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On the mean and variance of the estimated tangency portfolio weights for small samples

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Abstract

In this paper, we consider the sample estimator of the tangency portfolio (TP) weights, where the inverse of the sample covariance matrix plays an important role. We assume that the number of observations is less than the number of assets in the portfolio, and the returns are independent and identically multivariate normally distributed. Under these assumptions, the sample covariance matrix follows a singular Wishart distribution and, therefore, the regular inverse cannot be taken. This paper delivers bounds and approximations for the first two moments of the estimated TP weights, as well as exact results when the population covariance matrix is equal to the identity matrix, employing the Moore-Penrose inverse. Moreover, exact moments based on the reflexive generalized inverse are provided. The properties of the bounds are investigated in a simulation study, where they are compared to the sample moments. The difference between the moments based on the reflexive generalized inverse and the sample moments based the Moore-Penrose inverse is also studied.

Keywords: Tangency portfolio, Singular inverse Wishart, Moore-Penrose inverse, Reflexive generalized inverse, Estimator moments.

AMS Classification: Primary 62H12, secondary 91G10.

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1 Introduction

How to efficiently allocate capital lies in the heart of financial decision making. Portfolio theory, as developed by Markowitz (1952), provides a framework for this problem, based on the means, variances and covariances of the assets in the considered portfolio. The theory revolves around the trade-off between expected return and variance (risk), denoted by mean-variance optimization. In this setting investors allocate wealth in order to maximize expected return given a certain level of risk, or conversely allocate wealth to minimize the risk given a certain level of expected return.

In this paper, we consider the tangency portfolio (TP) which is one of the most important portfolios in the financial literature. The TP weights determine what proportions of the capital to invest in each asset, and are obtained by maximizing the expected quadratic utility function. For a portfolio of p risky assets, the TP weights are given by

$$\mathbf{w}_{TP} = \alpha^{-1}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_f\mathbf{1}_p), \tag{1}$$

where $\boldsymbol{\mu}$ is a p-dimensional mean vector of the asset returns, $\boldsymbol{\Sigma}$ is a $p \times p$ symmetric positive definite covariance matrix of the asset returns, the coefficient $\alpha > 0$ describes the investors' risk aversion, r_f denotes the rate of a risk-free asset and $\mathbf{1}_p$ is a p-dimensional vector of ones. We allow for short sales and, therefore, some weights can be negative. Let us also note that $\mathbf{w}_{TP}$ determines the structure of the portfolio which corresponds to risky assets and does in general not sum to 1. Consequently, the rest of the wealth $1 - \mathbf{w}'_{TP}\mathbf{1}_p$ needs to be invested into the risk-free asset.

Naturally, the TP weights $\mathbf{w}_{TP}$ depend on knowledge of the mean vector $\boldsymbol{\mu}$ and the covariance matrix $\boldsymbol{\Sigma}$. In general, these quantities are not known and need to be estimated from historical return data. Plugging sample estimates of the mean vector and covariance matrix into (1) leads us to the sample estimate of the TP weights expressed as

$$\hat{\mathbf{w}}_{TP} = \alpha^{-1}\mathbf{S}^{-1}(\bar{\mathbf{x}} - r_f\mathbf{1}_p), \tag{2}$$

where $\mathbf{S}$ is the sample covariance matrix and $\bar{\mathbf{x}}$ is the sample mean vector[1]. The statistical properties of $\hat{\mathbf{w}}_{TP}$ have been extensively studied throughout the literature. Britten-Jones (1999) derived an exact test of the weights in the multivariate normal case. Okhrin and Schmid (2006) obtained the univariate density for the TP weights as well as its asymptotic distribution, under the assumption that returns are independent and identically multivariate normally distributed. Further, Bodnar (2009) provided a procedure of monitoring the TP weights with a sequential approach. Bodnar and Okhrin (2011) obtained the density for, and several exact tests on, linear transformations of estimated TP weights, while Kotsiuba and Mazur (2015) provided approximate and asymptotic distributions for the weights. Bauder et al. (2018) studied the distribution of $\hat{\mathbf{w}}_{TP}$ from a Bayesian perspective[2]. Bodnar et al. (2019c) studied the TP weights in small and large dimensions when both the population and sample covariance matrix are singular. Analytical expressions of higher order moments of the estimated TP weights are derived in Javed et al. (2020), while Karlsson et al. (2020) delivered asymptotic distribution of the estimated TP weights as well as the asymptotic distribution of the statistical test about the elements of the TP under a high-dimensional asymptotic regime. Muhinyuza et al. (2020) derived a test for the location of the TP, while Muhinyuza (2020) extended this result to the high-dimensional setting. Furthermore, Bodnar et al. (2019b) derived central limit theorems for the TP weights estimator under the assumption that the matrix of observations has a matrix-variate location mixture of normal distributions.

The common scenario considered is that the number of observations available for the estimation, denoted by N, is greater than the portfolio size, denoted by p. In this case the sample covariance matrix $\mathbf{S}$ is positive definite, and $\hat{\mathbf{w}}_{TP}$ can be obtained as presented in (2). However, when the considered portfolio is large, it is possible that the number of available observations is less than the portfolio dimension. This can be due to a lack of data for all the assets in the portfolio, but it may also occur due to the fact that covariance of asset returns tend to change

[1]It worth to mention that a similar structure appears in the discriminant analysis. Namely, the coefficients of a discriminant function that maximizes the discrepancy between two datasets are expressed as a product of the inverse sample covariance matrix and the sample mean vector (see, for example, Bodnar and Okhrin (2011), Bodnar et al. (2019a)).

[2]In the Bayesian setting, the posterior distribution of TP weights is expressed as a product of the (singular) Wishart matrix and Gaussian vector. Statistical properties of those products are studied by Bodnar et al. (2013, 2014), Bodnar et al. (2018), Bodnar et al. (2019b).

over time. As such, the assumption of a constant covariance might only hold for limited periods of time, hence limiting the amount of data available for estimation. Any such situations, where $p > N$, would result in a singular sample covariance matrix $\mathbf{S}$, which in turn is non-invertible, in the standard sense.

This issue can be remedied by estimating $\mathbf{\Sigma}^{-1}$ in (1) with the Moore-Penrose inverse of $\mathbf{S}$, which we will denote by $\mathbf{S}^{+}$. Let us also note that the Moore-Penrose inverse is successfully employed in the portfolio theory by Bodnar et al. (2016, 2017), Tsukuma (2016), Bodnar et al. (2019c)[3]. Doing so, the TP weights are estimated as

$$\tilde{\mathbf{w}}_{TP} = \alpha^{-1}\mathbf{S}^{+}(\bar{\mathbf{x}} - r_f \mathbf{1}_p). \tag{3}$$

An attractive feature of applying the Moore-Penrose inverse $\mathbf{S}^{+}$ in (1) is that it is the least square solution to the system of equations described by

$$\mathbf{S}\mathbf{v} = \alpha^{-1}(\bar{\mathbf{x}} - r_f \mathbf{1}_p) \tag{4}$$

which in the singular case generally lack exact solution. That is, as shown in Planitz (1979), for any vector $\mathbf{v} \in \mathbb{R}^p$, we have that $\|\mathbf{S}\mathbf{v} - \alpha^{-1}(\bar{\mathbf{x}} - r_f \mathbf{1}_p)\|_2 \geq \|\mathbf{S}\mathbf{S}^{+}(\bar{\mathbf{x}} - r_f \mathbf{1}_p) - \alpha^{-1}(\bar{\mathbf{x}} - r_f \mathbf{1}_p)\|_2$, where $\|\cdot\|_2$ denotes the Euclidean norm of a vector. Phrased differently, (3) provides the best solution to equation (4), in the least square sense. In addition, when $p \leq N$, we have that $\mathbf{S}^{+} = \mathbf{S}^{-1}$ and $\tilde{\mathbf{w}}_{TP} = \hat{\mathbf{w}}_{TP}$, such that $\tilde{\mathbf{w}}_{TP}$ can be viewed as a general estimator for the TP weights, covering both the singular and non-singular case. For further properties on the Moore-Penrose inverse, see e.g. Boullion and Odell (1971).

The expectation and variance of an estimator are key quantities to describing its statistical properties. With the standard assumption of normally distributed asset returns, the stochastic components of $\tilde{\mathbf{w}}_{TP}$ consists of $\mathbf{S}^{+}$ and $\bar{\mathbf{x}}$, which are independent under the assumption of normally distributed data (see e.g. Bodnar et al. (2016)). Unfortunately, there exist no derivation of the expected value or variance of $\mathbf{S}^{+}$, when $p > N$. In Cook and Forzani (2011) however, these quantities

[3]Instead of using the Moore-Penrose inverse, one can consider regularization techniques such as the ridge-type method (Tikhonov and Arsenin, 1977), the Landweber-Fridman iteration approach (Kress, 1999), a form of Lasso (Brodie et al., 2009), or an iterative algorithm based on a second order damped dynamical systems (Gulliksson and Mazur, 2019).

are presented in the special case of $\mathbf{\Sigma} = \mathbf{I}_p$. The authors also provided approximate results, using moments of standard normal random variables, and exact results for moments of the generalized reflexive inverse, another quantity that can be applied as an inverse of $\mathbf{S}$. Further, in a recent paper, Imori and Rosen (2020) provided several bounds on the mean and variance of $\mathbf{S}^+$, based on the Poincaré separation theorem. Our paper build on the results presented in Cook and Forzani (2011) and Imori and Rosen (2020) in order to provide bounds and approximations for the moments of the TP weights, $\mathbb{E}[\tilde{\mathbf{w}}_{TP}]$ and $\mathbb{V}[\tilde{\mathbf{w}}_{TP}]$, where $\mathbb{E}[\cdot]$ and $\mathbb{V}[\cdot]$ denote the expected value and variance, respectively. A simulation study is also presented, where various measures compare the derived bounds with the equivalent sample quantities obtained from simulated data. It also compares the moments obtained applying the reflexive generalized inverse and the sample moments based on the Moore-Penrose inverse.

The rest of this paper is organized as follows. Section 2.1 provides exact moment results for the case $\mathbf{\Sigma} = \mathbf{I}_p$. Section 2.2 presents bounds for the moments of $\tilde{\mathbf{w}}_{TP}$ in the general case, while approximate moments are derived in Section 2.3. Exact moments applying the reflexive generalized inverse are derived in Section 3. The simulation study is presented in Section 4 while Section 5 summarizes.

2 Moments with the Moore-Penrose inverse

Let $\mathbf{X}$ be a $p \times N$ matrix with N asset return vectors of dimension $p \times 1$ stacked as columns, where $p > N$. Further, we assume that these return vectors are independent and normally distributed with mean vector $\boldsymbol{\mu}$ and positive definite covariance matrix $\mathbf{\Sigma}$. Thus $\mathbf{X} \sim \mathcal{MN}_{p,N}(\boldsymbol{\mu}\mathbf{1}_N, \mathbf{\Sigma}, \mathbf{I}_N)$, where $\mathcal{MN}_{p,n}(\mathbf{M}, \mathbf{\Sigma}, \mathbf{U})$ denotes the matrix-variate normal distribution with $p \times N$ mean matrix $\mathbf{M}$, $p \times p$ row-wise covariance matrix $\mathbf{\Sigma}$ and $N \times N$ column-wise covariance matrix $\mathbf{U}$. Further, let the $p \times 1$ vector $\bar{\mathbf{x}}$ be the row mean of $\mathbf{X}$. Now, define $\mathbf{Y} = \mathbf{X} - \bar{\mathbf{x}}\mathbf{1}'_N$, such that $\mathbf{Y} \sim \mathcal{MN}_{p,N}(\mathbf{0}, \mathbf{\Sigma}, \mathbf{I}_N)$. Further, let $\mathbf{S} = \mathbf{Y}\mathbf{Y}'/n$ such that $\text{rank}(\mathbf{S}) = n < p$ with $n = N - 1$, and $n\mathbf{S} \sim \mathcal{W}_p(n, \mathbf{\Sigma})$, i.e. $n\mathbf{S}$ follows a p-dimensional singular Wishart distribution with n degrees of freedom and the parameter matrix $\mathbf{\Sigma}$. Let $\mathbf{S} = \mathbf{Q}\mathbf{R}\mathbf{Q}'$ denotes the eigenvalue decomposition of $\mathbf{S}$, where $\mathbf{R}$ is the $n \times n$ diagonal matrix of positive eigenvalues and $\mathbf{Q}$ is the $p \times n$ matrix with corresponding eigenvectors as columns. Further, define

$$\mathbf{S}^{+} = \mathbf{Q}\mathbf{R}^{-1}\mathbf{Q}'.$$

As such, $\mathbf{S}^{+}$ constitutes the Moore-Penrose inverse of $\mathbf{Y}\mathbf{Y}'/n$, and $\mathbf{S}^{+}$ is independent of $\bar{\mathbf{x}}$ (see Bodnar et al. (2016)).

In the following, let $\boldsymbol{\eta} = \alpha^{-1}(\bar{\mathbf{x}} - r_f \mathbf{1}_p)$ and $\boldsymbol{\theta} = \mathbb{E}[\boldsymbol{\eta}] = \alpha^{-1}(\boldsymbol{\mu} - r_f \mathbf{1}_p)$. Consequently, from Corollary 3.2b.1 in Mathai and Provost (1992), together with the fact that $\mathbb{E}[\bar{\mathbf{x}}] = \boldsymbol{\mu}$ and $\mathbb{V}[\bar{\mathbf{x}}] = \boldsymbol{\Sigma}/(n+1)$, we obtain that

$$\mathbb{E}[\boldsymbol{\eta}\boldsymbol{\eta}'] = \boldsymbol{\theta}\boldsymbol{\theta}' + \frac{\boldsymbol{\Sigma}}{\alpha^2(n+1)}, \tag{5}$$

$$\mathbb{E}[\boldsymbol{\eta}'\boldsymbol{\eta}] = \boldsymbol{\theta}'\boldsymbol{\theta} + \frac{\mathrm{tr}(\boldsymbol{\Sigma})}{\alpha^2(n+1)}, \tag{6}$$

$$\mathbb{E}[\boldsymbol{\eta}'\boldsymbol{\Sigma}\boldsymbol{\eta}] = \boldsymbol{\theta}'\boldsymbol{\Sigma}\boldsymbol{\theta} + \frac{\mathrm{tr}(\boldsymbol{\Sigma}\boldsymbol{\Sigma})}{\alpha^2(n+1)}. \tag{7}$$

Further, let s^{ij} denotes the element on row i and column j of $\mathbf{S}^{+}$, and let σ^{ij} denotes the element on row i and column j of $\boldsymbol{\Sigma}^{-1}$. Also let $\mathbf{e}_i$ denotes a $p \times 1$ vector where all values are equal to zero, except the i :th element, which is equal to one. Moreover, it is assumed that $\lambda_1(\mathbf{M}) \geq \lambda_2(\mathbf{M}) \geq \cdots \geq \lambda_p(\mathbf{M})$ are the ordered eigenvalues of a symmetric matrix $\mathbf{M}$: $p \times p$, and that $\mathbf{A} \leq_L \mathbf{B}$ denotes the Löwner ordering of two positive semi-definite matrices $\mathbf{A}$ and $\mathbf{B}$.

2.1 Exact moments when $\boldsymbol{\Sigma} = \mathbf{I}_p$

When $\boldsymbol{\Sigma}$ is the identity matrix, it is possible to derive exact moments of the TP weights obtained from the Moore-Penrose inverse in the singular case. First, note the following results presented in Theorem 2.1 of Cook and Forzani (2011), which state that in the case $\boldsymbol{\Sigma} = \mathbf{I}_p$ and $p > n + 3$, we have that

$$\mathbb{E}[\mathbf{S}^{+}] = a_1 \mathbf{I}_p, \tag{8}$$

$$\mathbb{V}[\mathrm{vec}(\mathbf{S}^{+})] = a_2(\mathbf{I}_{p^2} + \mathbf{C}_{p^2}) + 2a_3 \mathrm{vec}(\mathbf{I}_p)\mathrm{vec}'(\mathbf{I}_p), \tag{9}$$

where $\mathbf{C}_{p^2}$ is the commutation matrix and

$$\begin{aligned} a_1 &= \frac{n^2}{p(p-n-1)}, && (10)\\ a_2 &= \frac{n^3[p(p-1)-n(p-n-2)-2]}{p(p-1)(p+2)(p-n)(p-n-1)(p-n-3)}, && (11)\\ a_3 &= \frac{n^3[n^2(n-1)+2n(p-2)(p-n)+2p(p-1)]}{p^2(p-1)(p+2)(p-n)(p-n-1)^2(p-n-3)}. && (12)\end{aligned}$$

Note that constants in (10)-(12) differ slightly to the constants presented in Cook and Forzani (2011), since our paper considers results for $n\mathbf{S} \sim \mathcal{W}_p(n, \mathbf{\Sigma})$, while Cook and Forzani (2011) derived the results for $\mathbf{W} \sim \mathcal{W}_p(n, \mathbf{\Sigma})$. The moments in (8) and (9) allow us to derive the following results.

Theorem 1. *If $p > n+3$ and $\mathbf{\Sigma} = \mathbf{I}_p$, then*

$$\begin{aligned} \mathbb{E}[\tilde{\mathbf{w}}_{TP}] &= a_1 \mathbf{w}_{TP}, \\ \mathbb{V}[\tilde{\mathbf{w}}_{TP}] &= (a_2 + 2a_3)\mathbf{w}_{TP}\mathbf{w}_{TP}' + \left[a_2 \mathbf{w}_{TP}'\mathbf{w}_{TP} + \frac{a_1^2 + (p+1)a_2 + 2a_3}{\alpha^2(n+1)}\right]\mathbf{I}_p \end{aligned}$$

with constants a_1, a_2 and a_3 that are defined in (10)-(12).

Proof. Since $\tilde{\mathbf{w}}_{TP} = \alpha^{-1}\mathbf{S}^+(\bar{\mathbf{x}} - r_f \mathbf{1}_p)$ the first result follows directly from (8) and the independence of $\mathbf{S}^+$ and $\bar{\mathbf{x}}$. For the second result, first note that as discussed in Cook and Forzani (2011), equation (9) can be written as

$$\mathrm{Cov}(s^{ij}, s^{kl}) = a_2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + 2a_3\delta_{ij}\delta_{kl},$$

where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise, such that δ_{ij}, $i, j = 1\ldots, p$ denotes the elements of $\mathbf{I}_p$. Hence, we have that

$$\mathbb{E}[s^{ij}s^{kl}] = a_2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + (a_1^2 + 2a_3)\delta_{ij}\delta_{kl}. \quad (13)$$

Also note the following element representations of matrix operations, where $\mathbf{A}$ and

$\mathbf{B}$ are $p \times p$ symmetric matrices:

$$[\mathbf{A}\text{tr}(\mathbf{BA})]_{ij} = a_{ij} \sum_{k=1}^{p} \sum_{l=1}^{p} a_{kl} b_{kl}, \tag{14}$$

$$\begin{aligned} [\mathbf{ABA}]_{ij} &= \sum_{k=1}^{p} \sum_{l=1}^{p} b_{kl} a_{ik} a_{jl} \\ &= \sum_{k=1}^{p} \sum_{l=1}^{p} b_{kl} a_{il} a_{jk}. \end{aligned} \tag{15}$$

Moreover, with $\boldsymbol{\eta} = \alpha^{-1}(\bar{\mathbf{x}} - r_f \mathbf{1}_p)$ and $\mathbb{E}[\boldsymbol{\eta}] = \boldsymbol{\theta}$,

$$\mathbb{V}[\tilde{\mathbf{w}}_{TP}] = \mathbb{V}[\mathbf{S}^+ \boldsymbol{\eta}] = \mathbb{E}\left[\mathbb{E}[\mathbf{S}^+ \boldsymbol{\eta}\boldsymbol{\eta}' \mathbf{S}^+ \mid \boldsymbol{\eta}]\right] - \mathbb{E}[\mathbf{S}^+]\boldsymbol{\theta}\boldsymbol{\theta}' \,\mathbb{E}[\mathbf{S}^+]. \tag{16}$$

By letting $\mathbf{H} = \boldsymbol{\eta}\boldsymbol{\eta}'$ and applying equations (13)-(15) we obtain

$$\begin{aligned} \mathbb{E}[\mathbf{S}^+ \mathbf{H} \mathbf{S}^+ \mid \boldsymbol{\eta}]_{ij} &= \sum_{k=1}^{p} \sum_{l=1}^{p} h_{kl} \,\mathbb{E}[s^{ik} s^{jl}] \\ &= \sum_{k=1}^{p} \sum_{l=1}^{p} h_{kl} [a_2(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{kj}) + (a_1^2 + 2a_3)\delta_{ik}\delta_{jl}] \\ &= a_2 \left[\mathbf{I}_p \text{tr}(\mathbf{H}\mathbf{I}_p)\right]_{ij} + a_2 [\mathbf{I}_p \mathbf{H} \mathbf{I}_p]_{ij} + (a_1^2 + 2a_3)[\mathbf{I}_p \mathbf{H} \mathbf{I}_p]_{ij}. \end{aligned}$$

Consequently,

$$\mathbb{E}[\mathbf{S}^+ \mathbf{H} \mathbf{S}^+ \mid \boldsymbol{\eta}] = (a_1^2 + a_2 + 2a_3)\mathbf{H} + a_2 \text{tr}(\mathbf{H})\mathbf{I}_p,$$

and inserting the above result into (16) together with (5) and (8) gives

$$\begin{aligned} \mathbb{V}[\mathbf{S}^+ \boldsymbol{\eta}] &= (a_1^2 + a_2 + 2a_3)\left(\boldsymbol{\theta}\boldsymbol{\theta}' + \alpha^{-2} N^{-1} \mathbf{I}_p\right) + \\ &\quad + a_2 \left(\text{tr}(\boldsymbol{\theta}\boldsymbol{\theta}') + \alpha^{-2} N^{-1} p\right) - a_1^2 \boldsymbol{\theta}\boldsymbol{\theta}' \end{aligned}$$

and the theorem follows noting that $\boldsymbol{\theta} = \mathbf{w}_{TP}$ when $\boldsymbol{\Sigma} = \mathbf{I}_p$. □

A direct consequence of Theorem 1 is that the estimator $\tilde{\mathbf{w}}_{TP}$ is biased, with bias factor a_1. Hence, in the case of $\boldsymbol{\Sigma} = \mathbf{I}_p$, we have that $a_1^{-1}\tilde{\mathbf{w}}_{TP}$ constitutes an unbiased estimator. Further, in accordance with Corollary 2.1 in Cook and Forzani (2011), as $n, p \to \infty$, assuming $n/p \to r$, with $0 < r < 1$, the constants of $\mathbb{V}[\tilde{\mathbf{w}}_{TP}]$ emits the following asymptotic magnitudes: $a_1 = \mathcal{O}(1)$, $a_2 = \mathcal{O}(n^{-1}) = \mathcal{O}(p^{-1})$ and $a_3 = \mathcal{O}(n^{-2}) = \mathcal{O}(p^{-2})$. Consequently, since $\text{tr}(\mathbf{w}_{TP}\mathbf{w}_{TP}') = \mathcal{O}(p)$ in the general case, we have that $a_2 \text{tr}(\mathbf{w}_{TP}\mathbf{w}'_{TP}) = \mathcal{O}(1)$. Hence, unless $\mathbf{w}_{TP}$ has some specific structure, $\mathbb{V}[\tilde{\mathbf{w}}_{TP}]$ does not vanish to zero under this asymptotic regime. This is not unique for the singular case, since the corresponding is also true for $\hat{\mathbf{w}}_{TP}$ in the non-singular case, when $n, p \to \infty$.

2.2 Bounds on the moments

This section aims to provide upper and lower bounds for the expected value of $\tilde{\mathbf{w}}_{TP}$ and upper bounds for the variance of $\tilde{\mathbf{w}}_{TP}$. First, define the following $p \times p$ matrices,

$$\begin{aligned}
\mathbf{D} &= a_1(\lambda_p(\mathbf{\Sigma}^{-1}))^2\mathbf{\Sigma}, \\
\mathbf{U}_a &= a_1(\lambda_1(\mathbf{\Sigma}^{-1}))^2\mathbf{\Sigma}, \\
\mathbf{U}_b &= \frac{n}{p-n-1}\lambda_1(\mathbf{\Sigma}^{-1})\mathbf{I}_p,
\end{aligned}$$

with elements d_{ij}, $u_{ij}^{(a)}$ and $u_{ij}^{(b)}$, respectively. Further denote e_{ij} the elements of $\mathbb{E}[\mathbf{S}^+]$ and let $u_{ii}^{(*)} = \min\{u_{ii}^{(a)}, u_{ii}^{(b)}\}$, $i = 1, \ldots, p$. Then we can derive the following result.

Theorem 2. *Suppose $p > n+3$ and $\mathbf{\Sigma} > 0$. Let w_i and θ_i, be the i:th element of the $p \times 1$ vectors $\mathbf{w} = \mathbb{E}[\tilde{\mathbf{w}}_{TP}]$ and $\boldsymbol{\theta} = \alpha^{-1}(\boldsymbol{\mu} - r_f\mathbf{1}_p)$, respectively. Then for $i = 1, \ldots, p$, it holds that*

$$v_{ii}\theta_i + \sum_{j \neq i}^{p} v_{ij}\theta_j \leq w_i \leq z_{ii}\theta_i + \sum_{j \neq i}^{p} z_{ij}\theta_j$$

where, for $i, j = 1, \ldots, p$,

$$\begin{aligned}
v_{ij} &= \begin{cases} g_{ij} & \text{if } \theta_j \geq 0, \\ h_{ij} & \text{if } \theta_j < 0, \end{cases} \\
z_{ij} &= \begin{cases} g_{ij} & \text{if } \theta_j < 0, \\ h_{ij} & \text{if } \theta_j \geq 0, \end{cases}
\end{aligned}$$

with $g_{ii} = d_{ii}$, $h_{ii} = u_{ii}^{(*)}$, while for $i \neq j$,

$$
\begin{aligned}
g_{ij} &= \max \left\{ \begin{array}{l} d_{ij} - \sqrt{(u_{ii}^{(*)} - d_{ii})(u_{jj}^{(*)} - d_{jj})}, \\ u_{ij}^{(a)} - \sqrt{(u_{ii}^{(a)} - d_{ii})(u_{jj}^{(a)} - d_{jj})}, \\ -\sqrt{(u_{ii}^{(b)} - d_{ii})(u_{jj}^{(b)} - d_{jj})}, \\ -\sqrt{u_{ii}^{(*)} u_{jj}^{(*)}} \end{array} \right\}, \\
h_{ij} &= \min \left\{ \begin{array}{l} d_{ij} + \sqrt{(u_{ii}^{(*)} - d_{ii})(u_{jj}^{(*)} - d_{jj})}, \\ u_{ij}^{(a)} + \sqrt{(u_{ii}^{(a)} - d_{ii})(u_{jj}^{(a)} - d_{jj})}, \\ \sqrt{(u_{ii}^{(b)} - d_{ii})(u_{jj}^{(b)} - d_{jj})}, \\ \sqrt{u_{ii}^{(*)} u_{jj}^{(*)}} \end{array} \right\}.
\end{aligned}
$$

Proof. The result follows directly from the element wise bounds in Lemma A2 and that due to the independence of $\mathbf{S}^+$ and $\bar{\mathbf{x}}$ we have $\mathbb{E}[\tilde{\mathbf{w}}_{TP}] = \mathbb{E}[\mathbf{S}^+]\boldsymbol{\theta}$. □

Note that when $\boldsymbol{\Sigma} = \mathbf{I}_p$, we have that $\lambda_1(\boldsymbol{\Sigma}^{-1})^2 = \lambda_p(\boldsymbol{\Sigma}^{-1})^2 = 1$, and hence $\mathbb{E}[\mathbf{S}^+] = \mathbf{D} = \mathbf{U}_a = a_1\mathbf{I}_p$. Consequently $g_{ij} = h_{ij} = 0$, $i \neq j$, and $g_{ii} = h_{ii} = a_1$, $i = 1, \ldots, p$, since $u_{ii}^{(a)} = a_1 < a_1 \frac{p}{n} = u_{ii}^{(b)}$, and $p > n$. Hence, Theorem 2 yields that $\mathbb{E}[\tilde{\mathbf{w}}_{TP}] = a_1\boldsymbol{\theta}$, consistent with the result of Theorem 1.

The following result provides two upper bounds for the variance of the TP weights estimate $\tilde{\mathbf{w}}_{TP}$.

Theorem 3. *Suppose $p > n + 3$ and $\boldsymbol{\Sigma} > 0$. Then*

$$
\mathbb{V}[\tilde{\mathbf{w}}_{TP}] \leq_L (2c_1 + c_2)(\lambda_1(\boldsymbol{\Sigma}^{-1}))^4 \left(k_1 \boldsymbol{\Sigma}\, \mathbb{E}[\boldsymbol{\eta}\boldsymbol{\eta}']\boldsymbol{\Sigma} + k_2 \boldsymbol{\Sigma}\, \mathbb{E}[\boldsymbol{\eta}'\boldsymbol{\Sigma}\boldsymbol{\eta}]\right), \quad (17)
$$

$$
\mathbb{V}[\tilde{\mathbf{w}}_{TP}] \leq_L (2c_1 + c_2)(\lambda_1(\boldsymbol{\Sigma}^{-1}))^4\, \mathbb{E}[(\boldsymbol{\eta}'\boldsymbol{\eta})]\mathbf{I}_p, \quad (18)
$$

with the expected values given in (5)-(7) and

$$
\begin{aligned}
c_1 &= n^2[(p-n)(p-n-1)(p-n-3)]^{-1}, \\
c_2 &= (p-n-2)c_1, \\
k_1 &= \left[1 + n - \frac{(p+1)(p(n+1)-2)}{p(p+1)-2}\right]\frac{n}{p}, \\
k_2 &= \left[1 - \frac{(p+1)(p-n)}{p(p+1)-2}\right]\frac{n}{p}.
\end{aligned}
$$

Proof. We are interested in bounds for the quantity $\boldsymbol{\alpha}'\,\mathbb{V}[\tilde{\mathbf{w}}_{TP}]\boldsymbol{\alpha} = \boldsymbol{\alpha}'\,\mathbb{V}[\mathbf{S}^+\boldsymbol{\eta}]\boldsymbol{\alpha}$, for all $\boldsymbol{\alpha} \in \mathbb{R}^p$. First, by the tower property we have

$$\mathbb{V}[\mathbf{S}^+\boldsymbol{\eta}] = \mathbb{E}\left[\mathbb{E}[\mathbf{S}^+\boldsymbol{\eta}\boldsymbol{\eta}'\mathbf{S}^+ \mid \boldsymbol{\eta}]\right] - \mathbb{E}[\mathbf{S}^+]\boldsymbol{\theta}\boldsymbol{\theta}'\,\mathbb{E}[\mathbf{S}^+].$$

Hence, we can obtain

$$\begin{aligned}
\boldsymbol{\alpha}'\,\mathbb{V}[\mathbf{S}^+\boldsymbol{\eta}]\boldsymbol{\alpha} &= \mathbb{E}\left[\mathbb{E}[\boldsymbol{\alpha}'\mathbf{S}^+\boldsymbol{\eta}\boldsymbol{\eta}'\mathbf{S}^+\boldsymbol{\alpha} \mid \boldsymbol{\eta}]\right] - \boldsymbol{\alpha}'\,\mathbb{E}[\mathbf{S}^+]\boldsymbol{\theta}\boldsymbol{\theta}'\,\mathbb{E}[\mathbf{S}^+]\boldsymbol{\alpha}' \\
&= \mathbb{E}\left[\mathbb{E}[(\boldsymbol{\alpha}'\mathbf{S}^+\boldsymbol{\eta})^2 \mid \boldsymbol{\eta}]\right] - (\boldsymbol{\alpha}'\,\mathbb{E}[\mathbf{S}^+]\boldsymbol{\theta})^2.
\end{aligned}$$

Then, by noting that $(\boldsymbol{\alpha}'\,\mathbb{E}[\mathbf{S}^+]\boldsymbol{\theta})^2 > 0$ and applying the bounds from Lemma A4 on $\mathbb{E}[(\boldsymbol{\alpha}'\mathbf{S}^+\boldsymbol{\eta})^2]$ we can derive

$$\begin{aligned}
\boldsymbol{\alpha}'\,\mathbb{V}[\mathbf{S}^+\boldsymbol{\eta}]\boldsymbol{\alpha} &\leq (2c_1 + c_2)(\lambda_1(\boldsymbol{\Sigma}^{-1}))^4 \\
&\quad \times \mathbb{E}\left[k_1(\boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\eta})^2 + k_2(\boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha})(\boldsymbol{\eta}'\boldsymbol{\Sigma}\boldsymbol{\eta})\right], \\
\boldsymbol{\alpha}'\,\mathbb{V}[\mathbf{S}^+\boldsymbol{\eta}]\boldsymbol{\alpha} &\leq (\lambda_1(\boldsymbol{\Sigma}^{-1}))^4(2c_1 + c_2)\,\mathbb{E}[(\boldsymbol{\alpha}'\boldsymbol{\alpha})(\boldsymbol{\eta}'\boldsymbol{\eta})],
\end{aligned}$$

and with the aid of (5)-(7) the result follows. □

2.3 Approximate moments

Regarding general $\boldsymbol{\Sigma}$, it is possible to provide approximate moments for $\tilde{\mathbf{w}}_{TP}$ using simulations of standard normal matrices. Following Section 3.1 in Cook and Forzani (2011), we denote the eigendecomposition of $\boldsymbol{\Sigma}$ as $\boldsymbol{\Sigma} = \boldsymbol{\Gamma}\boldsymbol{\Lambda}\boldsymbol{\Gamma}'$, with λ_i denoting the i:th diagonal element of $\boldsymbol{\Lambda}$, and let $\mathbf{Z} \sim \mathcal{MN}_{p,n}(\mathbf{0}, \mathbf{I}_p, \mathbf{I}_n)$, with $\mathbf{z}_i'$ denoting row i of $\mathbf{Z}$. Further denote $m_{ij}(\boldsymbol{\Lambda}) = \mathbb{E}[\mathbf{z}_i'(\mathbf{Z}'\boldsymbol{\Lambda}\mathbf{Z})^{-2}\mathbf{z}_j]$ and $v_{ij,kl}(\boldsymbol{\Lambda}) = \mathrm{Cov}[\mathbf{z}_i'(\mathbf{Z}'\boldsymbol{\Lambda}\mathbf{Z})^{-2}\mathbf{z}_j, \mathbf{z}_k'(\mathbf{Z}'\boldsymbol{\Lambda}\mathbf{Z})^{-2}\mathbf{z}_l]$, where $\mathrm{Cov}[X,Y]$ denote the covariance between X and Y.

Also define

$$\begin{aligned}
\mathbf{M}(\boldsymbol{\Lambda}) &= n\sum_{i=1}^{p}\lambda_i m_{ii}(\boldsymbol{\Lambda})\mathbf{e}_i\mathbf{e}_i',\\
\mathbf{V}(\boldsymbol{\Lambda}) &= n^2\left[\sum_{i=1}^{p}\sum_{j=1}^{p}\lambda_i\lambda_j v_{ii,jj}(\boldsymbol{\Lambda})(\mathbf{e}_i\mathbf{e}_j'\otimes\mathbf{e}_i\mathbf{e}_j')\right.\\
&\quad+\sum_{i=1}^{p}\sum_{j=1}^{p}\lambda_i\lambda_j v_{ij,ij}(\boldsymbol{\Lambda})(\mathbf{e}_j\mathbf{e}_j'\otimes\mathbf{e}_i\mathbf{e}_i')(\mathbf{I}_p+\mathbf{C}_{p^2})\\
&\quad\left.-2\sum_{i}^{p}\lambda_i^2 v_{ii,ii}(\boldsymbol{\Lambda})(\mathbf{e}_i\mathbf{e}_i'\otimes\mathbf{e}_i\mathbf{e}_i')\right],
\end{aligned}\tag{19}$$

and make the following decomposition

$$(\boldsymbol{\Gamma}\otimes\boldsymbol{\Gamma})\mathbf{V}(\boldsymbol{\Lambda})(\boldsymbol{\Gamma}'\otimes\boldsymbol{\Gamma}') = \begin{pmatrix}\boldsymbol{\Psi}_{11} & \cdots & \boldsymbol{\Psi}_{1p}\\ \vdots & \ddots & \vdots\\ \boldsymbol{\Psi}_{p1} & \cdots & \boldsymbol{\Psi}_{pp}\end{pmatrix},\tag{20}$$

where $\boldsymbol{\Psi}_{ij}$ are $p\times p$ matrices, $i,j=1,\ldots,p$. The following result can then be derived.

Theorem 4. *If $p > n+3$ and $\boldsymbol{\Sigma} > 0$ then*

$$\begin{aligned}
\mathbb{E}[\tilde{\mathbf{w}}_{TP}] &= \boldsymbol{\Gamma}\mathbf{M}(\boldsymbol{\Lambda})\boldsymbol{\Gamma}'\boldsymbol{\theta},\\
\mathbb{V}[\tilde{\mathbf{w}}_{TP}] &= \sum_{i=1}^{p}\sum_{j=1}^{p}\left(\theta_i\theta_j+\frac{\sigma_{ij}}{\alpha^2(n+1)}\right)\boldsymbol{\Psi}_{ij}+\frac{1}{\alpha^2(n+1)}\boldsymbol{\Gamma}\mathbf{M}(\boldsymbol{\Lambda})\boldsymbol{\Lambda}\mathbf{M}(\boldsymbol{\Lambda})\boldsymbol{\Gamma}'
\end{aligned}$$

with $\theta_i = \alpha^{-1}(\mu_i - r_f)$.

Proof. From Theorem 3.1 in Cook and Forzani (2011), we have that $\mathbb{E}[\mathbf{S}^+] = \boldsymbol{\Gamma}\mathbf{M}(\boldsymbol{\Lambda})\boldsymbol{\Gamma}'$. Then the first result follows due to the independence of $\mathbf{S}^+$ and $\bar{\mathbf{x}}$. For the second result, we have that

$$\mathbb{V}[\mathbf{S}^+\boldsymbol{\eta}] = \mathbb{E}\left[\mathbb{E}[\mathbf{S}^+\boldsymbol{\eta}\boldsymbol{\eta}'\mathbf{S}^+\mid\boldsymbol{\eta}]\right]-\mathbb{E}[\mathbf{S}^+]\boldsymbol{\theta}\boldsymbol{\theta}'\,\mathbb{E}[\mathbf{S}^+].\tag{21}$$

Again we let $\mathbf{H}=\boldsymbol{\eta}\boldsymbol{\eta}'$. Applying Theorem 3.1 in Cook and Forzani (2011) we have that

$$\mathbb{V}[\mathrm{vec}(\mathbf{S}^+)] = (\boldsymbol{\Gamma}\otimes\boldsymbol{\Gamma})\mathbf{V}(\boldsymbol{\Lambda})(\boldsymbol{\Gamma}'\otimes\boldsymbol{\Gamma}'),$$

and in accordance with equation (6.8) in Ghazal and Neudecker (2000), we get

$$\mathbb{E}[\mathbf{S}^{+}\mathbf{H}\mathbf{S}^{+}] = \sum_{i=1}^{p}\sum_{j=1}^{p} h_{ij}\boldsymbol{\Psi}_{ij} + \mathbb{E}[\mathbf{S}^{+}]\mathbf{H}\,\mathbb{E}[\mathbf{S}^{+}].$$

where $\boldsymbol{\Psi}_{ij}$ is obtained from the decomposition (20). Inserting the above into (21) gives

$$\begin{aligned}\mathbb{V}[\mathbf{S}^{+}\boldsymbol{\eta}] &= \sum_{i=1}^{p}\sum_{j=1}^{p}\mathbb{E}[h_{ij}]\boldsymbol{\Psi}_{ij} + \mathbb{E}[\mathbf{S}^{+}]\,\mathbb{E}[\mathbf{H}]\,\mathbb{E}[\mathbf{S}^{+}] - \mathbb{E}[\mathbf{S}^{+}]\boldsymbol{\theta}\boldsymbol{\theta}'\,\mathbb{E}[\mathbf{S}^{+}] \\ &= \sum_{i=1}^{p}\sum_{j=1}^{p}\left(\theta_i\theta_j + \frac{\sigma_{ij}}{\alpha^2 N}\right)\boldsymbol{\Psi}_{ij} + \frac{1}{\alpha^2 N}\boldsymbol{\Gamma}\mathbf{M}(\boldsymbol{\Lambda})\boldsymbol{\Lambda}\mathbf{M}(\boldsymbol{\Lambda})\boldsymbol{\Gamma}',\end{aligned}$$

due to (5) and since $\boldsymbol{\Gamma}'\boldsymbol{\Sigma}\boldsymbol{\Gamma} = \boldsymbol{\Lambda}$. The theorem is proved. □

In Cook and Forzani (2011) the authors note that the moments $m_{ij}(\boldsymbol{\Lambda})$ and $v_{ij,kl}(\boldsymbol{\Lambda})$ does not seem to have tractable closed-form representations. However, these quantities can be approximated by simulation of $\mathbf{Z}$, given the eigenvalues of $\boldsymbol{\Sigma}$.

3 Exact moments with reflexive generalized inverse

An alternative to using the Moore-Penrose inverse $\mathbf{S}^{+}$ to estimate $\boldsymbol{\Sigma}^{-1}$ is to apply the reflexive generalized inverse, defined as

$$\mathbf{S}^{\dagger} = \boldsymbol{\Sigma}^{-1/2}\left(\boldsymbol{\Sigma}^{-1/2}\mathbf{S}\boldsymbol{\Sigma}^{-1/2}\right)^{+}\boldsymbol{\Sigma}^{-1/2},$$

with the elements of $\mathbf{S}^{\dagger}$ are denoted $s_{ij}^{\dagger}$. As such, the TP weights vector can be estimated by

$$\mathbf{w}_{TP}^{\dagger} = \mathbf{S}^{\dagger}\boldsymbol{\eta},$$

as to which we derive the following result.

Theorem 5. *If $p > n+3$ and $\mathbf{\Sigma} > 0$ then*

$$\begin{aligned}\mathbb{E}[\mathbf{w}_{TP}^{\dagger}] &= a_1 \mathbf{w}_{TP}, \\ \mathbb{V}[\mathbf{w}_{TP}^{\dagger}] &= (a_2 + 2a_3)\mathbf{w}_{TP}\mathbf{w}_{TP}' \\ &\quad + \left[a_2 \mathbf{w}_{TP}' \mathbf{\Sigma} \mathbf{w}_{TP} + \frac{a_1^2 + (p+1)a_2 + 2a_3}{\alpha^2(n+1)} \right] \mathbf{\Sigma}^{-1}.\end{aligned}$$

Proof. The first result follows directly from Corollary 2.3 in Cook and Forzani (2011), and the independence of $\mathbf{S}$ and $\bar{\mathbf{x}}$. For the second result, we have that

$$\mathbb{V}[\mathbf{S}^{\dagger}\boldsymbol{\eta}] = \mathbb{E}\left[\mathbb{E}[\mathbf{S}^{\dagger}\boldsymbol{\eta}\boldsymbol{\eta}'\mathbf{S}^{\dagger} \mid \boldsymbol{\eta}] \right] - \mathbb{E}[\mathbf{S}^{\dagger}]\boldsymbol{\theta}\boldsymbol{\theta}'\,\mathbb{E}[\mathbf{S}^{\dagger}]. \tag{22}$$

Again we let $\mathbf{H} = \boldsymbol{\eta}\boldsymbol{\eta}'$, and note that by Corollary 2.3 in Cook and Forzani (2011) we have

$$\mathbb{E}[s_{ij}^{\dagger} s_{kl}^{\dagger}] = a_2(\sigma^{ik}\sigma^{jl} + \sigma^{il}\sigma^{jk}) + (a_1^2 + 2a_3)\sigma^{ij}\sigma^{kl}$$

which combined with (14)-(15) allows us to obtain

$$\begin{aligned}\mathbb{E}\left[\mathbf{S}^{\dagger}\mathbf{H}\mathbf{S}^{\dagger} \mid \boldsymbol{\eta} \right]_{ij} &= \sum_{k=1}^{p}\sum_{l=1}^{p} h_{kl}\,\mathbb{E}\left[s_{ik}^{\dagger} s_{lj}^{\dagger} \right] \\ &= \sum_{k=1}^{p}\sum_{l=1}^{p} h_{kl} \Big(a_2(\sigma^{ij}\sigma^{kl} + \sigma^{il}\sigma^{kj}) \\ &\quad + (a_1^2 + 2a_3)\sigma^{ik}\sigma^{jl} \Big) \\ &= (a_1^2 + a_2 + 2a_3)\left[\mathbf{\Sigma}^{-1}\mathbf{H}\mathbf{\Sigma}^{-1} \right]_{ij} + a_2 \mathrm{tr}(\mathbf{H}\mathbf{\Sigma}^{-1}) \left[\mathbf{\Sigma}^{-1} \right]_{ij}\end{aligned}$$

such that

$$\mathbb{E}[\mathbf{S}^{\dagger}\mathbf{H}\mathbf{S}^{\dagger} \mid \boldsymbol{\eta}] = (a_1^2 + a_2 + 2a_3)\mathbf{\Sigma}^{-1}\mathbf{H}\mathbf{\Sigma}^{-1} + a_2 \mathrm{tr}(\mathbf{H}\mathbf{\Sigma}^{-1})\mathbf{\Sigma}^{-1}.$$

Inserting this into equation (22) gives

$$\begin{aligned}\mathbb{V}[\mathbf{S}^{\dagger}\boldsymbol{\eta}] &= (a_1^2 + a_2 + 2a_3)\mathbf{\Sigma}^{-1}\,\mathbb{E}[\boldsymbol{\eta}\boldsymbol{\eta}']\mathbf{\Sigma}^{-1} + a_2 \mathrm{tr}(\mathbb{E}[\boldsymbol{\eta}\boldsymbol{\eta}']\mathbf{\Sigma}^{-1})\mathbf{\Sigma}^{-1} \\ &\quad - \mathbb{E}[\mathbf{S}^{\dagger}]\boldsymbol{\theta}\boldsymbol{\theta}'\,\mathbb{E}[\mathbf{S}^{\dagger}],\end{aligned}$$

and applying the first result on $\mathbb{E}[\mathbf{S}^{\dagger}]$ together with (5) concludes the proof. □

An obvious drawback of $\mathbf{w}_{TP}^{\dagger}$ is that $\mathbf{\Sigma}$ must be known in order to construct $\mathbf{S}^{\dagger}$. Moreover, in the case of $\mathbf{\Sigma} = \mathbf{I}_p$ the results in Theorem 5 coincides with the results in Theorem 1, since in this case $\mathbf{S}^{\dagger} = \mathbf{S}^{+}$.

4 Simulation study

The aim of this section is to compare the bounds on the moments of $\tilde{\mathbf{w}}_{TP}$ derived in Section 2.2 with the sample mean and sample variance of this estimator. It will also investigate the difference between the moments of $\mathbf{w}_{TP}^{\dagger}$ derived in Theorem 5 and the sample moments of $\tilde{\mathbf{w}}_{TP}$. Ideally, the bounds should not deviate from the obtained sample moments very much. To this end, define $\mathbf{b}^l$ and $\mathbf{b}^u$ as the $p \times 1$ vectors with elements

$$\begin{aligned} b_i^l &= v_{ii}\mu_i + \sum_{j \neq i}^{p} v_{ij}\mu_j, \\ b_i^u &= z_{ii}\mu_i + \sum_{j \neq i}^{p} z_{ij}\mu_j, \end{aligned}$$

such that $\mathbf{b}^l$ and $\mathbf{b}^u$ represents the element wise lower and upper bound for the expected TP weights vector presented in Theorem 2, where we set $\alpha = 1$ and $r_f = 0$. Let

$$\begin{aligned} \mathbf{B}_1 &= (2c_1 + c_2)(\lambda_1(\mathbf{\Sigma}^{-1}))^4 \left(k_1 \mathbf{\Sigma}\, \mathbb{E}[\boldsymbol{\eta}\boldsymbol{\eta}']\mathbf{\Sigma} + k_2 \mathbf{\Sigma}\, \mathbb{E}[\boldsymbol{\eta}'\mathbf{\Sigma}\boldsymbol{\eta}]\right), \\ \mathbf{B}_2 &= (2c_1 + c_2)(\lambda_1(\mathbf{\Sigma}^{-1}))^4\, \mathbb{E}[(\boldsymbol{\eta}'\boldsymbol{\eta})]\mathbf{I}_p, \end{aligned}$$

such that $\mathbf{B}_1$ and $\mathbf{B}_2$ represents the bounds in equation (17) and (18) in Theorem 3, respectively. Further, let $\mathbf{m}$ and $\mathbf{V}$ respectively denote the sample mean vector and sample covariance matrix of $\tilde{\mathbf{w}}_{TP}$ based on an observed matrix $\mathbf{X}$, as described in Section 2. Moreover, define

$$t_l = \frac{\mathbf{1}_p'|\mathbf{b}^l - \mathbf{m}|}{p} \tag{23}$$

$$t_u = \frac{\mathbf{1}_p'|\mathbf{b}^u - \mathbf{m}|}{p} \tag{24}$$

$$t^{\dagger} = \frac{\mathbf{1}_p'|\,\mathbb{E}[\mathbf{w}_{TP}^{\dagger}] - \mathbf{m}|}{p} \tag{25}$$

such that t_l, t_u and $t^\dagger$ measures the element wise difference between the sample mean vector and the lower and upper bounds on the mean, and mean of $\mathbf{w}_{TP}^\dagger$, respectively. Dividing by p allows comparing the measures between various portfolio sizes. Further, let

$$T_1 = \frac{\left|\mathbf{1}_p' \left(\mathbf{B}_1 - \mathbf{V}\right) \mathbf{1}_p\right|}{p^2} \tag{26}$$

$$T_2 = \frac{\left|\mathbf{1}_p' \left(\mathbf{B}_2 - \mathbf{V}\right) \mathbf{1}_p\right|}{p^2} \tag{27}$$

$$T^\dagger = \frac{\left|\mathbf{1}_p' \left(\mathbb{V}[\mathbf{w}_{TP}^\dagger] - \mathbf{V}\right) \mathbf{1}_p\right|}{p^2} \tag{28}$$

where $\mathbf{1}_p$ is a $p \times 1$ vector of ones. As such, T_1 and T_2 provide a measure of discrepancy between the sample covariance matrix and bounds presented in Theorem 3, while $T^\dagger$ measures the discrepancy between the variance of $\mathbf{w}_{TP}^\dagger$ presented in Theorem 5 and the sample covariance matrix of $\tilde{\mathbf{w}}_{TP}$. Since it is divided by p^2, the number of elements in $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbb{V}[\mathbf{w}_{TP}^\dagger]$ and $\mathbf{V}$, the measures T_1, $T^\dagger$ and T_2 again allow for comparison between different portfolio dimensions. Moreover, define

$$f_l = \|\mathbf{b}^l - \mathbf{m}\|_F^2 / \|\mathbf{m}\|_F^2 \tag{29}$$

$$f_u = \|\mathbf{b}^u - \mathbf{m}\|_F^2 / \|\mathbf{m}\|_F^2 \tag{30}$$

$$f^\dagger = \|\mathbb{E}[\mathbf{w}_{TP}^\dagger] - \mathbf{m}\|_F^2 / \|\mathbf{m}\|_F^2 \tag{31}$$

$$F_1 = \|\mathbf{B}_1 - \mathbf{V}\|_F^2 / \|\mathbf{V}\|_F^2 \tag{32}$$

$$F_2 = \|\mathbf{B}_2 - \mathbf{V}\|_F^2 \| / \mathbf{V}\|_F^2 \tag{33}$$

$$F^\dagger = \|\mathbb{V}[\mathbf{w}_{TP}^\dagger] - \mathbf{V} / \|_F^2 \|\mathbf{V}\|_F^2 \tag{34}$$

where $\|\mathbf{M}\|_F^2$ denotes the Frobenius norm of the matrix $\mathbf{M}$. Hence f_1, f_2, F_1 and F_2 represents the normalized Frobenius norm of differences between the bounds and the sample variance, while $f^\dagger$ and $F^\dagger$ denote the difference between the moments of $\mathbf{w}_{TP}^\dagger$ and the sample variance of $\tilde{\mathbf{w}}_{TP}$.

In the following, simulations of (23)-(34) will be studied for various parameter values. In order to account for a wide range of values of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, these values will be randomly generated in the simulation study. Each of the p elements in the mean vector $\boldsymbol{\mu}$ will be independently generated as $\mathcal{U}(-0.1, 0.1)$, where $\mathcal{U}(l, u)$ denotes the uniform distribution between l to u. The positive-definite covariance matrix $\boldsymbol{\Sigma}$ will be determined as $\mathbf{\Sigma} = \mathbf{\Gamma}\mathbf{\Lambda}\mathbf{\Gamma}'$, where the $p \times p$ matrix $\mathbf{\Gamma}$ represents the eigenvectors of $\mathbf{\Sigma}$ and is generated according to the Haar distribution. The $p \times p$ matrix $\mathbf{\Lambda}$ is diagonal, and its elements represents the ordered eigenvalues of $\mathbf{\Sigma}$. Here we let the p eigenvalues be equally spaced from d to 1, for various values of d. As such, the parameter d represents a measure of dependency between the p assets in the portfolio, where $d = 1$ represents no dependency and larger d represents a stronger dependency structure. Consequently, the simulation procedure can be described as follows:

1) Generate $\boldsymbol{\mu}$, with $\mu_i \sim \mathcal{U}(-0.1, 0.1)$, $i = 1 \ldots, p$.
2) Generate $\mathbf{\Gamma}$ according to the Haar distribution, and compute $\mathbf{\Sigma} = \mathbf{\Gamma}\mathbf{\Lambda}\mathbf{\Gamma}'$, where $\text{diag}(\mathbf{\Lambda}) = d \ldots, 1$.
3) Independently generate $\bar{\mathbf{x}} \sim \mathcal{N}_{p,1}(\boldsymbol{\mu}, \mathbf{\Sigma}/N)$ and $n\mathbf{S} \sim \mathcal{W}_p(n, \mathbf{\Sigma})$.
4) Compute $\tilde{\mathbf{w}}_{TP}$.
5) Repeat step 3)-4) above $s = 10000$ times.
6) Based on the s samples of $\tilde{\mathbf{w}}_{TP}$, compute $\mathbf{m}$ and $\mathbf{V}$.
7) Given $\mathbf{m}$ and $\mathbf{V}$, compute (23)-(34).

The above procedure is repeated $r = 10$ times to give r values of (23)-(34) for a given combination of p, N and d. Figures 1-12 display the mean value, for the r simulations, of each respective measure, for $p = \{25, 50, 75, 100\}$, $d = \{1, \ldots, 10\}$ and $N = \{2, 0.4p, 0.7p, p-3\}$. For easier reading, the values are displayed on a logarithmic scale and are connected with a solid line. First, it is noticeable that most measures seem to increase with increasing dependency measure d. Further, t_l, t_u, $t^\dagger$, T_1, T_2, $T^\dagger$ increase with increasing sample size N. However, F_2, the measure on the discrepancy between the sample variance of $\tilde{\mathbf{w}}_{TP}$ and the variance bound $\mathbf{B}_2$ instead decrease with increasing N. Regarding the bounds on the expected value of $\tilde{\mathbf{w}}_{TP}$, t_l and t_u appear very similar, as do f_1 and f_2. The measures on the difference between $\mathbb{E}[\tilde{\mathbf{w}}_{TP}]$ and $\mathbb{E}[\mathbf{w}_{TP}^\dagger]$, $t^\dagger$ and $f^\dagger$ obtains as fairly small for most the considered simulation parameters. This suggests that $\mathbb{E}[\mathbf{w}_{TP}^\dagger]$ can serve as a rough approximation of $\mathbb{E}[\tilde{\mathbf{w}}_{TP}]$, especially for $N \in (0.4p, 0.7p)$. Furthermore, when $d = 1$ we have $\mathbf{\Sigma} = \mathbf{I}_p$, and hence both the bounds $\mathbf{b}_1$ and $\mathbf{b}_2$, as well as $\mathbb{E}[\mathbf{w}_{TP}^\dagger]$ provide equality with $\mathbb{E}[\tilde{\mathbf{w}}_{TP}]$. As such, for $d = 1$, these measures simply capture sample variance for the mean of $\mathbf{m}$. Similarly, when $d = 1$, $T^\dagger$ and $F^\dagger$ capture the sample variance of $\mathbf{V}$. Further, for $N < p - 3$ and low values of d, $T^\dagger$ and $F^\dagger$ obtains as fairly small, suggesting that $\mathbb{V}[\mathbf{w}_{TP}^\dagger]$ could be applied as a rough approximation of $\mathbb{V}[\tilde{\mathbf{w}}_{TP}]$ in these cases. Finally, it is noticeable that the measures F_1 and F_2 appear very large for most of the combinations of p, N and d. It is however important to note that the Frobenius norm of differences, that these measures are based on, captures element-wise squared discrepancies, while $\mathbf{B}_1$ and $\mathbf{B}_2$ are not element-wise bounds, but rather bounds in the Löwner order sense.

5 Summary

The TP is an important strategy in mean-variance asset optimization, and the statistical properties of the typical TP weight estimator have been thoroughly studied. However, when the portfolio dimension is greater than the sample size, this estimator is not applicable since standard inversion of the now singular sample covariance matrix is not possible. This issue can be solved by applying the Moore-Penrose inverse, to which a general TP weights estimator can be provided, covering both the singular and non-singular case. Unfortunately, there exists no derivation of the moments for the Moore-Penrose inverse of a singular Wishart matrix, and consequently the moments of the general TP estimator cannot be obtained.

This paper provides bounds on the mean and variance of the TP weights estimator in the singular case. Further, approximate results are provided, as well as exact moment results in the case when the population covariance is equal to the identity matrix. It also provides exact moment results when the reflexive generalized inverse is applied in the TP weights equation.

Finally, the properties of the derived bounds, and the estimator based on the reflexive generalized inverse, are investigated in a simulation study. The difference between the various bounds and the sample counterparts are measured by several quantities, and studied for numerous dimensions, samples sizes and levels of dependencies of the population covariance matrix. The results suggest that many of the derived bounds are closest to the sample moments when the population covariance matrix suggest low dependency between the considered assets. Finally, the study implies that in some cases the moments of TP weights based on the reflexive generalized inverse can be used as a rough approximation for the moments of TP weights based on the Moore-Penrose inverse.

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Appendix

Lemma A1. *The elements of* $\mathbb{E}[\mathbf{S}^+]$ *has the following bounds,* $i = 1, \ldots, p$

$$0 < d_{ii} \leq e_{ii} \leq u_{ii}^{(a)},$$

and, for $i, j = 1, \ldots, p$, $i \neq j$,

$$\begin{aligned} e_{ij} &\leq \min\{d_{ij}, u_{ij}^{(a)}\} + \sqrt{(u_{ii}^{(a)} - d_{ii})(u_{jj}^{(a)} - d_{jj})}, \\ e_{ij} &\geq \max\{d_{ij}, u_{ij}^{(a)}\} - \sqrt{(u_{ii}^{(a)} - d_{ii})(u_{jj}^{(a)} - d_{jj})}. \end{aligned}$$

Proof. First note that in accordance with Theorem 3.2 and Theorem 3.3 of Imori and Rosen (2020), we have that

$$\begin{aligned} \mathbf{D} \leq_L \mathbb{E}[\mathbf{S}^+] &\leq_L \mathbf{U}_a, \\ \mathbb{E}[\mathbf{S}^+] &\leq_L \mathbf{U}_b. \end{aligned}$$

Further, by definition of the Löwner order we have, with $\boldsymbol{\alpha} \in \mathbb{R}^p$, that

$$\boldsymbol{\alpha}'\mathbf{D}\boldsymbol{\alpha} \leq \boldsymbol{\alpha}' \, \mathbb{E}[\mathbf{S}^+]\boldsymbol{\alpha} \leq \boldsymbol{\alpha}'\mathbf{U}\boldsymbol{\alpha}. \tag{35}$$

Thus, since $\boldsymbol{\alpha}'(\mathbb{E}[\mathbf{S}^+] - \mathbf{D})\boldsymbol{\alpha} \geq 0$, we have that $\mathbb{E}[\mathbf{S}^+] - \mathbf{D}$ is a positive semi-definite matrix, and the same holds for $\mathbf{U} - \mathbb{E}[\mathbf{S}^+]$. This gives that $0 < d_{ii} \leq e_{ii} \leq u_{ii}^{(a)}$, $i = 1, \ldots, p$.

Moreover, note that every principal submatrix of a positive definite matrix is also positive definite. Combined with (35) it provides the following inequalities, for any $i, j = 1, \ldots, p$, and with arbitrary non-zero scalars x_1 and x_2,

$$\begin{aligned} x_1^2 u_{ii}^{(a)} + 2x_1 x_2 u_{ij}^{(a)} + x_2^2 u_{jj}^{(a)} &\geq \\ x_1^2 e_{ii} + 2x_1 x_2 e_{ij} + x_2^2 e_{jj} &\geq \\ x_1^2 d_{ii} + 2x_1 x_2 d_{ij} + x_2^2 d_{jj} &> 0. \end{aligned}$$

Now, first assume $x_1 > 0$, $x_2 > 0$. As such the above expressions can be applied to obtain

$$\begin{aligned}
x_1^2 e_{ii} + 2x_1x_2e_{ij} + x_2^2 e_{ii} &\geq x_1^2 d_{ii} + 2x_1x_2d_{ij} + x_2^2 d_{jj} \qquad (36)\\
x_1^2 u_{ii}^{(a)} + 2x_1x_2e_{ij} + x_2^2 u_{jj}^{(a)} &\geq x_1^2 d_{ii} + 2x_1x_2d_{ij} + x_2^2 d_{jj}\\
e_{ij} &\geq -\frac{x_1^2(u_{ii}^{(a)} - d_{ii}) + x_2^2(u_{jj}^{(a)} - d_{jj}) - 2x_1x_2d_{ij}}{2x_1x_2}\\
&= -\frac{x_1(u_{ii}^{(a)} - d_{ii})}{2x_2} - \frac{x_2(u_{jj}^{(a)} - d_{jj})}{2x_1} + d_{ij} \qquad (37)
\end{aligned}$$

for any $i, j = \ldots, p$, $i \neq j$. As the right-hand side is a lower bound, we would like to obtain values x_1 and x_2 that maximizes this concave function. Deriving and setting the expression equal to zero, we obtain that it has its maximum at

$$x_1^2(u_{ii}^{(a)} - d_{ii}) = x_2^2(u_{jj}^{(a)} - d_{jj}).$$

Without loss of generality we can set $x_1 = 1$ and thus obtain the maximum at

$$\begin{aligned}
x_1 &= 1,\\
x_2 &= \sqrt{\frac{(u_{ii}^{(a)} - d_{ii})}{(u_{jj}^{(a)} - d_{jj})}}.
\end{aligned}$$

Applying this result to equation (37) yields

$$e_{ij} \geq d_{ij} - \sqrt{(u_{ii}^{(a)} - d_{ii})(u_{jj}^{(a)} - d_{jj})}. \qquad (38)$$

With an equivalent approach, again with $x_1 > 0$, $x_2 > 0$, we can utilize that

$$\begin{aligned}
x_1^2 d_{ii} + 2x_1x_2e_{ij} + x_2^2 d_{jj} &\leq x_1^2 u_{ii}^{(a)} + 2x_1x_2u_{ij}^{(a)} + x_2^2 u_{jj}^{(a)}\\
e_{ij} &\leq \frac{x_1^2(u_{ii}^{(a)} - d_{ii}) + x_2^2(u_{jj}^{(a)} - d_{jj}) + 2x_1x_2u_{ij}^{(a)}}{2x_1x_2}
\end{aligned}$$

in order to obtain the upper bound

$$e_{ij} \leq u_{ij}^{(a)} + \sqrt{(u_{ii}^{(a)} - d_{ii})(u_{jj}^{(a)} - d_{jj})}. \qquad (39)$$

Instead considering $x_1 < 0$ and $x_2 > 0$ (or $x_1 > 0$ and $x_2 < 0$) we can with a similar approach obtain the bounds

$$\begin{aligned} e_{ij} &\leq d_{ij} + \sqrt{(u_{ii}^{(a)} - d_{ii})(u_{jj}^{(a)} - d_{jj})}, \\ e_{ij} &\geq u_{ij}^{(a)} - \sqrt{(u_{ii}^{(a)} - d_{ii})(u_{jj}^{(a)} - d_{jj})}. \end{aligned}$$

Letting $x_1 < 0$ and $x_2 < 0$ again yield bounds (38) and (39). Expressed differently, the above bounds can be written as

$$\begin{aligned} e_{ij} &\leq \min\{d_{ij}, u_{ij}^{(a)}\} + \sqrt{(u_{ii}^{(a)} - d_{ii})(u_{jj}^{(a)} - d_{jj})}, \\ e_{ij} &\geq \max\{d_{ij}, u_{ij}^{(a)}\} - \sqrt{(u_{ii}^{(a)} - d_{ii})(u_{jj}^{(a)} - d_{jj})}, \end{aligned}$$

concluding the proof. □

The results in Lemma A1 can be further extended, by also considering the bounding matrix $\mathbf{U}_b$. The following lemma summarizes this result.

Lemma A2. *The elements of* $\mathbb{E}[\mathbf{S}^+]$ *has the following bounds,* $i = 1, \ldots, p$

$$0 < g_{ii} := d_{ii} \leq e_{ii} \leq h_{ii} := u_{ii}^{(*)}$$

where $u_{ii}^{(*)} = \min\{u_{ii}^{(a)}, u_{ii}^{(b)}\}$*. Further, for* $i, j = 1, \ldots, p$*,* $i \neq j$*,*

$$g_{ij} \leq e_{ij} \leq h_{ij}$$

with

$$\begin{aligned} g_{ij} &= \max \left\{ \begin{array}{l} d_{ij} - \sqrt{(u_{ii}^{(*)} - d_{ii})(u_{jj}^{(*)} - d_{jj})}, \\ u_{ij}^{(a)} - \sqrt{(u_{ii}^{(a)} - d_{ii})(u_{jj}^{(a)} - d_{jj})}, \\ -\sqrt{(u_{ii}^{(b)} - d_{ii})(u_{jj}^{(b)} - d_{jj})}, \\ -\sqrt{u_{ii}^{(*)} u_{jj}^{(*)}} \end{array} \right\}, \\ h_{ij} &= \min \left\{ \begin{array}{l} d_{ij} + \sqrt{(u_{ii}^{(*)} - d_{ii})(u_{jj}^{(*)} - d_{jj})}, \\ u_{ij}^{(a)} + \sqrt{(u_{ii}^{(a)} - d_{ii})(u_{jj}^{(a)} - d_{jj})}, \\ \sqrt{(u_{ii}^{(b)} - d_{ii})(u_{jj}^{(b)} - d_{jj})}, \\ \sqrt{u_{ii}^{(*)} u_{jj}^{(*)}} \end{array} \right\}. \end{aligned}$$

Proof. First, we have that

$$d_{ij} - \sqrt{(u_{ii}^{(*)} - d_{ii})(u_{jj}^{(*)} - d_{jj})} \leq e_{ij} \leq d_{ij} + \sqrt{(u_{ii}^{(*)} - d_{ii})(u_{jj}^{(*)} - d_{jj})},$$

since e_{ii} (and e_{jj}) in (36) can be replaced by either $u_{ii}^{(a)}$ or $u_{ii}^{(b)}$, whichever is smallest. Then

$$\begin{aligned} u_{ij}^{(a)} - \sqrt{(u_{ii}^{(a)} - d_{ii})(u_{jj}^{(a)} - d_{jj})} &\leq e_{ij} \leq u_{ij}^{(a)} + \sqrt{(u_{ii}^{(a)} - d_{ii})(u_{jj}^{(a)} - d_{jj})} \\ -\sqrt{(u_{ii}^{(b)} - d_{ii})(u_{jj}^{(b)} - d_{jj})} &\leq e_{ij} \leq \sqrt{(u_{ii}^{(b)} - d_{ii})(u_{jj}^{(b)} - d_{jj})} \end{aligned}$$

follows directly from Lemma A1 and the fact that $\mathbf{U}_b$ is diagonal and thus $u_{ij}^{(b)} = 0$. Finally

$$-\sqrt{u_{ii}^{(*)} u_{jj}^{(*)}} \leq e_{ij} \leq \sqrt{u_{ii}^{(*)} u_{jj}^{(*)}}$$

follows from

$$-\sqrt{u_{ii}^{(*)} u_{jj}^{(*)}} \leq -\sqrt{e_{ii} e_{jj}} \leq e_{ij} \leq \sqrt{e_{ii} e_{jj}} \leq \sqrt{u_{ii}^{(*)} u_{jj}^{(*)}}.$$

The lemma is proved. □

In the following, let

$$\begin{aligned} k_3 &= \frac{n[p(n+1) - 2]}{p[p(p+1) - 2]}, \\ k_4 &= \frac{n(p-n)}{p[p(p+1) - 2]}. \end{aligned}$$

Further define $g(\mathbf{L}) = \prod_{i=1}^{n} |\mathbf{L}_i|_+$ and $c(n,p) = (2\pi)^{np/2} 2^n s(n,p)$ where $|\mathbf{L}_i|_+$ and $s(n,p)$ is defined as on page 128 and 129 in Imori and Rosen (2020).

Lemma A3. *Let* $\mathbf{L} : n \times p$ *satisfy* $\mathbf{L}\mathbf{L}' = \mathbf{I}_n$. *Then, for all* $\boldsymbol{\alpha}, \mathbf{x} \in \mathbb{R}^p$,

(i) $\int (\boldsymbol{\alpha}'\mathbf{L}'\mathbf{L}\boldsymbol{\alpha})(\mathbf{x}'\mathbf{L}'\mathbf{L}\mathbf{x}) g(\mathbf{L}) d\mathbf{L} = k_1 c(n,p)(\boldsymbol{\alpha}'\mathbf{x})^2 + k_2 c(n,p)(\boldsymbol{\alpha}'\boldsymbol{\alpha})(\mathbf{x}'\mathbf{x})$

(ii) $\int (\boldsymbol{\alpha}'\mathbf{L}'\mathbf{L}\mathbf{x})^2 g(\mathbf{L}) d\mathbf{L} = k_3 c(n,p)(\boldsymbol{\alpha}'\mathbf{x})^2 + k_4 c(n,p)(\boldsymbol{\alpha}'\boldsymbol{\alpha})(\mathbf{x}'\mathbf{x})$

Proof. In accordance with page 130 in Imori and Rosen (2020) we have

$$n(\mathbf{I}_{p^2}+\mathbf{K}_{p,p})+n^2\mathrm{vec}(\mathbf{I}_p)\mathrm{vec}'(\mathbf{I}_p)$$
$$=c(n,p)^{-1}\int(\mathbf{L}\otimes\mathbf{L})'\left\{p(\mathbf{I}_{n^2}+\mathbf{K}_{n,n})+p^2\mathrm{vec}(\mathbf{I}_n)\mathrm{vec}'(\mathbf{I}_n)\right\}\times$$
$$\times(\mathbf{L}\otimes\mathbf{L})g(\mathbf{L})d\mathbf{L}, \tag{40}$$

where $\mathbf{K}_{\cdot,\cdot}$ is the commutation matrix. Now note that

$$(\boldsymbol{\alpha}\otimes\mathbf{x})'\mathbf{I}_{p^2}(\boldsymbol{\alpha}\otimes\mathbf{x})=(\boldsymbol{\alpha}'\boldsymbol{\alpha})(\mathbf{x}'\mathbf{x}) \tag{41}$$
$$(\boldsymbol{\alpha}\otimes\mathbf{x})'\mathbf{K}_{p,p}(\boldsymbol{\alpha}\otimes\mathbf{x})=(\boldsymbol{\alpha}'\mathbf{x})^2 \tag{42}$$
$$(\boldsymbol{\alpha}\otimes\mathbf{x})'\mathrm{vec}(\mathbf{I}_p)\mathrm{vec}'(\mathbf{I}_p)(\boldsymbol{\alpha}\otimes\mathbf{x})=(\boldsymbol{\alpha}'\mathbf{x})^2 \tag{43}$$
$$(\boldsymbol{\alpha}\otimes\mathbf{x})'(\mathbf{L}\otimes\mathbf{L})'\mathbf{I}_{n^2}(\mathbf{L}\otimes\mathbf{L})(\boldsymbol{\alpha}\otimes\mathbf{x})=(\boldsymbol{\alpha}'\mathbf{L}'\mathbf{L}\boldsymbol{\alpha})(\mathbf{x}'\mathbf{L}'\mathbf{L}\mathbf{x}) \tag{44}$$
$$(\boldsymbol{\alpha}\otimes\mathbf{x})'(\mathbf{L}\otimes\mathbf{L})'\mathbf{K}_{n,n}(\mathbf{L}\otimes\mathbf{L})(\boldsymbol{\alpha}\otimes\mathbf{x})=(\boldsymbol{\alpha}'\mathbf{L}'\mathbf{L}\mathbf{x})^2 \tag{45}$$
$$(\boldsymbol{\alpha}\otimes\mathbf{x})'(\mathbf{L}\otimes\mathbf{L})'\mathrm{vec}(\mathbf{I}_n)\mathrm{vec}'(\mathbf{I}_n)(\mathbf{L}\otimes\mathbf{L})(\boldsymbol{\alpha}\otimes\mathbf{x})=(\boldsymbol{\alpha}'\mathbf{L}'\mathbf{L}\mathbf{x})^2 \tag{46}$$

and

$$(\boldsymbol{\alpha}\otimes\boldsymbol{\alpha})'\mathbf{I}_{p^2}(\mathbf{x}\otimes\mathbf{x})=(\boldsymbol{\alpha}'\mathbf{x})^2$$
$$(\boldsymbol{\alpha}\otimes\boldsymbol{\alpha})'\mathbf{K}_{p,p}(\mathbf{x}\otimes\mathbf{x})=(\boldsymbol{\alpha}'\mathbf{x})^2$$
$$(\boldsymbol{\alpha}\otimes\boldsymbol{\alpha})'\mathrm{vec}(\mathbf{I}_p)\mathrm{vec}'(\mathbf{I}_p)(\mathbf{x}\otimes\mathbf{x})=(\boldsymbol{\alpha}'\boldsymbol{\alpha})(\mathbf{x}'\mathbf{x})$$
$$(\boldsymbol{\alpha}\otimes\boldsymbol{\alpha})'(\mathbf{L}\otimes\mathbf{L})'\mathbf{I}_{n^2}(\mathbf{L}\otimes\mathbf{L})(\mathbf{x}\otimes\mathbf{x})=(\boldsymbol{\alpha}'\mathbf{L}'\mathbf{L}\mathbf{x})^2$$
$$(\boldsymbol{\alpha}\otimes\boldsymbol{\alpha})'(\mathbf{L}\otimes\mathbf{L})'\mathbf{K}_{n,n}(\mathbf{L}\otimes\mathbf{L})(\mathbf{x}\otimes\mathbf{x})=(\boldsymbol{\alpha}'\mathbf{L}'\mathbf{L}\mathbf{x})^2$$
$$(\boldsymbol{\alpha}\otimes\boldsymbol{\alpha})'(\mathbf{L}\otimes\mathbf{L})'\mathrm{vec}(\mathbf{I}_n)\mathrm{vec}'(\mathbf{I}_n)(\mathbf{L}\otimes\mathbf{L})(\mathbf{x}\otimes\mathbf{x})=(\boldsymbol{\alpha}'\mathbf{L}'\mathbf{L}\boldsymbol{\alpha})(\mathbf{x}'\mathbf{L}'\mathbf{L}\mathbf{x}).$$

As such, from equation (40) we can obtain the following two expressions:

$$(\boldsymbol{\alpha}\otimes\mathbf{x})'\left\{n(\mathbf{I}_{p^2}+\mathbf{K}_{p,p})+n^2\mathrm{vec}(\mathbf{I}_p)\mathrm{vec}'(\mathbf{I}_p)\right\}(\boldsymbol{\alpha}\otimes\mathbf{x})$$
$$=c(n,p)^{-1}(\boldsymbol{\alpha}\otimes\mathbf{x})'\left[\int(\mathbf{L}\otimes\mathbf{L})'\left\{p(\mathbf{I}_{n^2}+\mathbf{K}_{n,n})+p^2\mathrm{vec}(\mathbf{I}_n)\mathrm{vec}'(\mathbf{I}_n)\right\}\right.$$
$$\left.\times(\mathbf{L}\otimes\mathbf{L})g(\mathbf{L})d\mathbf{L}\right](\boldsymbol{\alpha}\otimes\mathbf{x}),$$
$$n(\boldsymbol{\alpha}'\boldsymbol{\alpha})(\mathbf{x}'\mathbf{x})+(n+n^2)(\boldsymbol{\alpha}'\mathbf{x})^2$$
$$=c(n,p)^{-1}\int\left\{p(\boldsymbol{\alpha}'\mathbf{L}'\mathbf{L}\boldsymbol{\alpha})(\mathbf{x}'\mathbf{L}'\mathbf{L}\mathbf{x})+(p+p^2)(\boldsymbol{\alpha}'\mathbf{L}'\mathbf{L}\mathbf{x})^2\right\}g(\mathbf{L})d\mathbf{L}, \tag{47}$$

and

$$(\boldsymbol{\alpha}\otimes\boldsymbol{\alpha})'\left\{n(\mathbf{I}_{p^2}+\mathbf{K}_{p,p})+n^2\text{vec}(\mathbf{I}_p)\text{vec}'(\mathbf{I}_p)\right\}(\mathbf{x}\otimes\mathbf{x})$$
$$= c(n,p)^{-1}(\boldsymbol{\alpha}\otimes\boldsymbol{\alpha})'\left[\int(\mathbf{L}\otimes\mathbf{L})'\left\{p(\mathbf{I}_{n^2}+\mathbf{K}_{n,n})+p^2\text{vec}(\mathbf{I}_n)\text{vec}'(\mathbf{I}_n)\right\}\right.$$
$$\left.\times(\mathbf{L}\otimes\mathbf{L})g(\mathbf{L})d\mathbf{L}\right](\mathbf{x}\otimes\mathbf{x}),$$
$$2n(\boldsymbol{\alpha}'\mathbf{x})^2+n^2(\boldsymbol{\alpha}'\boldsymbol{\alpha})(\mathbf{x}'\mathbf{x})$$
$$= c(n,p)^{-1}\int\left\{p^2(\boldsymbol{\alpha}'\mathbf{L}'\mathbf{L}\boldsymbol{\alpha})(\mathbf{x}'\mathbf{L}'\mathbf{L}\mathbf{x})+2p(\boldsymbol{\alpha}'\mathbf{L}'\mathbf{L}\mathbf{x})^2\right\}g(\mathbf{L})d\mathbf{L}. \tag{48}$$

From equation (47) we can then derive

$$\int(\boldsymbol{\alpha}'\mathbf{L}'\mathbf{L}\boldsymbol{\alpha})(\mathbf{x}'\mathbf{L}'\mathbf{L}\mathbf{x})g(\mathbf{L})d\mathbf{L} = \frac{c(n,p)n}{p}\left[(\boldsymbol{\alpha}'\boldsymbol{\alpha})(\mathbf{x}'\mathbf{x})+(n+1)(\boldsymbol{\alpha}'\mathbf{x})^2\right] -(1+p)\int(\boldsymbol{\alpha}'\mathbf{L}'\mathbf{L}\mathbf{x})^2g(\mathbf{L})d\mathbf{L}.$$

Inserting this expression into equation (48) yields

$$\int(\boldsymbol{\alpha}'\mathbf{L}'\mathbf{L}\mathbf{x})^2g(\mathbf{L})d\mathbf{L}=\frac{n}{p}\frac{(p(n+1)-2)(\boldsymbol{\alpha}'\mathbf{x})^2+(p-n)(\boldsymbol{\alpha}'\boldsymbol{\alpha})(\mathbf{x}'\mathbf{x})}{c(n,p)^{-1}(p(p+1)-2)},$$

and as such we finally obtain

$$\begin{aligned}\int(\boldsymbol{\alpha}'\mathbf{L}'\mathbf{L}\boldsymbol{\alpha})(\mathbf{x}'\mathbf{L}'\mathbf{L}\mathbf{x})g(\mathbf{L})d\mathbf{L} &= \frac{n}{p}\frac{(\boldsymbol{\alpha}'\boldsymbol{\alpha})(\mathbf{x}'\mathbf{x})+(n+1)(\boldsymbol{\alpha}'\mathbf{x})^2}{c(n,p)^{-1}} \\ &\quad -(p+1)\frac{n}{p}\frac{(p(n+1)-2)(\boldsymbol{\alpha}'\mathbf{x})^2+(p-n)(\boldsymbol{\alpha}'\boldsymbol{\alpha})(\mathbf{x}'\mathbf{x})}{c(n,p)^{-1}(p(p+1)-2))} \\ &= \frac{c(n,p)n}{p}\left(1-\frac{(p+1)(p-n)}{p(p+1)-2}\right)(\boldsymbol{\alpha}'\boldsymbol{\alpha})(\mathbf{x}'\mathbf{x}) \\ &\quad +\frac{c(n,p)n}{p}\left(1+n-\frac{(p+1)(p(n+1)-2)}{p(p+1)-2}\right) \\ &\quad \times(\boldsymbol{\alpha}'\mathbf{x})^2,\end{aligned}$$

completing the proof. □

Lemma A4. *Let* $n\mathbf{S}\sim\mathcal{W}_p(n,\boldsymbol{\Sigma}), p>n+3$ *and* $\boldsymbol{\Sigma}>0$. *Then, for all* $\boldsymbol{\alpha},\mathbf{x}\in\mathbb{R}^p$,

(i) $\mathbb{E}[(\boldsymbol{\alpha}\mathbf{S}^+\mathbf{x})^2]\leq(2c_1+c_2)(\lambda_1(\boldsymbol{\Sigma}^{-1}))^4\left[k_1(\boldsymbol{\alpha}'\boldsymbol{\Sigma}\mathbf{x})^2+k_2(\boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha})(\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x})\right]$

(ii) $\mathbb{E}[(\boldsymbol{\alpha}\mathbf{S}^+\mathbf{x})^2]\leq(\lambda_1(\boldsymbol{\Sigma}^{-1}))^4(2c_1+c_2)(\boldsymbol{\alpha}'\boldsymbol{\alpha})(\mathbf{x}'\mathbf{x})$,

Proof. First, let $\mathbf{Y}'\boldsymbol{\Sigma}^{-1/2}=\mathbf{T}\mathbf{L}$, where $\mathbf{L}\mathbf{L}'=\mathbf{I}_n$, $\mathbf{L}:n\times p$ and $\mathbf{T}:n\times n$ is lower triangular with positive elements. Further, note that in accordance with page 131 in Imori and Rosen (2020), for $p > n+3$,

$$\begin{aligned}\mathbb{E}[\text{vec}(\mathbf{S}^{+})\text{vec}'(\mathbf{S}^{+})] &= c(n,p)^{-1}\int(c_1(\mathbf{I}_{p^2}+\mathbf{K}_{p,p})(\mathbf{P}\otimes\mathbf{P}) \\ &\quad +c_2\text{vec}(\mathbf{P})\text{vec}'(\mathbf{P}))g(\mathbf{L})d\mathbf{L},\end{aligned}$$

where

$$\mathbf{P} = \boldsymbol{\Sigma}^{1/2}\mathbf{L}'(\mathbf{L}\boldsymbol{\Sigma}\mathbf{L}')^{-1}(\mathbf{L}\boldsymbol{\Sigma}\mathbf{L}')^{-1}\mathbf{L}\boldsymbol{\Sigma}^{1/2}.$$

As such, with equalities similar to (41)-(46),

$$\begin{aligned}(\boldsymbol{\alpha}\otimes\mathbf{x})'\,\mathbb{E}[\text{vec}(\mathbf{S}^{+})\text{vec}'(\mathbf{S}^{+})](\boldsymbol{\alpha}\otimes\mathbf{x}) &= c(n,p)^{-1}(\boldsymbol{\alpha}\otimes\mathbf{x})'\int(c_1(\mathbf{I}_{p^2}+\mathbf{K}_{p,p})(\mathbf{P}\otimes\mathbf{P}) \\ &\quad +c_2\text{vec}(\mathbf{P})\text{vec}'(\mathbf{P}))g(\mathbf{L})d\mathbf{L}(\boldsymbol{\alpha}\otimes\mathbf{x}) \\ \mathbb{E}[(\boldsymbol{\alpha}\mathbf{S}^{+}\mathbf{x})^2] &= c(n,p)^{-1}\left[(c_1+c_2)\int(\mathbf{x}'\mathbf{P}\boldsymbol{\alpha})^2 g(\mathbf{L})d\mathbf{L}\right. \\ &\quad \left. + c_1\int(\mathbf{x}'\mathbf{P}\mathbf{x})(\boldsymbol{\alpha}'\mathbf{P}\boldsymbol{\alpha})g(\mathbf{L})d\mathbf{L}\right]. \qquad (49)\end{aligned}$$

Now, by Lemma A5, we have that $(\mathbf{x}'\mathbf{P}\mathbf{x})(\boldsymbol{\alpha}'\mathbf{P}\boldsymbol{\alpha}) \geq (\mathbf{x}'\mathbf{P}\boldsymbol{\alpha})^2$. As such, combining this inequality with (49) and Lemma 2.4 (i) in Imori and Rosen (2020), we have

$$\begin{aligned}\mathbb{E}[(\boldsymbol{\alpha}\mathbf{S}^{+}\mathbf{x})^2] &= c(n,p)^{-1}\left[(c_1+c_2)\int(\mathbf{x}'\mathbf{P}\boldsymbol{\alpha})^2 g(\mathbf{L})d\mathbf{L}\right. \\ &\quad \left. + c_1\int(\mathbf{x}'\mathbf{P}\mathbf{x})(\boldsymbol{\alpha}'\mathbf{P}\boldsymbol{\alpha})g(\mathbf{L})d\mathbf{L}\right] \\ &\leq c(n,p)^{-1}(2c_1+c_2)\int(\mathbf{x}'\mathbf{P}\mathbf{x})(\boldsymbol{\alpha}'\mathbf{P}\boldsymbol{\alpha})g(\mathbf{L})d\mathbf{L} \\ &\leq c(n,p)^{-1}(2c_1+c_2)(\lambda_1(\boldsymbol{\Sigma}^{-1}))^4 \\ &\quad \times\int(\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{1/2}\mathbf{L}'\mathbf{L}\boldsymbol{\Sigma}^{1/2}\boldsymbol{\alpha})(\mathbf{x}'\boldsymbol{\Sigma}^{1/2}\mathbf{L}'\mathbf{L}\boldsymbol{\Sigma}^{1/2}\mathbf{x})g(\mathbf{L})d\mathbf{L} \\ &= (2c_1+c_2)(\lambda_1(\boldsymbol{\Sigma}^{-1}))^4\left[k_1(\boldsymbol{\alpha}'\boldsymbol{\Sigma}\mathbf{x})^2+k_2(\boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha})(\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x})\right],\end{aligned}$$

where Lemma A3 (i) has been applied in the last equality. On the other hand, if we instead apply the inequality in Lemma 2.4 (ii) of Imori and Rosen (2020) we instead obtain

$$\begin{aligned}\mathbb{E}[(\boldsymbol{\alpha}\mathbf{S}^{+}\mathbf{x})^2] &\leq c(n,p)^{-1}(\lambda_1(\boldsymbol{\Sigma}^{-1}))^4\left[(2c_1+c_2)(\boldsymbol{\alpha}'\boldsymbol{\alpha})(\mathbf{x}'\mathbf{x})\right]\int g(\mathbf{L})d\mathbf{L} \\ &= (\lambda_1(\boldsymbol{\Sigma}^{-1}))^4\left[(2c_1+c_2)(\boldsymbol{\alpha}'\boldsymbol{\alpha})(\mathbf{x}'\mathbf{x})\right]\end{aligned}$$

where Lemma 3.1 (i) in Imori and Rosen (2020) gives the equality and concludes the proof. □

Lemma A5. *Let* $\mathbf{A}$ *be a* $p \times p$ *symmetric positive-definite matrix. Then for any* $\mathbf{c}, \mathbf{d} \in \mathbb{R}^p$,

$$(\mathbf{c}'\mathbf{A}\mathbf{c})(\mathbf{d}'\mathbf{A}\mathbf{d}) \geq (\mathbf{c}'\mathbf{A}\mathbf{d})^2.$$

Proof. Let $\mathbf{A} = \mathbf{Q}\mathbf{R}\mathbf{Q}'$ denote the eigenvalue decomposition of $\mathbf{A}$, such that $\mathbf{Q}$ is orthogonal and $\mathbf{R}$ is a diagonal matrix with positive elements. Make the substitutions

$$\mathbf{f} = \mathbf{R}^{1/2}\mathbf{Q}'\mathbf{c}$$
$$\mathbf{g} = \mathbf{R}^{1/2}\mathbf{Q}'\mathbf{d},$$

such that the inequality $(\mathbf{c}'\mathbf{A}\mathbf{c})(\mathbf{d}'\mathbf{A}\mathbf{d}) \geq (\mathbf{c}'\mathbf{A}\mathbf{d})^2$ can be written

$$(\mathbf{f}'\mathbf{f})(\mathbf{g}'\mathbf{g}) \geq (\mathbf{f}'\mathbf{g})^2. \tag{50}$$

Further, since $(\mathbf{f}'\mathbf{g}) = \|\mathbf{f}\|\|\mathbf{g}\|\cos(\theta)$, where $\|\cdot\|$ denote the Euclidean norm and θ is the angle between the vectors $\mathbf{f}$ and $\mathbf{g}$, the inequality (50) becomes

$$\|\mathbf{f}\|^2\|\mathbf{g}\|^2 \geq \|\mathbf{f}\|^2\|\mathbf{g}\|^2\cos(\theta)^2$$

which holds since $\cos(\theta)^2 \leq 1$. The lemma is proved. □

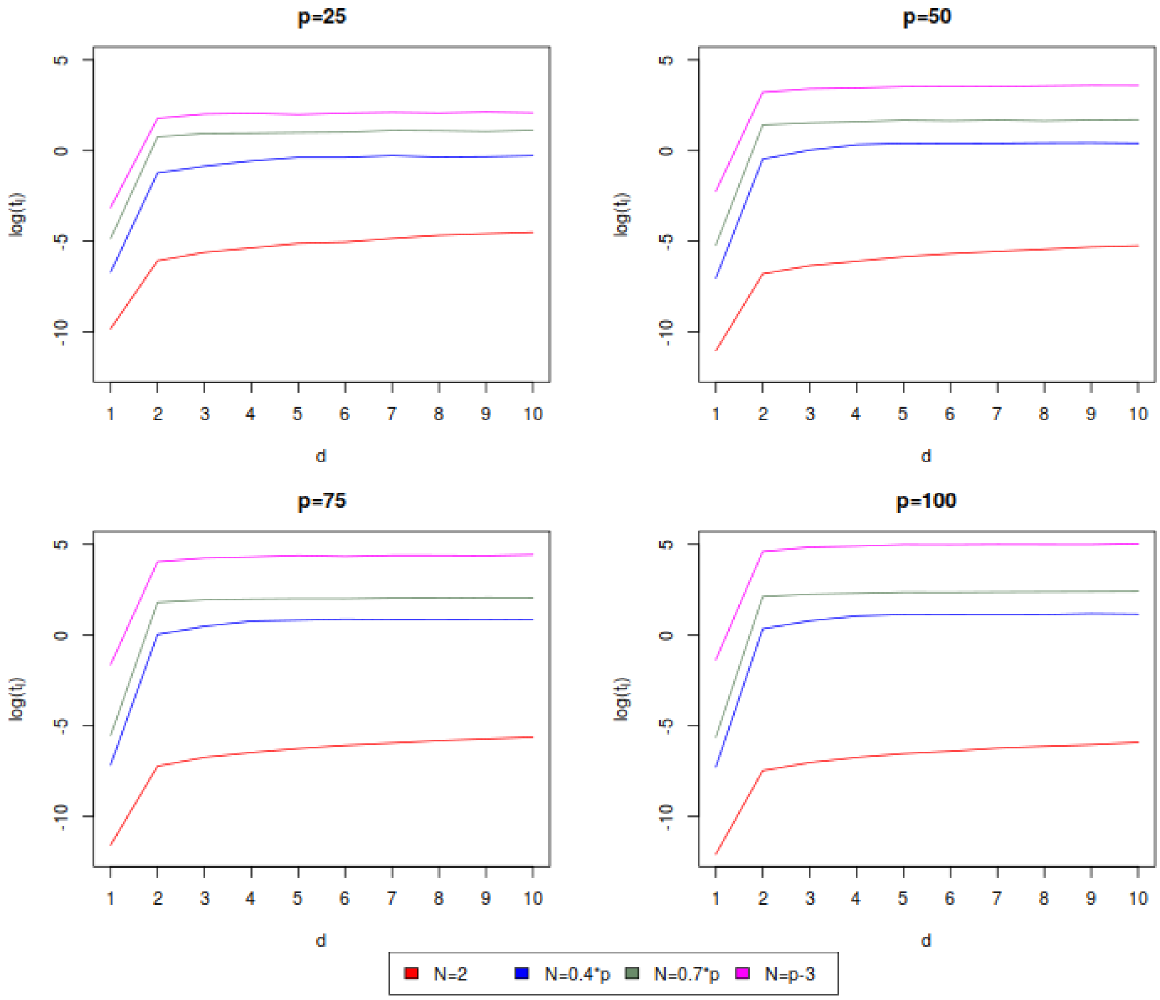

Figure 1: *The logarithm of t_l plotted for various values of p, N and d.*

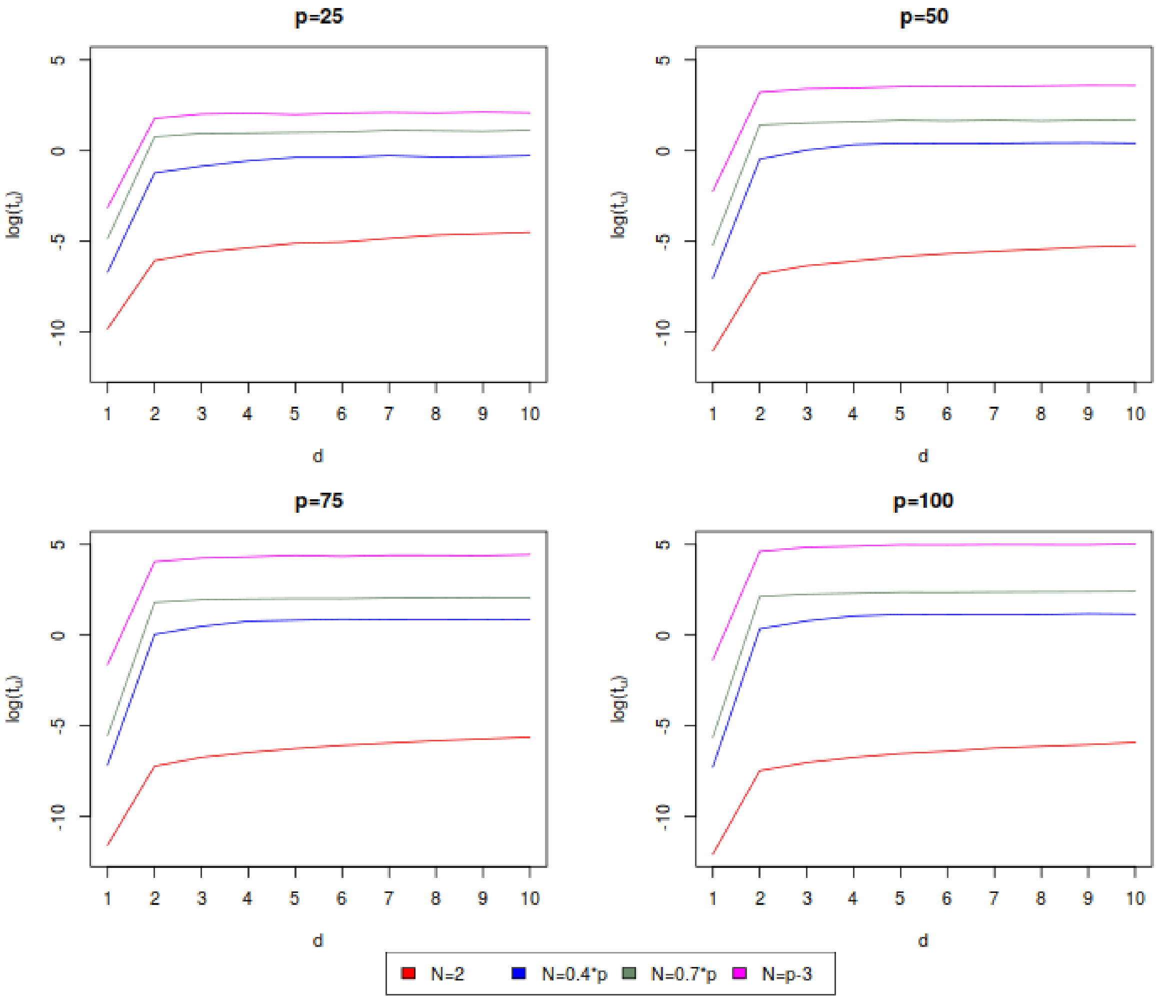

Figure 2: *The logarithm of t_u plotted for various values of p, N and d.*

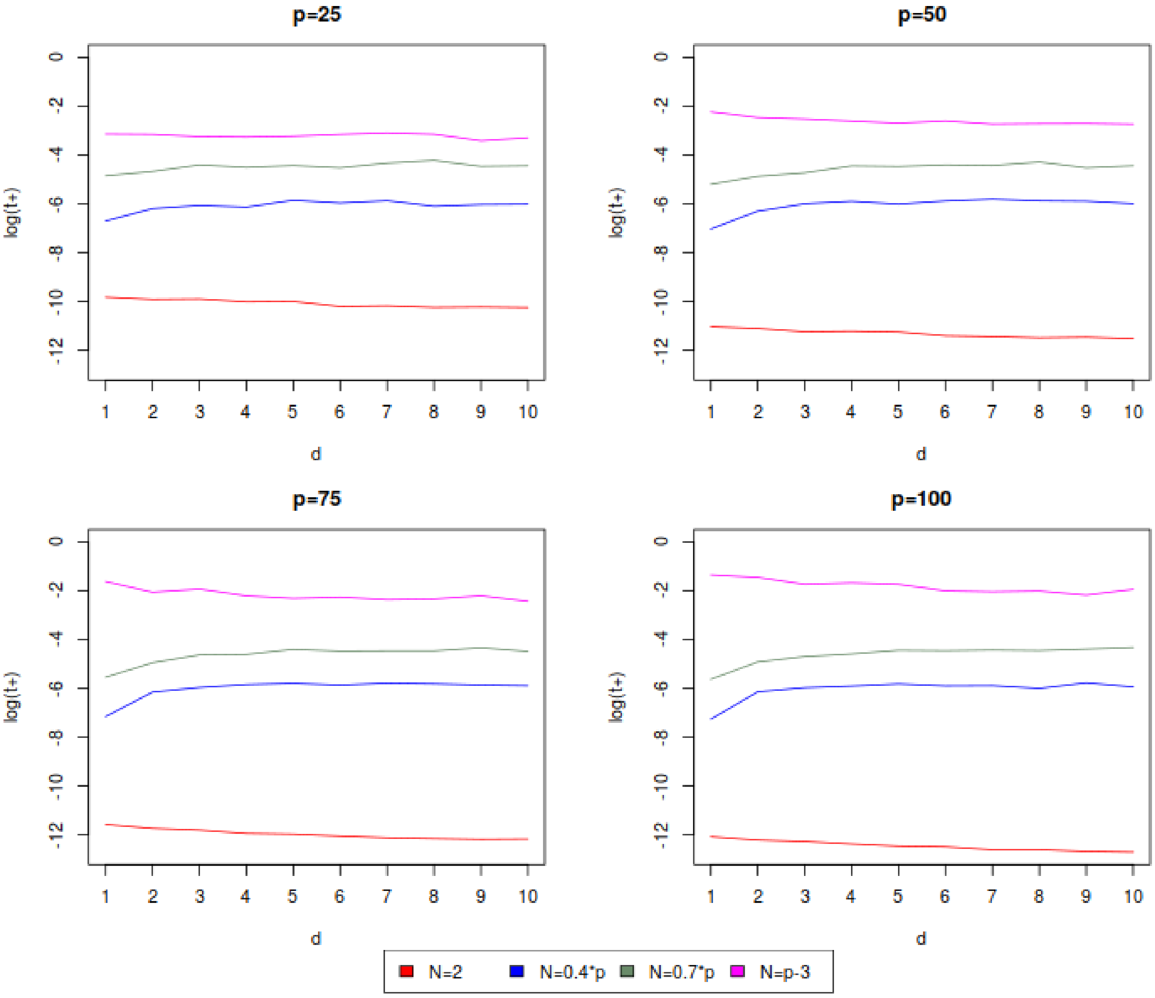

Figure 3: *The logarithm of* $t^{\dagger}$ *plotted for various values of* p*,* N *and* d*.*

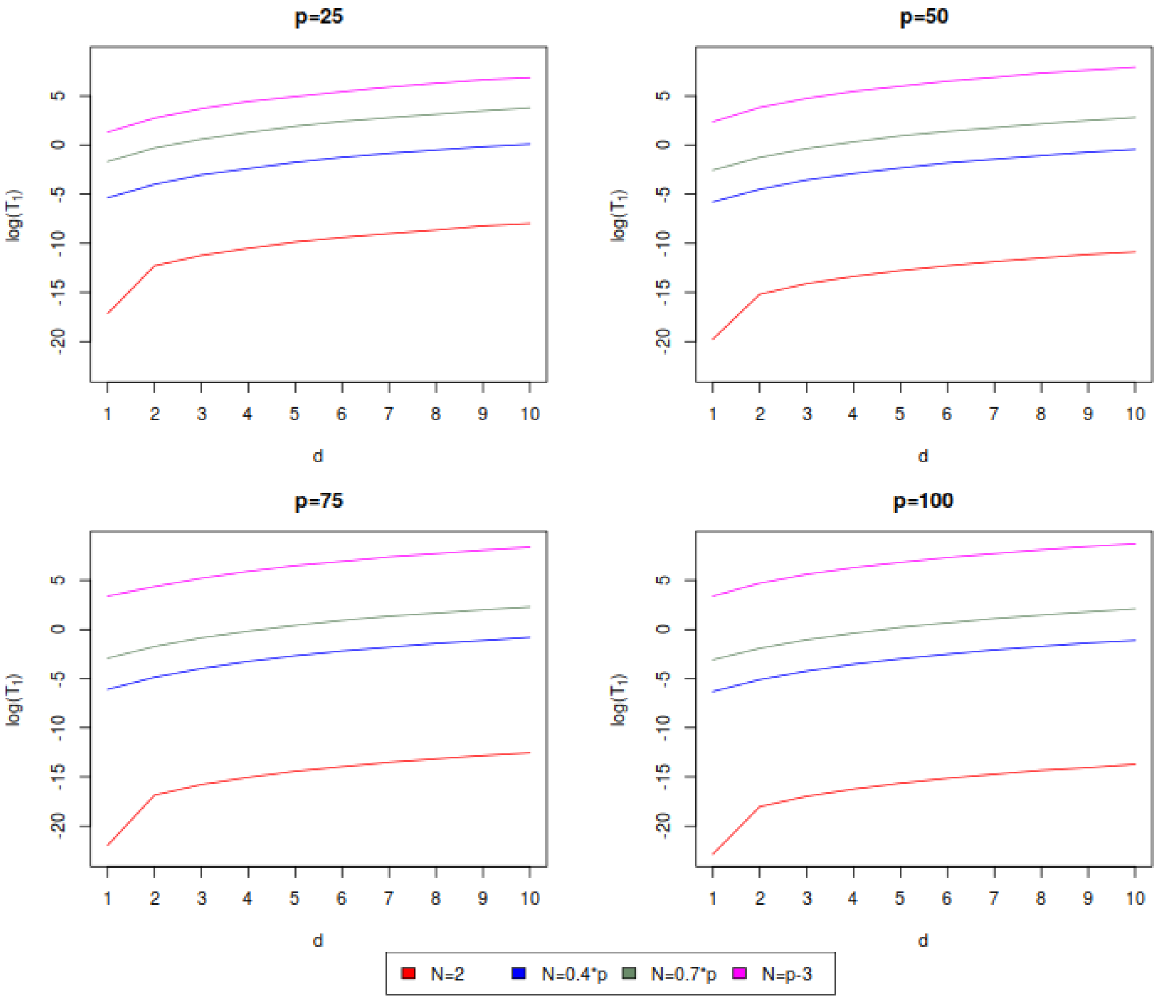

Figure 4: *The logarithm of T_1 plotted for various values of p, N and d.*

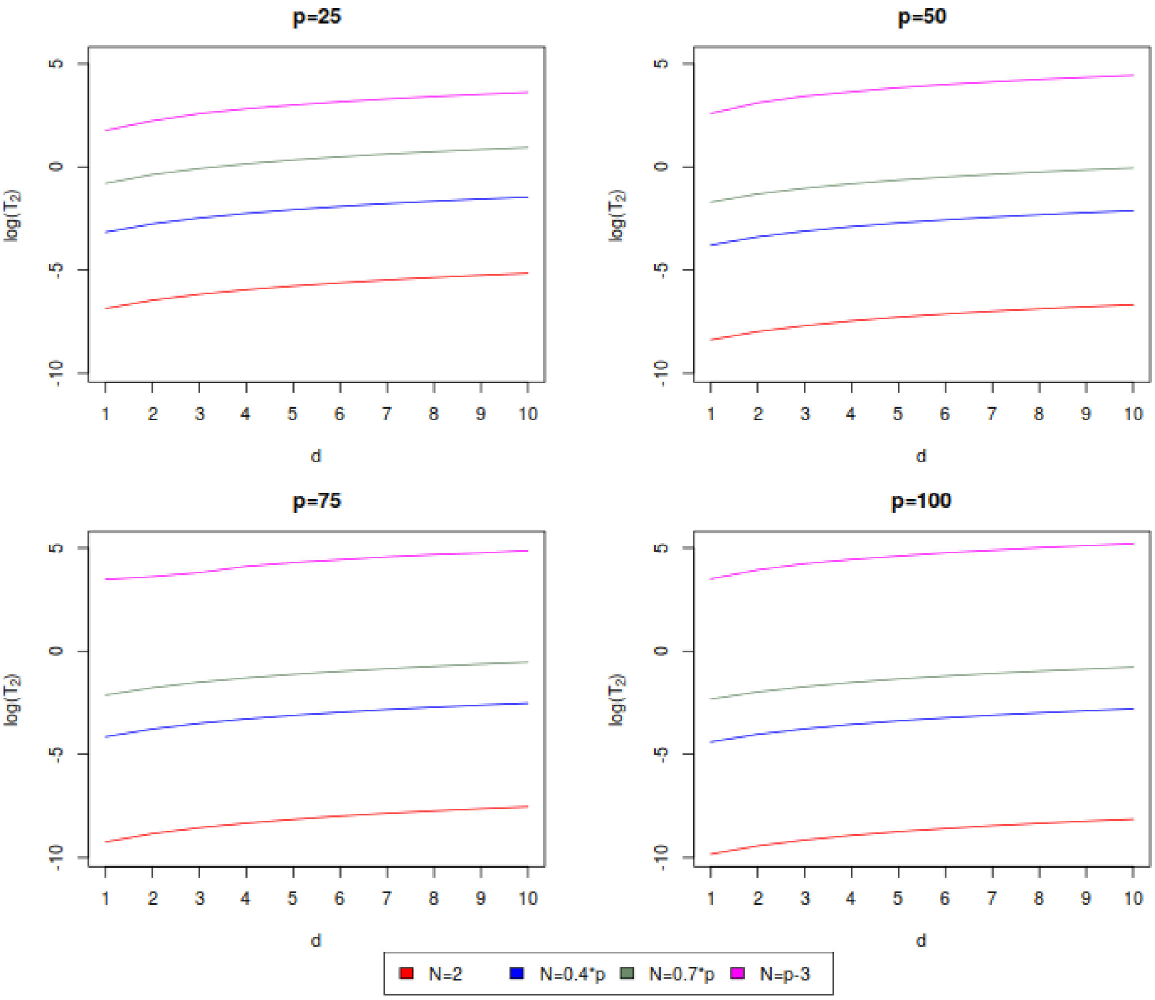

Figure 5: *The logarithm of T_2 plotted for various values of p, N and d.*

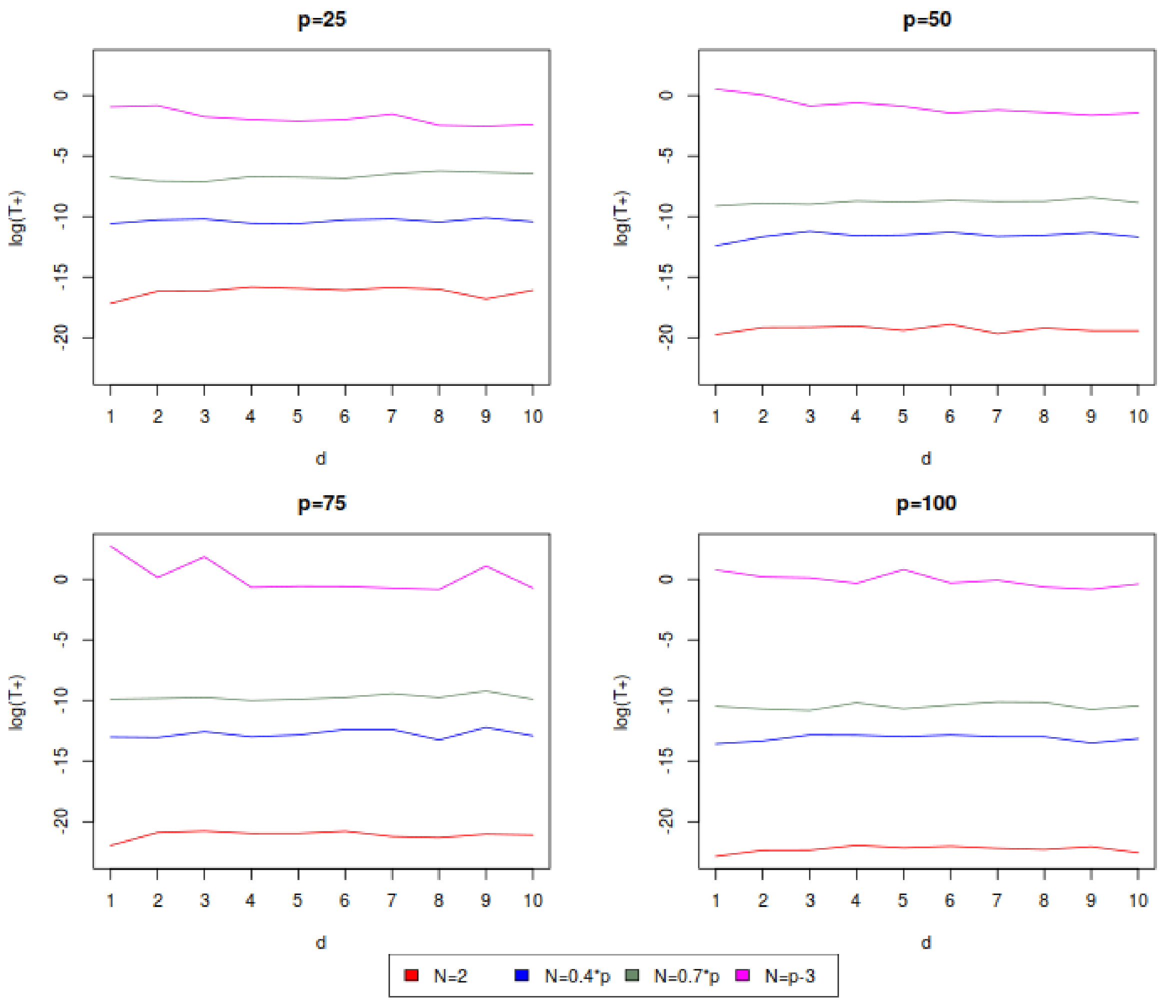

Figure 6: *The logarithm of $T^{\dagger}$ plotted for various values of p, N and d.*

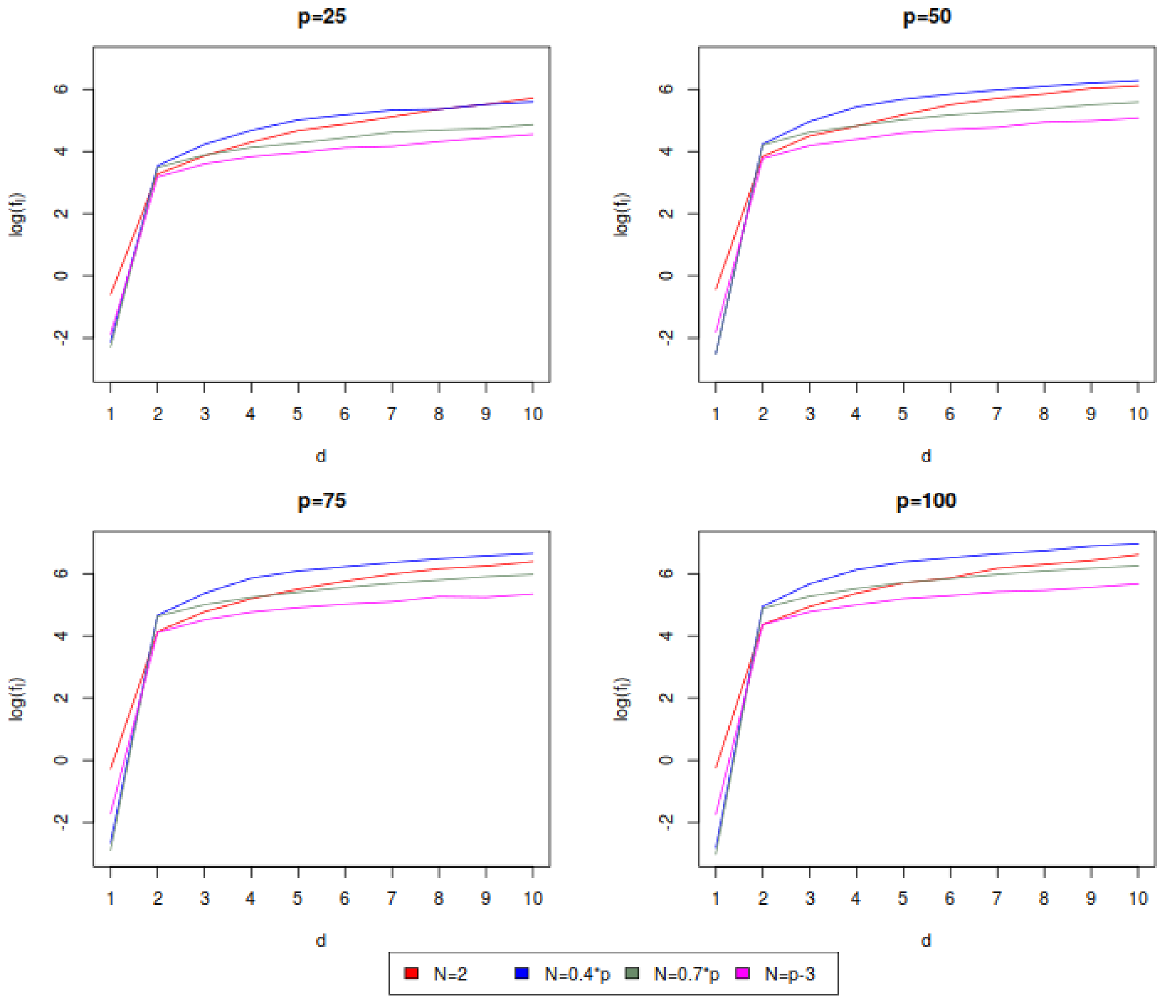

Figure 7: *The logarithm of f_l plotted for various values of p, N and d.*

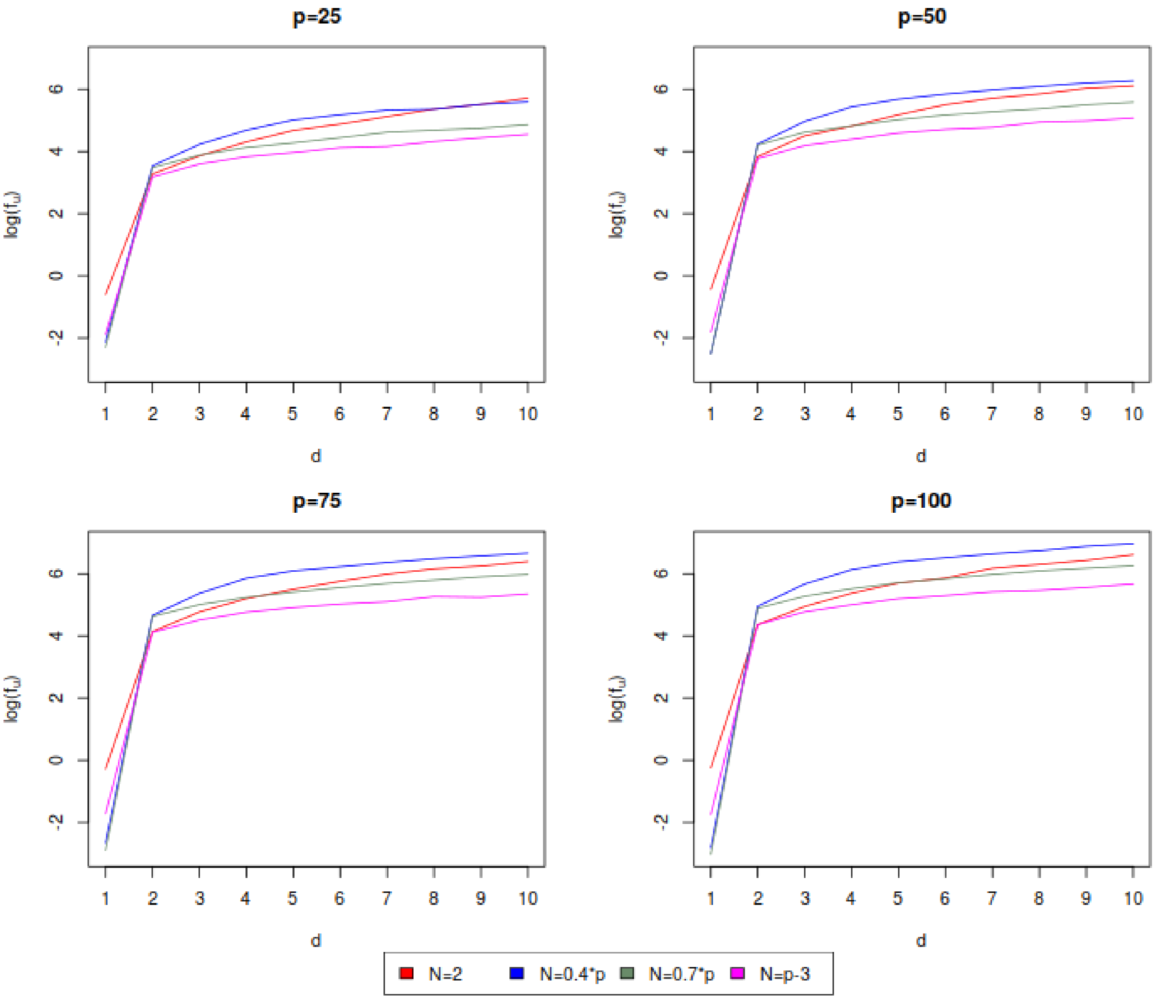

Figure 8: *The logarithm of f_u plotted for various values of p, N and d.*

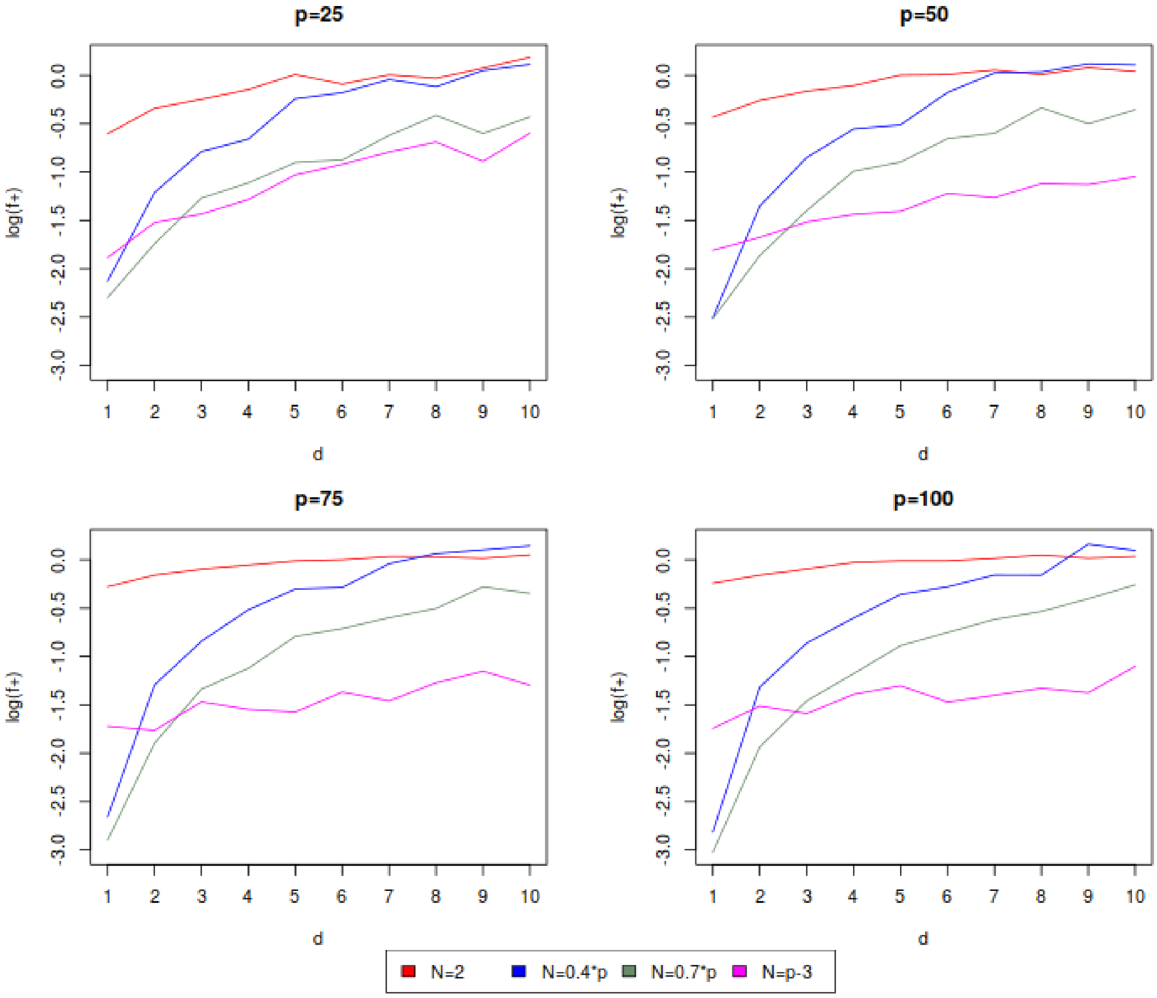

Figure 9: *The logarithm of* $f^{\dagger}$ *plotted for various values of* p, N *and* d.

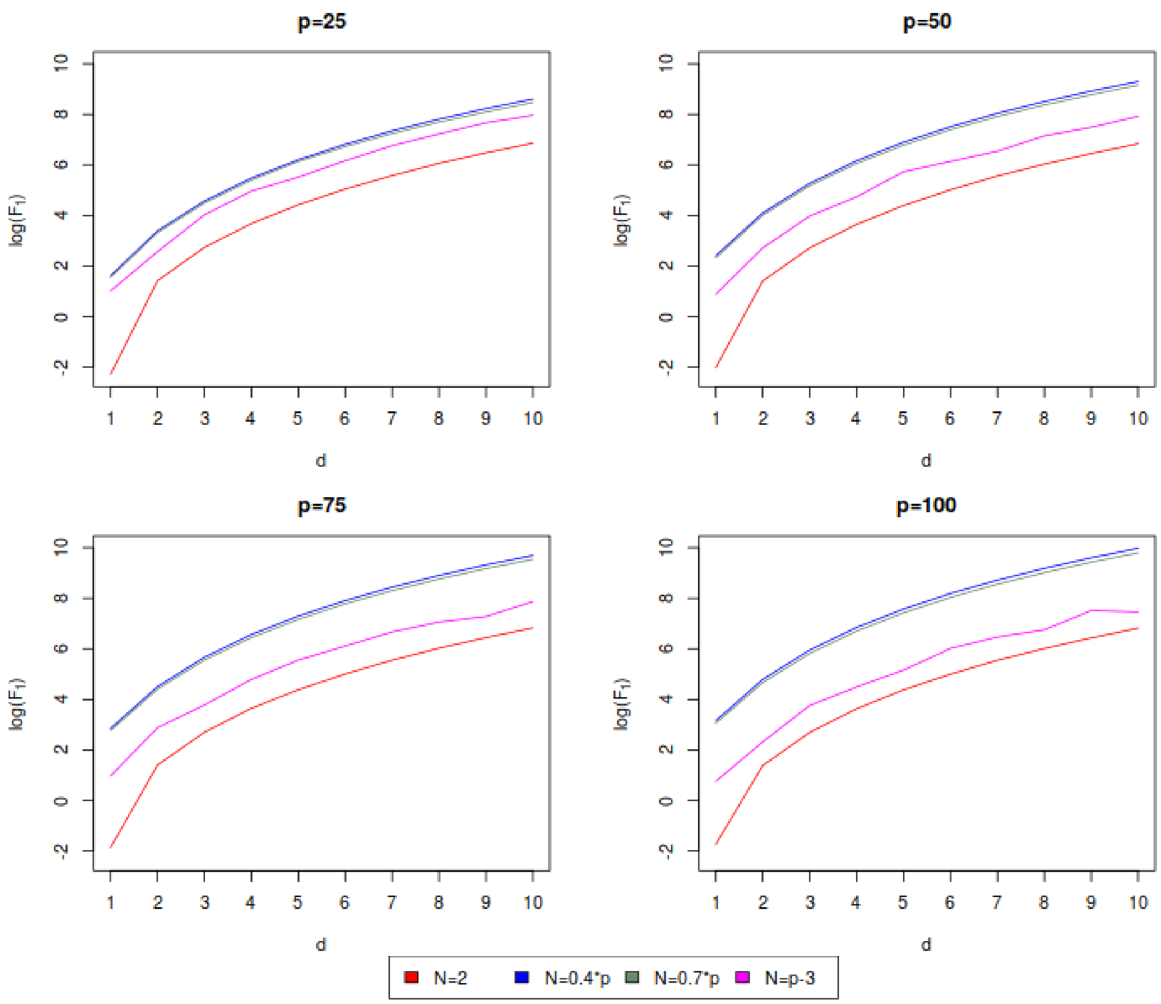

Figure 10: *The logarithm of F_1 plotted for various values of p, N and d.*

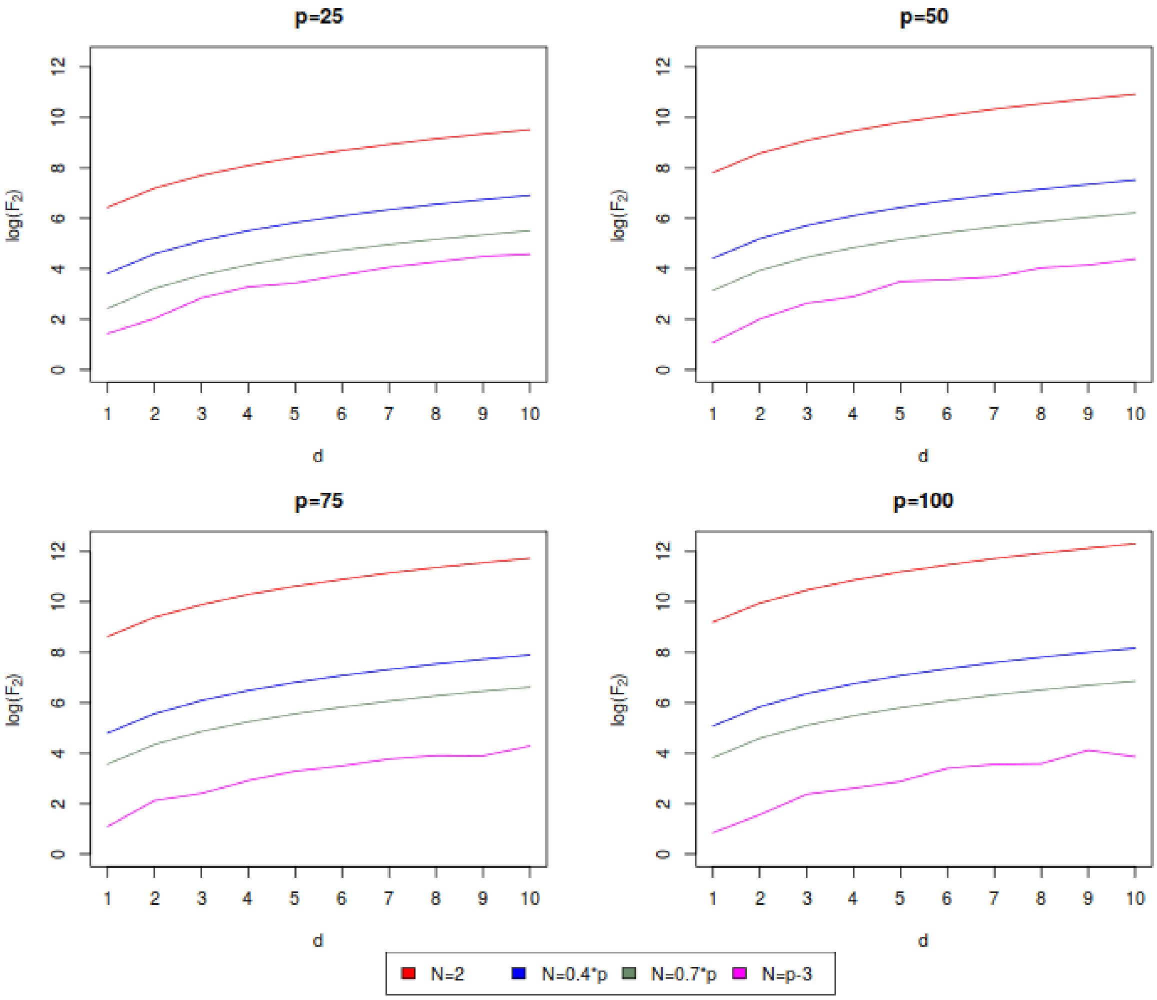

Figure 11: *The logarithm of F_2 plotted for various values of p, N and d.*

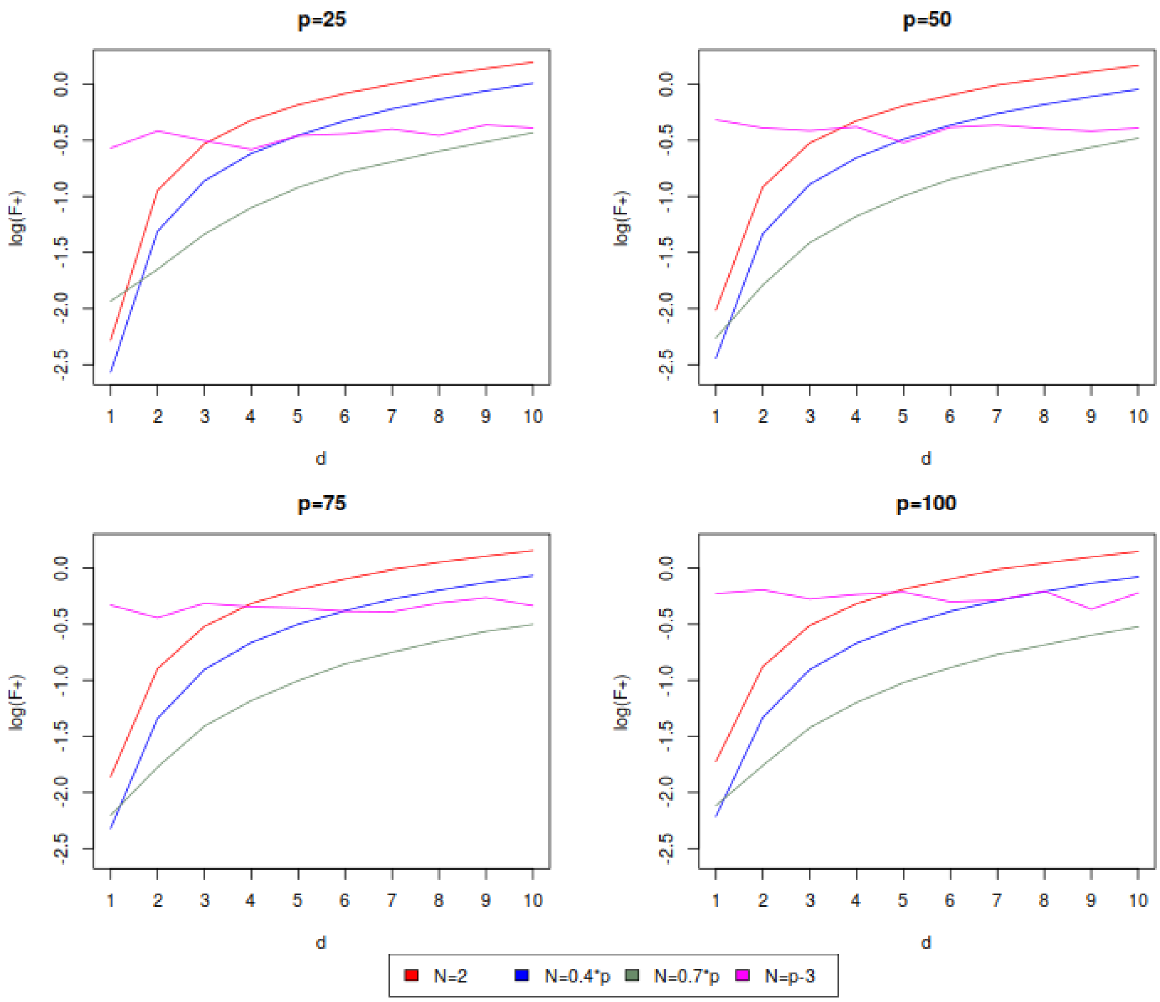

Figure 12: *The logarithm of $F^{\dagger}$ plotted for various values of p, N and d.*