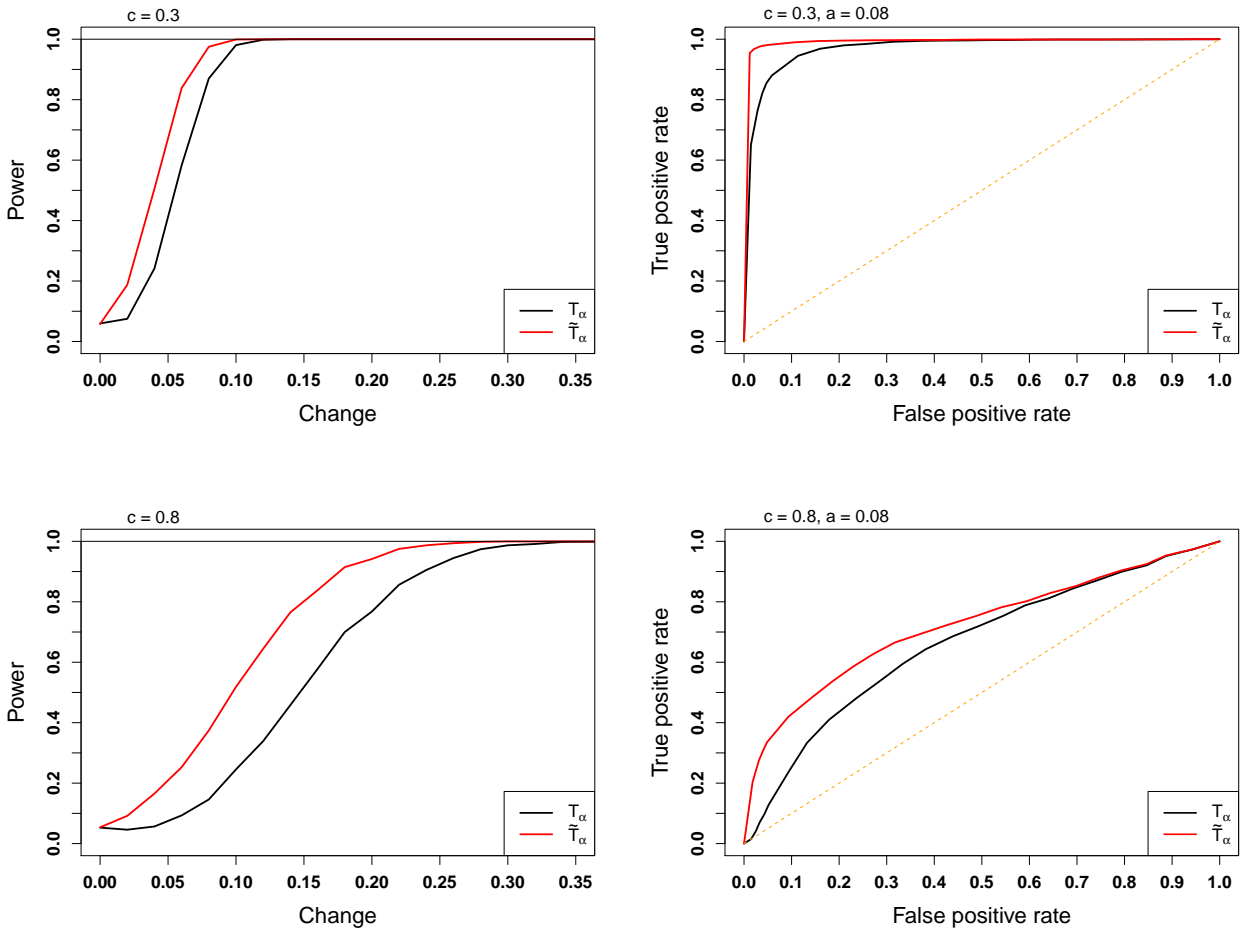

Figure 3: Empirical power functions of the proposed tests as a function of the change a (left) and ROC curves of two tests for $a = 0.08$ (right) for different values of c according to the scenario given in (49) and $p = 300$.

4.2   Empirical study

In this section, we apply the derived theoretical results to real data. The objective is to determine the periods where the shrinkage intensity is significantly different from zero and thus the EU optimal portfolio is significantly different from the target or the benchmark portfolio $\boldsymbol{b}$. This
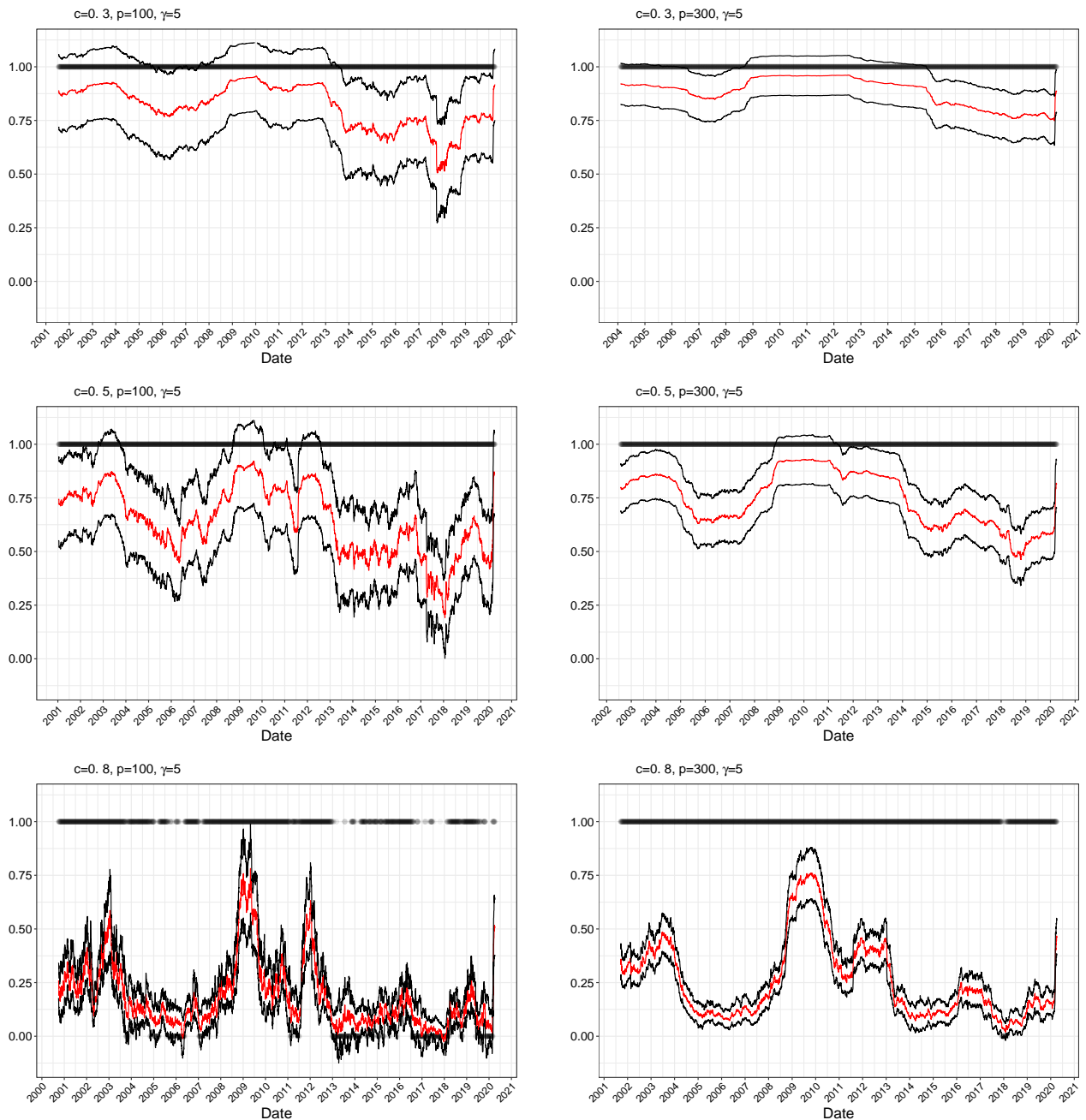
Figure 4: Estimated shrinkage intensities for the equally weighted portfolio as the target portfolio ($p = 100$ on the right and $p = 300$ on the left) with 95% pointwise confidence intervals. The black dots indicate the periods with rejected H_0 (1-values) and not rejected H_0 (0- values).

study is based on daily return data of all companies listed in the S&P 500 index for the period from April 1999 to March 2020. We assume that the investor allocates her wealth to portfolios of size $p \in \{100, 300\}$ with daily reallocation. She selects the first p assets in alphabetic order from the available data. The sample size n is chosen to attain $c \in \{0.3, 0.5, 0.8\}$, i.e. $n = p/c$. We put $\gamma = 5$ which is a common value for the risk aversion coefficient in financial literature. As the target portfolio we consider the equally weighted portfolio with all weights equal $1/p$. Despite of its simplicity this portfolio appears to show a superior long-run performance and dominates many more sophisticated trading strategies (see DeMiguel et al. (2009)).

Figure 4 shows the time series of estimated shrinkage intensities together with 95% confidence intervals as defined in (47). If $c = 0.3$, then the shrinkage intensity is close to one indicating

that the EU portfolio clearly dominates both benchmarks in the convex combination. This is due to the fact, that the investor has more historical data to estimate the unknown parameters and the estimation risk is relatively low. If c increases, the sample available for a portfolio of a fixed size gets smaller and the shrinkage intensity shifts towards zero. The benchmark portfolio gets higher weight and for $c = 0.8$ it even becomes dominant. The same reasoning applies if we analyse the impact of increase in p from 100 to 300. Fixed c and larger p increase the sample size n and has a stabilizing impact on the shrinkage intensity.

We cannot reject the null hypothesis of the test based on $\tilde{T}_\alpha$ in (45) that the shrinkage intensity is zero if the confidence intervals cover the zero value (see Remark 3 above). The figures reveal that we never opt for H_0 if $c = 0.3$ or 0.5. Thus for this parameter constellation the portfolio weights of the EU portfolio are always significantly different from the weights of the equally weighted portfolio. The situation changes for $c = 0.8$ where we do have periods with not rejected H_0 in (39). Similar behavior is observed for $p = 300$ too, however, here the intensities and their variances are more stable leading to less periods with not rejected H_0.
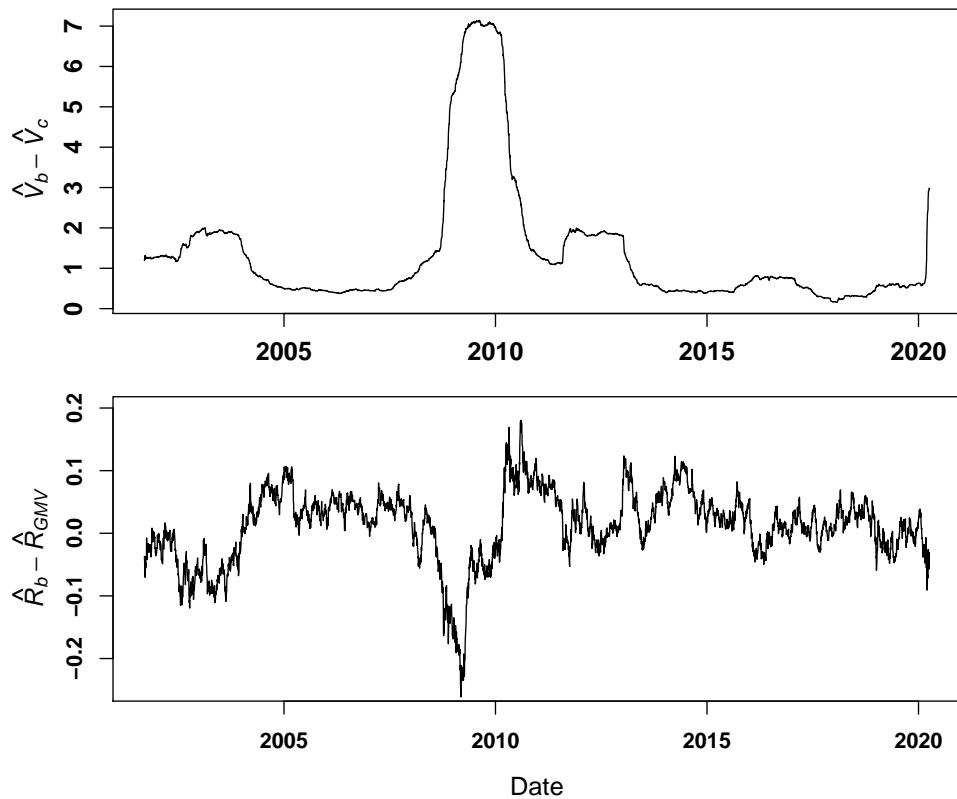

Figure 5: Components of the estimated shrinkage intensity given in (25) using equally weighted target for $c = 0.8$, $p = 300$ and $\gamma = 5$.

Recall that a non-rejection of H_0 in (39) does not guarantee that the weights of the EU portfolio coincide with the weights of the target portfolio. To elaborate on the difference between the two portfolios and to get more economic insight into the dynamics of the intensities we consider Figure 5. Here we plot the difference between the means and variances of the GMV and the equally weighted benchmark. These quantities determine the behavior of the empirical

shrinkage intensity in (23). On the one hand, we observe in Figure 4 that the shrinkage intensity increases during a crisis period, e.g. 2002-2003 and 2008-2010. This seems to be surprising since the volatility of returns is high in this period and the equally weighted portfolio is believed to reduce the risk. However, Figure 5 shows that the variance of the benchmark portfolio is much higher (i.e. $\hat{V}_b > \hat{V}_c$) and its return is much lower (i.e. $\hat{R}_b < \hat{R}_{GMV}$) compared to the GMV portfolio in the crisis period leading to a higher relative precision and efficiency of the EU portfolio. On another hand, the mean returns and the variances are almost indistinguishable in calm periods leading to shrinkage intensities closer to zero and even insignificant for larger c's. Thus we conclude that non-rejecting H_0 is driven by high similarity between the mean and the variance of the target and GMV portfolios.

5   Summary

This paper is dedicated to portfolio selection problems driven by high-dimensional financial data sets. In particular, we deal with optimal asset allocation in a high-dimensional asymptotic regime, namely when the number of assets and the sample size tend to infinity at the same rate. Due to the curse of dimensionality in the parameter estimation process, asset allocation for such portfolios becomes a challenging task. Using the techniques from the theory of random matrices, new inferential procedures based on the optimal shrinkage intensity for testing the efficiency of the high-dimensional EU portfolio are developed and the asymptotic distributions of the proposed test statistics are derived. In extensive simulations, we show that the suggested tests have excellent performance characteristics for various values of c. The practical advantage of the proposed procedures are demonstrated in en empirical study based on stocks included into the S&P 500 index.

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6 Appendix

In this section the proofs of the theoretical results are given. The proof of Theorem 1 is based on Lemmas 1-2.

Lemma 1 *Let* $\mathbf{z}_1$, *...*, $\mathbf{z}_n$ *be an independent sample from the* p*-dimensional standard normal distribution and let*

$$\mathbf{S}_n = \frac{1}{n-1} \sum_{j=1}^{n} (\mathbf{z}_j - \bar{\mathbf{z}})(\mathbf{z}_j - \bar{\mathbf{z}})' \tag{50}$$

be the corresponding sample covariance matrix. Let $\mathbf{m}_1$, $\mathbf{m}_2$, *and* $\mathbf{m}_3$ *be the* p*-dimensional vector of constants with the Euclidean norms equal to one. Then*

$$\sqrt{n} \begin{pmatrix} \mathbf{m}_1' \mathbf{S}_n \mathbf{m}_1 - 1 \\ \mathbf{m}_2' \mathbf{S}_n^{-1} \mathbf{m}_2 - \frac{1}{1-c_n} \\ \mathbf{m}_2' \mathbf{S}_n^{-1} \mathbf{m}_3 - \frac{1}{1-c_n} \mathbf{m}_2' \mathbf{m}_3 \\ \mathbf{m}_3' \mathbf{S}_n^{-1} \mathbf{m}_3 - \frac{1}{1-c_n} \end{pmatrix} \xrightarrow{d} \mathcal{N}_4 \left(\mathbf{0}, \frac{2}{c} \mathbf{\Theta}(\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3) \circ \mathbf{\Lambda} \right), \tag{51}$$

23

with

$$\Theta(\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3) = \begin{pmatrix} 1 & \lim_{n\to\infty}(\mathbf{m}_1'\mathbf{m}_2)^2 & \lim_{n\to\infty}(\mathbf{m}_1'\mathbf{m}_2)(\mathbf{m}_1'\mathbf{m}_3) & \lim_{n\to\infty}(\mathbf{m}_1'\mathbf{m}_3)^2 \\ \lim_{n\to\infty}(\mathbf{m}_1'\mathbf{m}_2)^2 & 1 & \lim_{n\to\infty}(\mathbf{m}_2'\mathbf{m}_3) & \lim_{n\to\infty}(\mathbf{m}_2'\mathbf{m}_3)^2 \\ \lim_{n\to\infty}(\mathbf{m}_1'\mathbf{m}_2)(\mathbf{m}_1'\mathbf{m}_3) & \lim_{n\to\infty}(\mathbf{m}_2'\mathbf{m}_3) & 0.5 + 0.5\lim_{n\to\infty}(\mathbf{m}_2'\mathbf{m}_3)^2 & \lim_{n\to\infty}(\mathbf{m}_2'\mathbf{m}_3) \\ \lim_{n\to\infty}(\mathbf{m}_1'\mathbf{m}_3)^2 & \lim_{n\to\infty}(\mathbf{m}_2'\mathbf{m}_3)^2 & \lim_{n\to\infty}(\mathbf{m}_2'\mathbf{m}_3) & 1 \end{pmatrix} \tag{52}$$

and

$$\Lambda = \begin{pmatrix} c & -\frac{c}{1-c} & -\frac{c}{1-c} & -\frac{c}{1-c} \\ -\frac{c}{1-c} & \frac{c}{(1-c)^3} & \frac{c}{(1-c)^3} & \frac{c}{(1-c)^3} \\ -\frac{c}{1-c} & \frac{c}{(1-c)^3} & \frac{c}{(1-c)^3} & \frac{c}{(1-c)^3} \\ -\frac{c}{1-c} & \frac{c}{(1-c)^3} & \frac{c}{(1-c)^3} & \frac{c}{(1-c)^3} \end{pmatrix}, \tag{53}$$

where the symbol $\circ$ denotes the Hadamard (elementwise) product of matrices.

Proof of Lemma 1: Since $(n-1)\mathbf{S}_n$ has a p-dimensional Wishart distribution with the identity covariance matrix, we get that there exists a $p\times(n-1)$ matrix $\tilde{\mathbf{Z}}$ whose entries are independent and standard normally distributed such that $(n-1)\mathbf{S}_n = \tilde{\mathbf{Z}}\tilde{\mathbf{Z}}'$. The application of Theorem 2 in Bai et al. (2011) leads to (51) with Θ as in (52) and Λ given by

$$\Lambda = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_2 & \lambda_2 \\ \lambda_2 & \lambda_3 & \lambda_3 & \lambda_3 \\ \lambda_2 & \lambda_3 & \lambda_3 & \lambda_3 \\ \lambda_2 & \lambda_3 & \lambda_3 & \lambda_3 \end{pmatrix}$$

with

$$\lambda_1 = \int_{a_-}^{a_+} z^2 \mathrm{d}F_c(z) - \left(\int_{a_-}^{a_+} z \mathrm{d}F_c(z)\right)^2,$$

$$\lambda_2 = 1 - \int_{a_-}^{a_+} z \mathrm{d}F_c(z) \int_{a_-}^{a_+} \frac{1}{z} \mathrm{d}F_c(z),$$

$$\lambda_3 = \int_{a_-}^{a_+} \frac{1}{z^2} \mathrm{d}F_c(z) - \left(\int_{a_-}^{a_+} \frac{1}{z} \mathrm{d}F_c(z)\right)^2$$

where the function $F_c(z)$ denotes the cumulative distribution function of the Marchenko-Pastur law (see, Bai and Silverstein (2010)) for $c < 1$ expressed as

$$\mathrm{d}F_c(z) = \frac{1}{2\pi z c}\sqrt{(a_+ - z)(z - a_-)}\mathbb{1}_{[a_-, a_+]}(z)dz,$$

where $a_\pm = (1 \pm \sqrt{c})^2$. The moments of $F_c(z)$ present in Λ can be found in Glombeck (2014, Lemma 14). This completes the proof of the lemma. $\square$

Lemma 2 *Under the conditions of Theorem 1 it holds that*

$$\sqrt{n}\mathbf{h} = \sqrt{n} \begin{pmatrix} \mathbf{1}_p'\hat{\boldsymbol{\Sigma}}_n^{-1}\bar{\mathbf{x}}_n - \frac{1}{1-c_n}\mathbf{1}_p'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} \\ \mathbf{1}_p'\hat{\boldsymbol{\Sigma}}_n^{-1}\mathbf{1}_p - \frac{1}{1-c_n}\mathbf{1}_p'\boldsymbol{\Sigma}^{-1}\mathbf{1}_p \\ \bar{\mathbf{x}}_n'\hat{\boldsymbol{\Sigma}}_n^{-1}\bar{\mathbf{x}}_n - \frac{1}{1-c_n}\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - \frac{c_n}{1-c_n} \\ \mathbf{b}'\bar{\mathbf{x}}_n - \mathbf{b}'\boldsymbol{\mu} \\ \mathbf{b}'\hat{\boldsymbol{\Sigma}}_n\mathbf{b} - \mathbf{b}'\boldsymbol{\Sigma}\mathbf{b} \end{pmatrix} \xrightarrow{d} \mathcal{N}_5\left(\mathbf{0}, \boldsymbol{\Xi}\right) \tag{54}$$

for $c_n = p/n \to c \in [0,1)$ *as* $n \to \infty$ *with*

$$\boldsymbol{\Xi} = \begin{pmatrix} \frac{1}{(1-c)^3}\frac{1}{V_{GMV}}\left(s^* + \frac{R_{GMV}^2}{V_{GMV}}\right) & \frac{2}{(1-c)^3}\frac{R_{GMV}}{V_{GMV}^2} & \frac{2}{(1-c)^3}\frac{R_{GMV}s^*}{V_{GMV}} & \frac{1}{1-c} & -\frac{2}{1-c}R_b \\ \frac{2}{(1-c)^3}\frac{R_{GMV}}{V_{GMV}^2} & \frac{2}{(1-c)^3}\frac{1}{V_{GMV}^2} & \frac{2}{(1-c)^3}\frac{R_{GMV}^2}{V_{GMV}^2} & 0 & -\frac{2}{1-c} \\ \frac{2}{(1-c)^3}\frac{R_{GMV}s^*}{V_{GMV}} & \frac{2}{(1-c)^3}\frac{R_{GMV}^2}{V_{GMV}^2} & \frac{2}{(1-c)^3}\left((s^*)^2 + c - 1\right) & \frac{2R_b}{1-c} & -\frac{2}{1-c}R_b^2 \\ \frac{1}{1-c} & 0 & \frac{2R_b}{1-c} & V_b & 0 \\ -\frac{2}{1-c}R_b & -\frac{2}{1-c} & -\frac{2}{1-c}R_b^2 & 0 & 2V_b^2 \end{pmatrix}, \tag{55}$$

where $s^* = s + \dfrac{R_{GMV}^2}{V_{GMV}} + 1$.

Proof of Lemma 2: Let $\mathbf{a}' = (a_1, a_2, a_3, a_4, a_5)$ be an arbitrary vector of constants. Next, we show that $\sqrt{n}\mathbf{a}'\mathbf{h} \xrightarrow{d} \mathcal{N}\left(0, \mathbf{a}'\boldsymbol{\Xi}\mathbf{a}\right)$, which will prove the statement of the lemma.

Since $\mathbf{x}_1, ..., \mathbf{x}_n$ are independent and identically distributed with $\mathbf{x}_i \sim \mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, we get that $\mathbf{x}_i = \boldsymbol{\mu} + \boldsymbol{\Sigma}\mathbf{z}_i$ where $\mathbf{z}_1, ..., \mathbf{z}_n$ are independent standard normally distributed and $\boldsymbol{\Sigma}^{1/2}$ is the symmetric square root of $\boldsymbol{\Sigma}$. Moreover, it holds that

$$\bar{\mathbf{x}}_n = \boldsymbol{\mu} + \boldsymbol{\Sigma}\bar{\mathbf{z}}_n \quad \text{and} \quad \hat{\boldsymbol{\Sigma}}_n = \boldsymbol{\Sigma}^{1/2}\mathbf{S}_n\boldsymbol{\Sigma}^{1/2},$$

where

$$\bar{\mathbf{z}}_n = \frac{1}{n}\sum_{i=1}^n \mathbf{z}_i \quad \text{and} \quad \mathbf{S}_n = \frac{1}{n-1}\sum_{i=1}^n (\mathbf{z}_i - \bar{\mathbf{z}}_n)(\mathbf{z}_i - \bar{\mathbf{z}}_n)'.$$

To this end, we have that $\bar{\mathbf{z}}_n$ and $\mathbf{S}_n$ are independent with $\sqrt{n}\bar{\mathbf{z}}_n$ standard normally distributed and $(n-1)\mathbf{S}_n$ standard Wishart distributed.

Let $\boldsymbol{\nu} = \boldsymbol{\Sigma}^{-1/2}\boldsymbol{\mu}$. We get

$$\sqrt{n}\mathbf{a}'\mathbf{h} = H_1(\bar{\mathbf{z}}_n, \mathbf{S}_n) + H_2(\bar{\mathbf{z}}_n),$$

with

$$
\begin{aligned}
H_1(\bar{\mathbf{z}}_n, \mathbf{S}_n) &= a_1\sqrt{\mathbf{1}_p'\boldsymbol{\Sigma}^{-1}\mathbf{1}_p}\sqrt{\bar{\mathbf{x}}_n'\boldsymbol{\Sigma}^{-1}\bar{\mathbf{x}}_n}\sqrt{n}\left(\frac{\mathbf{1}_p'\hat{\boldsymbol{\Sigma}}_n^{-1}\bar{\mathbf{x}}_n}{\sqrt{\mathbf{1}_p'\boldsymbol{\Sigma}^{-1}\mathbf{1}_p}\sqrt{\bar{\mathbf{x}}_n'\boldsymbol{\Sigma}^{-1}\bar{\mathbf{x}}_n}} - \frac{\frac{1}{1-c_n}\mathbf{1}_p'\boldsymbol{\Sigma}^{-1}\bar{\mathbf{x}}_n}{\sqrt{\mathbf{1}_p'\boldsymbol{\Sigma}^{-1}\mathbf{1}_p}\sqrt{\bar{\mathbf{x}}_n'\boldsymbol{\Sigma}^{-1}\bar{\mathbf{x}}_n}}\right) \\
&+ a_2\mathbf{1}_p'\boldsymbol{\Sigma}^{-1}\mathbf{1}_p\sqrt{n}\left(\frac{\mathbf{1}_p'\hat{\boldsymbol{\Sigma}}_n^{-1}\mathbf{1}_p}{\mathbf{1}_p'\boldsymbol{\Sigma}^{-1}\mathbf{1}_p} - \frac{1}{1-c_n}\right) + a_3\bar{\mathbf{x}}_n'\boldsymbol{\Sigma}^{-1}\bar{\mathbf{x}}_n\sqrt{n}\left(\frac{\bar{\mathbf{x}}_n'\hat{\boldsymbol{\Sigma}}_n^{-1}\bar{\mathbf{x}}_n}{\bar{\mathbf{x}}_n'\boldsymbol{\Sigma}^{-1}\bar{\mathbf{x}}_n} - \frac{1}{1-c_n}\right) \\
&+ a_5\mathbf{b}'\boldsymbol{\Sigma}\mathbf{b}\sqrt{n}\left(\frac{\mathbf{b}'\hat{\boldsymbol{\Sigma}}_n\mathbf{b}}{\mathbf{b}'\boldsymbol{\Sigma}\mathbf{b}} - 1\right) = \mathbf{d}_1'(\bar{\mathbf{z}}_n)\sqrt{n}\mathbf{h}_1(\bar{\mathbf{z}}_n, \mathbf{S}_n)
\end{aligned}
$$

and

$$
\begin{aligned}
H_2(\bar{\mathbf{z}}_n) &= a_1\frac{1}{1-c_n}\sqrt{n}\left(\mathbf{1}_p'\boldsymbol{\Sigma}^{-1}\bar{\mathbf{x}}_n - \mathbf{1}_p'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\right) \\
&+ a_3\frac{1}{1-c_n}\sqrt{n}\left(\bar{\mathbf{x}}_n'\boldsymbol{\Sigma}^{-1}\bar{\mathbf{x}}_n - \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - c_n\right) + a_4\sqrt{n}(\mathbf{b}'\bar{\mathbf{x}}_n - \mathbf{b}'\boldsymbol{\mu}) \\
&= \frac{a_3}{1-c_n}\sqrt{n}\left((\bar{\mathbf{z}}_n + \mathbf{d}_2)'(\bar{\mathbf{z}}_n + \mathbf{d}_2) - \mathbf{d}_2'\mathbf{d}_2 - c_n\right)
\end{aligned}
$$

with

$$
\mathbf{d}_1(\bar{\mathbf{z}}_n) = \begin{pmatrix} a_5\mathbf{b}'\boldsymbol{\Sigma}\mathbf{b} \\ a_2\mathbf{1}_p'\boldsymbol{\Sigma}^{-1}\mathbf{1}_p \\ a_1\sqrt{\mathbf{1}_p'\boldsymbol{\Sigma}^{-1}\mathbf{1}_p}\sqrt{(\boldsymbol{\nu}+\bar{\mathbf{z}}_n)'(\boldsymbol{\nu}+\bar{\mathbf{z}}_n)} \\ a_3(\boldsymbol{\nu}+\bar{\mathbf{z}}_n)'(\boldsymbol{\nu}+\bar{\mathbf{z}}_n) \end{pmatrix},
$$

$$
\mathbf{d}_2 = \frac{1-c_n}{a_3}\left(\frac{a_3}{1-c_n}\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\mu} + \frac{a_1}{2(1-c_n)}\boldsymbol{\Sigma}^{-1/2}\mathbf{1}_p + \frac{a_4}{2}\boldsymbol{\Sigma}^{1/2}\mathbf{b}\right),
$$

and

$$
\mathbf{h}_1(\bar{\mathbf{z}}_n, \mathbf{S}_n) = \begin{pmatrix} \frac{\mathbf{b}'\boldsymbol{\Sigma}^{1/2}\mathbf{S}_n\boldsymbol{\Sigma}^{1/2}\mathbf{b}}{\mathbf{b}'\boldsymbol{\Sigma}\mathbf{b}} - 1 \\ \frac{\mathbf{1}_p'\boldsymbol{\Sigma}^{-1/2}\mathbf{S}_n^{-1}\boldsymbol{\Sigma}^{-1/2}\mathbf{1}_p}{\mathbf{1}_p'\boldsymbol{\Sigma}^{-1}\mathbf{1}_p} - \frac{1}{1-c_n} \\ \frac{\mathbf{1}_p'\boldsymbol{\Sigma}^{-1/2}\mathbf{S}_n^{-1}(\boldsymbol{\nu}+\bar{\mathbf{z}}_n)}{\sqrt{\mathbf{1}_p'\boldsymbol{\Sigma}^{-1}\mathbf{1}_p}\sqrt{(\boldsymbol{\nu}+\bar{\mathbf{z}}_n)'(\boldsymbol{\nu}+\bar{\mathbf{z}}_n)}} - \frac{1}{1-c_n}\frac{\mathbf{1}_p'\boldsymbol{\Sigma}^{-1/2}(\boldsymbol{\nu}+\bar{\mathbf{z}}_n)}{\sqrt{\mathbf{1}_p'\boldsymbol{\Sigma}^{-1}\mathbf{1}_p}\sqrt{(\boldsymbol{\nu}+\bar{\mathbf{z}}_n)'(\boldsymbol{\nu}+\bar{\mathbf{z}}_n)}} \\ \frac{(\boldsymbol{\nu}+\bar{\mathbf{z}}_n)'\mathbf{S}_n^{-1}(\boldsymbol{\nu}+\bar{\mathbf{z}}_n)}{(\boldsymbol{\nu}+\bar{\mathbf{z}}_n)'(\boldsymbol{\nu}+\bar{\mathbf{z}}_n)} - \frac{1}{1-c_n} \end{pmatrix}.
$$

Since $\mathbf{S}_n$ and $\bar{\mathbf{z}}_n$ are independent the conditional distribution of $H_1(\bar{\mathbf{z}}_n, \mathbf{S}_n)$ given $\bar{\mathbf{z}}_n = \mathbf{v}$ coincides with $H_1(\mathbf{v}, \mathbf{S}_n)$. Furthermore, the application of Lemma 1 to $\sqrt{n}\mathbf{h}_1(\mathbf{v}, \mathbf{S}_n)$ proves that it is asymptotically normally distributed and, thus, the asymptotic stochastic representation of $H_1(\bar{\mathbf{z}}_n, \mathbf{S}_n)$ is given by

$$
H_1(\bar{\mathbf{z}}_n\mathbf{v}, \mathbf{S}_n) \stackrel{d}{=} \sqrt{\frac{2}{c}}\sqrt{\mathbf{d}_1'\left(\boldsymbol{\Theta}\left(\frac{\boldsymbol{\Sigma}^{1/2}\mathbf{b}}{\sqrt{\mathbf{b}'\boldsymbol{\Sigma}\mathbf{b}}}, \frac{\boldsymbol{\Sigma}^{-1/2}\mathbf{1}_p}{\sqrt{\mathbf{1}_p'\boldsymbol{\Sigma}^{-1}\mathbf{1}_p}}, \frac{(\boldsymbol{\nu}+\bar{\mathbf{z}}_n)}{\sqrt{(\boldsymbol{\nu}+\bar{\mathbf{z}}_n)'(\boldsymbol{\nu}+\bar{\mathbf{z}}_n)}}\right)\circ\boldsymbol{\Lambda}\right)\mathbf{d}_1}\,\omega_1, \qquad (56)
$$

where $\omega_1 \stackrel{d}{\to} \mathcal{N}(0,1)$ and is independent of $\bar{\mathbf{z}}_n$ and hence of $H_2(\bar{\mathbf{z}}_n)$. Finally, we have

that $n\left(\bar{\mathbf{z}}_n + \mathbf{d}_2\right)'\left(\bar{\mathbf{z}}_n + \mathbf{d}_2\right)$ has a non-central χ^2 distribution with p degrees of freedom and noncentrality parameter $n\mathbf{d}_2'\mathbf{d}_2$. The application of Bodnar and Reiß (2016) leads to

$$\sqrt{p}\left(\frac{n\left(\bar{\mathbf{z}}_n + \mathbf{d}_2\right)'\left(\bar{\mathbf{z}}_n + \mathbf{d}_2\right)}{p} - \frac{n\mathbf{d}_2'\mathbf{d}_2}{p} - 1\right) \xrightarrow{d} \mathcal{N}\left(0, 2 + 4\frac{\mathbf{d}_2'\mathbf{d}_2}{c}\right)$$

and, consequently,

$$H_2(\bar{\mathbf{z}}_n) \stackrel{d}{=} \frac{\sqrt{c_n}}{1 - c_n}a_3\sqrt{2 + 4\frac{\mathbf{d}_2'\mathbf{d}_2}{c_n}}\,\omega_2. \tag{57}$$

where $\omega_2 \xrightarrow{d} \mathcal{N}(0,1)$.

Using that $\boldsymbol{\nu}'\bar{\mathbf{z}}_n \xrightarrow{a.s.} 0$ and $\bar{\mathbf{z}}_n'\bar{\mathbf{z}}_n \xrightarrow{a.s.} c$, the application of Slutsky's lemma (c.f., DasGupta (2008, Theorem 1.5)) leads to

$$\sqrt{n}\mathbf{a}'\mathbf{h} \xrightarrow{d} \mathcal{N}(0, \mathbf{a}'\boldsymbol{\Xi}\mathbf{a})$$

for $p/n \to c + o(n^{-1/2})$ as $n \to \infty$ where $\boldsymbol{\Xi}$ is given in (55). Since $\mathbf{a}$ is an arbitrary vector, the statement of Lemma 2 is proved. □

Proof of Theorem 1: It holds that

$$\hat{R}_{GMV} - R_{GMV} = \hat{V}_{GMV}\left(\left(\mathbf{1}_p'\hat{\boldsymbol{\Sigma}}_n^{-1}\bar{\mathbf{x}}_n - \frac{1}{1 - c_n}\mathbf{1}_p'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\right) - R_{GMV}\left(\mathbf{1}_p'\hat{\boldsymbol{\Sigma}}_n^{-1}\mathbf{1}_p - \frac{1}{1 - c_n}\mathbf{1}_p'\boldsymbol{\Sigma}^{-1}\mathbf{1}_p\right)\right),$$

$$\hat{V}_c - V_{GMV} = -V_{GMV}\hat{V}_{GMV}\left(\mathbf{1}_p'\hat{\boldsymbol{\Sigma}}_n^{-1}\mathbf{1}_p - \frac{1}{1 - c_n}\mathbf{1}_p'\boldsymbol{\Sigma}^{-1}\mathbf{1}_p\right)$$

and

$$\begin{aligned}
\hat{s}_c - s_{GMV} &= (1 - c_n)\left(\bar{\mathbf{x}}_n'\hat{\boldsymbol{\Sigma}}_n^{-1}\bar{\mathbf{x}}_n - \frac{1}{1 - c_n}\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - \frac{c_n}{1 - c_n}\right) \\
&\quad - (1 - c_n)\left(\frac{(\mathbf{1}_p'\hat{\boldsymbol{\Sigma}}_n^{-1}\bar{\mathbf{x}}_n)^2}{\mathbf{1}_p'\hat{\boldsymbol{\Sigma}}_n^{-1}\mathbf{1}_p} - \frac{1}{1 - c_n}\frac{(\mathbf{1}_p'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})^2}{\mathbf{1}_p'\boldsymbol{\Sigma}^{-1}\mathbf{1}_p}\right) \\
&= (1 - c_n)\left(\bar{\mathbf{x}}_n'\hat{\boldsymbol{\Sigma}}_n^{-1}\bar{\mathbf{x}}_n - \frac{1}{1 - c_n}\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - \frac{c_n}{1 - c_n}\right) \\
&\quad - (1 - c_n)\hat{V}_{GMV}\left(\left(\mathbf{1}_p'\hat{\boldsymbol{\Sigma}}_n^{-1}\bar{\mathbf{x}}_n + \frac{1}{1 - c_n}\frac{R_{GMV}}{V_{GMV}}\right)\left(\mathbf{1}_p'\hat{\boldsymbol{\Sigma}}_n^{-1}\bar{\mathbf{x}}_n - \frac{1}{1 - c_n}\mathbf{1}_p'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\right)\right. \\
&\quad \left. - \frac{1}{1 - c_n}\frac{R_{GMV}^2}{V_{GMV}}\left(\mathbf{1}_p'\hat{\boldsymbol{\Sigma}}_n^{-1}\mathbf{1}_p - \frac{1}{1 - c_n}\mathbf{1}_p'\boldsymbol{\Sigma}^{-1}\mathbf{1}_p\right)\right).
\end{aligned}$$

Hence,

$$\sqrt{n}\begin{pmatrix} \hat{R}_{GMV} - R_{GMV} \\ \hat{V}_c - V_{GMV} \\ \hat{s}_c - s \\ \hat{R}_b - R_b \\ \hat{V}_b - V_b \end{pmatrix} = \mathbf{D}\sqrt{n}\mathbf{h},$$

with $\mathbf{h}$ is defined in (54) and

$$\mathbf{D} = \begin{pmatrix} (1-c_n)\hat{V}_c & -(1-c_n)\hat{V}_c R_{GMV} & 0 & 0 & 0 \\ 0 & -(1-c_n)\hat{V}_c V_{GMV} & 0 & 0 & 0 \\ (1-c_n)\hat{V}_c \left(\frac{R_{GMV}}{V_{GMV}} - \frac{\hat{R}_{GMV}}{\hat{V}_c} \right) & (1-c_n)\hat{V}_c \frac{R^2_{GMV}}{V_{GMV}} & (1-c_n) & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The application of $\hat{R}_{GMV} \overset{a.s.}{\to} R_{GMV}$ and $\hat{V}_c \overset{a.s.}{\to} V_{GMV}$ for $p/n \to c \in [0,1)$ as $n \to \infty$, the results of Lemma 2, and Slutsky's lemma (c.f., DasGupta (2008, Theorem 1.5)) completes the proof of the theorem. $\qquad \square$