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Sequential monitoring of high-dimensional time series

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Abstract

In the paper we derive new types of multivariate EWMA control charts which are based on the Euclidean distance and on the distance defined by using the inverse of the diagonal matrix consisting of the variances. The design of the proposed control schemes does not involve the computation of the inverse covariance matrix and, thus, it can be used in the high-dimensional setting. The distributional properties of the control statistics are obtained and are used in the determination of the new control procedures. Within an extensive simulation study, the new approaches are compared with the multivariate EWMA control charts which are based on the Mahalanobis distance.

***Keywords*:** high-dimensional time series; sequential surveillance; vector autoregressive process; MEWMA control chart; maximum expected delay

1 Introduction

Multivariate statistical process control (SPC) is used to detect simultaneously changes in several model characteristics after their occurrence which may take place at unknown time points. The methods of multivariate SPC are widely used in many fields of science, like engineering, economics, medicine, chemistry, biology, and finance (see, e.g., Frisén (1992), Sonesson and Bock (2003), Lawson and Kleinman (2005), Schipper and Schmid (2001), Andersson et al. (2004), Schmid and Tzotchev (2004), Messaoud et al. (2008), Bodnar (2007), Golosnoy et al. (2011), Bodnar (2009), among others).

The first multivariate control chart was introduced by Hotelling (1947). It is based on the Mahalanobis distance between the vector of observations and the target vector of the characteristics. A control chart based on an MEWMA (multivariate exponentially weighted moving average) recursion was suggested by Lowry et al. (1992). Crosier (1988), Pignatiello and Runger (1990), and Ngai and Zhang (2001) proposed several multivariate CUSUM (cumulative sum) control charts.

The mentioned above multivariate control procedures were designed for independent observations, which appears to be a rather restrictive assumption in practice (see, Alwan and Roberts (1988)). Theodossiou (1993) proposed an extension of the multivariate CUSUM control chart designed to detect changes in the mean of vector autoregressive moving average (VARMA) processes. The extension of the MEWMA control chart to dependent observations was given in Kramer and Schmid (1997), while Bodnar and Schmid (2007, 2011) derived several CUSUM schemes for detecting changes in the mean vector of multivariate time series.

The problem of sequential monitoring of process parameters becomes extremely challenging, when the process dimension is very large. The classical approaches usually cannot be used in such a situation since the control statistics depend on the inverse covariance matrix. The computation of the inverse covariance matrix can be extremely time consuming in practice and also numerical issues can take place. As a result, modifications of the existing approaches are necessary when the parameters of a high-dimensional time series should be monitored.

Several statistical control procedures based on the high-dimensional data have been proposed recently. Wang and Jiang (2009) considered a variable-selection-based multivariate statistical process control procedure for monitoring the process parameters and fault diagnosis in high dimension. A forward-selection algorithm was utilized to screen out potential out-of-control variables followed by control charts to monitor the suspicious variables. A high-dimensional control chart approach for profile monitoring was suggested by Chen and Nembhard (2011) which is based on the adaptive Neyman test statistic for the coefficients of the discrete Fourier transform of profiles. Li et al. (2014) proposed a chart that starts monitoring with the second observation regardless of the dimensionality and reduces the average run length in detecting early shifts in high-dimensionality measurements. More recently, Wang et al. (2017b) constructed a hybrid control chart in the case of independent multivariate Poisson data which is based on a goodness-of-fit test. A number of challenges appear when these approaches are adopted to the case of dependent data. Although several high-dimensional time series models exist in the literature (see, e.g., Lam et al. (2012); Han and Liu (2013); Chudik and Pesaran (2013); Han et al. (2015); Kock and Callot (2015); Gupta and Robinson (2015); Liu et al. (2015); Hall et al. (2016); Aue et al. (2017a,b); Dias and Kapetanios (2017); Gupta and Robinson (2017); Wang et al. (2017a)), it seems that the problem of sequential monitoring of their parameters have not been treated up to now.

The rest of the paper is organized as follows. The change-point model is presented in Section 2, while Section 3 provides the results for the multivariate EWMA control chart based on the Mahalanobis distance. The new control schemes designed for high-dimensional time series are proposed in Section 4. Section 5 shows the results of the comparison study. Section 6 summarizes the obtained results, while technical proofs are moved to the appendix (see, Section 7).

2 Change-point model

We denote the observed process by $\{\mathbf{X}_t\}$. It is assumed to be a p-dimensional time series. The target process is denoted by $\{\mathbf{Y}_t\}$. The relationship between both processes is given by the change point model

$$\mathbf{X}_t = \begin{cases} \mathbf{Y}_t & \text{for} \quad t < \tau \\ \mathbf{Y}_t + \mathbf{a} & \text{for} \quad t \geq \tau \end{cases}, \qquad t \in \mathbb{Z}, \tag{1}$$

where $\mathbf{a} \neq \mathbf{0}$ and $\tau \in \mathbb{N} \cup \{\infty\}$. Thus both processes may differ by a change in the mean behaviour. We say that the observed process is in control if $\tau = \infty$, else it is said to be out of control. In the following the symbols $\mathbb{E}_\infty(.)$, $\mathbb{V}ar_\infty(.)$, and $\mathbb{C}ov_\infty(.)$ will denote the mean, the variance, and the covariance matrix, respectively, computed under the assumption of no change, that is in the in-control state.

Throughout the paper it is supposed that $\{\mathbf{Y}_t\}$ is a weakly stationary time series with $\mathbb{E}(\mathbf{Y}_t) = \boldsymbol{\mu} = \mathbb{E}_\infty(\mathbf{X}_t)$ and $\mathbb{C}ov(\mathbf{Y}_{t+h}, \mathbf{Y}_t) = \boldsymbol{\Gamma}(h) = \mathbb{C}ov_\infty(\mathbf{X}_{t+h}, \mathbf{X}_t)$.

3 The multivariate EWMA chart for time series

We consider the multivariate EWMA control chart for time series introduced by Kramer and Schmid (1997). It is based on the recursion

$$\mathbf{Z}_t = (\mathbf{I} - \mathbf{R})\mathbf{Z}_{t-1} + \mathbf{R}\mathbf{X}_t, \quad t \geq 1 \tag{2}$$

with $\mathbf{Z}_0 = \boldsymbol{\mu}$ where $\mathbf{R} = diag(r_1, ..., r_p)$ with $r_1, ..., r_p \in (0, 1]$. The recursion can also be written as

$$\mathbf{Z}_t - \boldsymbol{\mu} = (\mathbf{I} - \mathbf{R})(\mathbf{Z}_{t-1} - \boldsymbol{\mu}) + \mathbf{R}(\mathbf{X}_t - \boldsymbol{\mu}), \quad t \geq 1$$

and thus

$$\begin{aligned} \mathbf{Z}_t &= \boldsymbol{\mu} + \sum_{i=0}^{t-1} (\mathbf{I} - \mathbf{R})^i \mathbf{R}(\mathbf{X}_{t-i} - \boldsymbol{\mu}) = \sum_{i=0}^{t-1} (\mathbf{I} - \mathbf{R})^i \mathbf{R}\mathbf{X}_{t-i} + (\mathbf{I} - \mathbf{R})^t \boldsymbol{\mu} \\ &= \boldsymbol{\mu} + \sum_{i=0}^{t-1} (\mathbf{I} - \mathbf{R})^i \mathbf{R}(\mathbf{Y}_{t-i} - \boldsymbol{\mu}) + \sum_{i=0}^{t-\tau} (\mathbf{I} - \mathbf{R})^i \mathbf{R}\mathbf{a} \end{aligned} \tag{3}$$

for $t = 1, 2, ...$ Then it holds for p fixed that (cf., Kramer and Schmid (1997))

$$\begin{aligned} \mathbb{E}(\mathbf{Z}_t) &= \boldsymbol{\mu} + (\mathbf{I} - (\mathbf{I} - \mathbf{R})^{t-\tau+1})\mathbf{a} I_{\{\tau, \tau+1, ...\}}(t) \\ &\xrightarrow[t \to \infty]{} \boldsymbol{\mu} + \mathbf{a} \end{aligned}$$

and

$$\begin{aligned} \mathbb{C}ov(\mathbf{Z}_t) &= \mathbf{R} \sum_{i,j=0}^{t-1} (\mathbf{I} - \mathbf{R})^i \boldsymbol{\Gamma}(j - i)(\mathbf{I} - \mathbf{R})^j \mathbf{R} \\ &\xrightarrow[t \to \infty]{} \mathbf{R} \sum_{i,j=0}^{\infty} (\mathbf{I} - \mathbf{R})^i \boldsymbol{\Gamma}(j - i)(\mathbf{I} - \mathbf{R})^j \mathbf{R}, \end{aligned}$$

provided that $\{\boldsymbol{\Gamma}(v)\}$ is absolutely convergent. Note that the mean change does not influence the covariance matrix of $\mathbf{Z}_t$, i.e. $\mathbb{C}ov(\mathbf{Z}_t) = \mathbb{C}ov_\infty(\mathbf{Z}_t)$.

Now let

$$\boldsymbol{\Sigma}_{t,p} = \mathbb{C}ov_\infty(\mathbf{Z}_t), \quad \boldsymbol{\Sigma}_{l;p} = \lim_{t \to \infty} \mathbb{C}ov_\infty(\mathbf{Z}_t) = \mathbf{R} \sum_{i,j=0}^{\infty} (\mathbf{I} - \mathbf{R})^i \boldsymbol{\Gamma}(j - i)(\mathbf{I} - \mathbf{R})^j \mathbf{R}. \tag{4}$$

The MEWMA control chart is constructed by determining the control statistics at each time point t as the Mahalanobis distance of the vector $\mathbf{Z}_t$ given by

$$T_{Mah,t} = (\mathbf{Z}_t - \boldsymbol{\mu})'\boldsymbol{\Sigma}_{t,p}^{-1}(\mathbf{Z}_t - \boldsymbol{\mu}) \quad \text{or} \quad T_{MahInf,t} = (\mathbf{Z}_t - \boldsymbol{\mu})'\boldsymbol{\Sigma}_{l;p}^{-1}(\mathbf{Z}_t - \boldsymbol{\mu}), \tag{5}$$

depending on whether the exact or asymptotic covariance matrix of $\mathbf{Z}_t$ is used (cf., Kramer and Schmid (1997)). It is concluded that there is a change at time t if the control statistics are sufficiently large.

Note that contrary to independent samples the MEWMA chart for stationary processes is in general not directionally invariant. Kramer and Schmid (1997) proved an invariance property of the MEWMA chart for a VAR(1) process. They gave a sufficient condition such that the in-control ARL does not depend on the covariance matrix of the white noise process. If this holds, then the analysis of the MEWMA chart turns out to be much simpler.

Next we will analyze the distributional behavior of $T_{Mah,t}$ and $T_{MahInf,t}$, in particular in the high-dimensional setting. The following notations will be used throughout the paper.

$$\mathbf{a}_{t-\tau} = (\mathbf{I} - (\mathbf{I} - \mathbf{R})^{t-\tau+1})\mathbf{a}I_{\{0,1,2,\ldots\}}(t-\tau), \quad \zeta_{\tau,t,p} = \mathbf{a}_{t-\tau}'\boldsymbol{\Sigma}_{t,p}^{-1}\mathbf{a}_{t-\tau}, \quad \zeta_{l;p} = \mathbf{a}'\boldsymbol{\Sigma}_{l;p}^{-1}\mathbf{a}. \tag{6}$$

First, we focus on the control statistic based on the exact covariance matrix.

Lemma 3.1. *Let $\{\mathbf{Y}_t\}$ be a stationary Gaussian process with $\mathbb{E}(\mathbf{Y}_t) = \boldsymbol{\mu}$ and $\mathbb{C}ov(\mathbf{Y}_{t+h}, \mathbf{Y}_t) = \boldsymbol{\Gamma}(h)$. Let τ be fixed.*

a) Then $(\mathbf{Z}_t - \boldsymbol{\mu})'\boldsymbol{\Sigma}_{t,p}^{-1}(\mathbf{Z}_t - \boldsymbol{\mu}) \sim \chi^2_{p,\zeta_{\tau,t,p}}$ and

$$\mathbb{E}((\mathbf{Z}_t - \boldsymbol{\mu})'\boldsymbol{\Sigma}_{t,p}^{-1}(\mathbf{Z}_t - \boldsymbol{\mu})) = p + \zeta_{\tau,t,p}, \mathbb{V}ar((\mathbf{Z}_t - \boldsymbol{\mu})'\boldsymbol{\Sigma}_{t,p}^{-1}(\mathbf{Z}_t - \boldsymbol{\mu})) = 2(p + 2\zeta_{\tau,t,p}).$$

b) Suppose that $\{\boldsymbol{\Gamma}(v)\}$ is absolutely convergent. Let p be fixed, then

$$(\mathbf{Z}_t - \boldsymbol{\mu})'\boldsymbol{\Sigma}_{t,p}^{-1}(\mathbf{Z}_t - \boldsymbol{\mu}) \xrightarrow[t\to\infty]{d} \chi^2_{p,\zeta_{l;p}}.$$

c) Suppose that $\lim_{p\to\infty} \frac{\zeta_{\tau,t,p}}{p} < \infty$. Let t be fixed, then

$$\frac{(\mathbf{Z}_t - \boldsymbol{\mu})'\boldsymbol{\Sigma}_{t,p}^{-1}(\mathbf{Z}_t - \boldsymbol{\mu}) - p - \zeta_{\tau,t,p}}{\sqrt{2(p + 2\zeta_{\tau,t,p})}} \xrightarrow[p\to\infty]{d} N(0,1).$$

Proof. It holds that $\mathbf{Z}_t - \boldsymbol{\mu} \sim \mathcal{N}_p(\mathbf{a}_{t-\tau}, \boldsymbol{\Sigma}_{t,p})$. Thus the proofs of part a) and b) follow immediately for $(\mathbf{Z}_t - \boldsymbol{\mu})'\boldsymbol{\Sigma}_{t,p}^{-1}(\mathbf{Z}_t - \boldsymbol{\mu})$. In order to prove part c) for $(\mathbf{Z}_t - \boldsymbol{\mu})'\boldsymbol{\Sigma}_{t,p}^{-1}(\mathbf{Z}_t - \boldsymbol{\mu})$ we apply Lemma 3 of Bodnar and Reiß (2016). □

In the next lemma the behavior of the control statistic based on the asymptotic covariance matrix for t tending to infinity is analyzed. Since this quantity is easier to determine it is frequently used in applications. However, in the high-dimensional situation it is questionable whether such an approach makes sense. If t is small and p is large the approximation to the exact covariance matrix may be weak.

Lemma 3.2. *Let $\{\mathbf{Y}_t\}$ be a stationary Gaussian process with $\mathbb{E}(\mathbf{Y}_t) = \boldsymbol{\mu}$ and $\mathbb{C}ov(\mathbf{Y}_{t+h}, \mathbf{Y}_t) = \boldsymbol{\Gamma}(h)$.*

a) Then

$$\mathbb{E}((\mathbf{Z}_t - \boldsymbol{\mu})'\boldsymbol{\Sigma}_{l;p}^{-1}(\mathbf{Z}_t - \boldsymbol{\mu})) = tr(\boldsymbol{\Sigma}_{l;p}^{-1}\boldsymbol{\Sigma}_{t,p}) + \mathbf{a}_{t-\tau}'\boldsymbol{\Sigma}_{l;p}^{-1}\mathbf{a}_{t-\tau},$$

$$\mathbb{V}ar((\mathbf{Z}_t - \boldsymbol{\mu})'\boldsymbol{\Sigma}_{l;p}^{-1}(\mathbf{Z}_t - \boldsymbol{\mu})) = 2tr((\boldsymbol{\Sigma}_{l;p}^{-1}\boldsymbol{\Sigma}_{t,p})^2) + 4\mathbf{a}_{t-\tau}'\boldsymbol{\Sigma}_{l;p}^{-1}\boldsymbol{\Sigma}_{t,p}\boldsymbol{\Sigma}_{l;p}^{-1}\mathbf{a}_{t-\tau}.$$

b) Suppose that $\{\boldsymbol{\Gamma}(v)\}$ is absolutely convergent. Let p be fixed, then

$$(\mathbf{Z}_t - \boldsymbol{\mu})'\boldsymbol{\Sigma}_{l;p}^{-1}(\mathbf{Z}_t - \boldsymbol{\mu}) \xrightarrow[t\to\infty]{d} \chi^2_{p,\zeta_{l;p}}.$$

c) *Suppose that* $\lim_{p\to\infty} \frac{\zeta_{\tau,t,p}}{p} < \infty$. *Let* t *be fixed, then*

$$\frac{(\mathbf{Z}_t-\boldsymbol{\mu})'\boldsymbol{\Sigma}_{l;p}^{-1}(\mathbf{Z}_t-\boldsymbol{\mu}) - tr\left(\boldsymbol{\Sigma}_{l;p}^{-1}\boldsymbol{\Sigma}_{t,p}\right) - \mathbf{a}_{t-\tau}'\boldsymbol{\Sigma}_{l;p}^{-1}\mathbf{a}_{t-\tau}}{\sqrt{2tr(\boldsymbol{\Sigma}_{l;p}^{-1}\boldsymbol{\Sigma}_{t,p}\boldsymbol{\Sigma}_{l;p}^{-1}\boldsymbol{\Sigma}_{t,p}) + 4\mathbf{a}_{t-\tau}'\boldsymbol{\Sigma}_{l;p}^{-1}\boldsymbol{\Sigma}_{t,p}\boldsymbol{\Sigma}_{l;p}^{-1}\mathbf{a}_{t-\tau}}} \xrightarrow[p\to\infty]{d} \mathcal{N}(0,1).$$

Proof. Since $\mathbf{Z}_t-\boldsymbol{\mu} = \mathbf{Z}_t - \mathbb{E}(\mathbf{Z}_t) + \mathbb{E}(\mathbf{Z}_t) - \boldsymbol{\mu}$ we get that

$$\mathbb{E}((\mathbf{Z}_t-\boldsymbol{\mu})(\mathbf{Z}_t-\boldsymbol{\mu})') = \boldsymbol{\Sigma}_{t,p} + (\mathbb{E}(\mathbf{Z}_t)-\boldsymbol{\mu})(\mathbb{E}(\mathbf{Z}_t)-\boldsymbol{\mu})'$$

and thus the first part of a) follows. Now $\mathbf{Z}_t-\boldsymbol{\mu} \sim \mathcal{N}_p(\mathbf{a}_{t-\tau}, \boldsymbol{\Sigma}_{t,p})$ and the second part of a) follows with (3.2b.10) of Mathai and Provost (1992).

Applying that

$$(\mathbf{Z}_t-\boldsymbol{\mu})'\boldsymbol{\Sigma}_{l;p}^{-1}(\mathbf{Z}_t-\boldsymbol{\mu}) = (\boldsymbol{\Sigma}_{t,p}^{-1/2}(\mathbf{Z}_t-\boldsymbol{\mu}))'\boldsymbol{\Sigma}_{t,p}^{1/2}\boldsymbol{\Sigma}_{l;p}^{-1}\boldsymbol{\Sigma}_{t,p}^{1/2}(\boldsymbol{\Sigma}_{t,p}^{-1/2}(\mathbf{Z}_t-\boldsymbol{\mu}))$$

part b) is obtained.

Finally, part (c) of the lemma follows from $\mathbf{Z}_t-\boldsymbol{\mu} \sim \mathcal{N}_p(\mathbf{a}_{t-\tau}, \boldsymbol{\Sigma}_{t,p})$ and Lemma 7.1 given in the appendix. □

Since $(\mathbf{Z}_t-\boldsymbol{\mu})'\boldsymbol{\Sigma}_{l;p}^{-1}(\mathbf{Z}_t-\boldsymbol{\mu})$ is a quadratic form its exact distribution can be written as a series expression (cf., Mathai and Provost (1992)). Furthermore, from part b) it holds that for p fixed

$$P\left(\frac{(\mathbf{Z}_t-\boldsymbol{\mu})'\boldsymbol{\Sigma}_{t,p}^{-1}(\mathbf{Z}_t-\boldsymbol{\mu}) - p - \zeta_{\tau,t,p}}{\sqrt{2(p+2\zeta_{\tau,t,p})}} \le x\right) \xrightarrow[t\to\infty]{} \chi^2_{p,\zeta_{l;p}}(\sqrt{2(p+2\zeta_{l;p})}x + p + \zeta_{l;p}).$$

The practical calculation of $\boldsymbol{\Sigma}_{t,p}$ and $\boldsymbol{\Sigma}_{l;p}$ turns out to be difficult. Explicit formulas can only be obtained for special cases.

Lemma 3.3. *Let* $\{\mathbf{Y}_t\}$ *be a stationary VAR(1) process given by*

$$\mathbf{Y}_t = \boldsymbol{\Phi}\mathbf{Y}_{t-1} + \boldsymbol{\varepsilon}_t$$

with $\mathbb{C}ov(\mathbf{Y}_{t+h}, \mathbf{Y}_t) = \boldsymbol{\Gamma}(h)$, *where*

1) $\{\boldsymbol{\varepsilon}_t\}$ *is i.i.d.*, $\mathbb{E}(\boldsymbol{\varepsilon}_t) = \mathbf{0}$ *and* $\mathbb{C}ov(\boldsymbol{\varepsilon}_t, \boldsymbol{\varepsilon}_t) = \boldsymbol{\Sigma}$;

2) $\boldsymbol{\Phi} = \varphi\,\mathbf{I}$, *with* $|\varphi| < 1$.

Moreover, let $\mathbf{R} = r\,\mathbf{I}$ *with* $r \in (0,1]$. *Then, it holds that*

a) *if* $1-r \neq \varphi$, *then*

$$\boldsymbol{\Sigma}_{t,p} = c_t(r,\varphi)\,\boldsymbol{\Gamma}(0) \tag{7}$$

with

$$\begin{aligned} c_t(r,\varphi) &= \frac{r}{2-r}\,\frac{1}{1-(1-r)\varphi}\Bigg(1+\varphi(1-r) \\ &+ \varphi\frac{\varphi^t-(1-r)^t}{\varphi-(1-r)}\,(\varphi^t-(1-r)^t+(1-r)^{t+2}) \\ &- \frac{\varphi^{2t+1}-(1-r)^{2t+1}}{\varphi-(1-r)} + \varphi^2(1-r)^{t+2}\,\frac{\varphi^{t-1}-(1-r)^{t-1}}{\varphi-(1-r)}\Bigg) \end{aligned} \tag{8}$$

and

$$\boldsymbol{\Sigma}_{l;p} = c(r,\varphi)\,\boldsymbol{\Gamma}(0)$$

with

$$c(r,\varphi) = \lim_{t\to\infty} c_t(r,\varphi) = \frac{r}{2-r}\,\frac{1+\varphi(1-r)}{1-(1-r)\varphi}.$$

b) If $1 - r = \varphi$*, then*

$$\mathbf{\Sigma}_{t,p} = \frac{\mathbf{\Gamma}(0)}{(1+\varphi)^2}[(1+\varphi^2)(1-\varphi^{2t}) - 2t\varphi^{2t}(1-\varphi^2)] \tag{9}$$

and

$$\mathbf{\Sigma}_{l;p} = \frac{1+\varphi^2}{(1+\varphi)^2}\mathbf{\Gamma}(0).$$

The proof of Lemma 3.3 is given in the appendix. In the case $1 - r = \varphi$ we interpret the value of $c_t(r, \varphi)$ as the limit if φ converges to $1 - r$ and the limit coincides with the quantity given in (9).

The MEWMA chart was mainly applied in low-dimensional spaces. However, recently we are faced with situations where the amount of collected data is huge and thus it is of interest to monitor high-dimensional processes. Thus the question arises how good this chart behaves in a high-dimensional context. Following this approach, we might face a number of difficulties in a high-dimensional context. One of them is the computation of the inverse matrix $\mathbf{\Sigma}_{t,p}^{-1}$, which in general depends on t and has to be inverted at each time point when it is checked whether the process is still in-control. Even the application of the asymptotic covariance matrix $\mathbf{\Sigma}_{l;p}^{-1}$ might lead to some difficulties. For example, due to the dimensionality issue, the resulting covariance matrix might be ill-conditioned or its computation might be time consuming.

4 Control charts based on Euclidean type distances

4.1 Introduction of the control schemes

Recently, several authors considered 2-sample tests for high-dimensions (e.g., Bai and Saranadasa (1996), Chen and Qin (2010)). They proposed in principle to make use of the Euclidean distance or approximations to this distance in a high-dimensional setting. Motivated by the high-dimensional discriminant analysis (see, e.g., Bodnar et al. (2016)), we consider three types of control statistics which are based on $(\mathbf{Z}_t - \boldsymbol{\mu})'(\mathbf{Z}_t - \boldsymbol{\mu})$, $(\mathbf{Z}_t - \boldsymbol{\mu})'\mathbf{\Sigma}_{d;t,p}^{-1}(\mathbf{Z}_t - \boldsymbol{\mu})$, and $(\mathbf{Z}_t - \boldsymbol{\mu})'\mathbf{\Sigma}_{d;l;p}^{-1}(\mathbf{Z}_t - \boldsymbol{\mu})$ where $\mathbf{\Sigma}_{d;t,p}$ and $\mathbf{\Sigma}_{d;l;p}$ are obtained from $\mathbf{\Sigma}_{t,p}$ and $\mathbf{\Sigma}_{l;p}$ by setting all their non-diagonal elements equal to zero. The process is concluded to be out of control if the value of a control statistic is larger than a control limit $c > 0$.

The choice of the control limit c plays a central role in the analysis. If c is chosen small the chart will trigger many signals while for a large value of c signals will rarely occur. Frequently, c is chosen such that the in-control ARL is equal to a predetermined value. In engineering, the in-control ARL is often taken to be 500, in finance equal to 60. Essentially, this depends on the frequency of the obtained observations.

The calculation of the average run length turns out to be quite challenging in the present case. Note that for univariate time series an explicit formula for the ARL is only known for special type of processes as, e.g., exchangeable variables (cf., Schmid (1995)). Thus mostly simulations are used to determine the control limit. This approach is of course time consuming since for each parameter constellation and dimension the value must be calculated. Here we are confronted again with the problem of dimensionality.

Practitioners often choose a more simpler procedure. They are working with 3σ control limits. In order to apply such a procedure it is first necessary to determine characteristic quantities of the control statistics.

4.2 Determination of the control design

Following the proof of Lemma 3.2 we get the following two results.

Theorem 4.1. *Let* $\{\mathbf{Y}_t\}$ *be a stationary Gaussian process with* $\mathbb{E}(\mathbf{Y}_t) = \boldsymbol{\mu}$ *and* $\mathbb{C}ov(\mathbf{Y}_{t+h}, \mathbf{Y}_t) = \mathbf{\Gamma}(h)$*.*

a) Then

$$\mathbb{E}((\mathbf{Z}_t-\boldsymbol{\mu})'(\mathbf{Z}_t-\boldsymbol{\mu})) = tr(\boldsymbol{\Sigma}_{t,p}) + \mathbf{a}'_{t-\tau}\mathbf{a}_{t-\tau},$$

$$\mathbb{V}ar((\mathbf{Z}_t-\boldsymbol{\mu})'(\mathbf{Z}_t-\boldsymbol{\mu})) = 2tr(\boldsymbol{\Sigma}^2_{t,p}) + 4\mathbf{a}'_{t-\tau}\boldsymbol{\Sigma}_{t,p}\mathbf{a}_{t-\tau}.$$

b) Suppose that $\{\boldsymbol{\Gamma}(v)\}$ is absolutely convergent. Let p be fixed, then

$$(\mathbf{Z}_t-\boldsymbol{\mu})'(\mathbf{Z}_t-\boldsymbol{\mu}) \xrightarrow[t\to\infty]{d} \sum_{i=1}^{p}\lambda_{Eu;l;i}\chi^2_{1,\zeta_{Eu;l;i}},$$

where $\lambda_{Eu;l;1},...,\lambda_{Eu;l;p}$ are the eigenvalues of $\boldsymbol{\Sigma}_{t,p}$ and $\zeta_{Eu;l;1},...,\zeta_{Eu;l;p}$ are the components of the vector $\boldsymbol{\zeta}_{Eu;l}=\mathbf{U}'_{Eu}\boldsymbol{\Sigma}^{-1/2}_{t,p}\mathbf{a}_{t-\tau}$ with $\mathbf{U}_{Eu}$ the matrix of eigenvectors of $\boldsymbol{\Sigma}_{t,p}$.

c) Suppose that $\lim_{p\to\infty}\frac{\zeta_{\tau,t,p}}{p}<\infty$. Let t be fixed, then

$$\frac{(\mathbf{Z}_t-\boldsymbol{\mu})'(\mathbf{Z}_t-\boldsymbol{\mu}) - tr(\boldsymbol{\Sigma}_{t,p}) - \mathbf{a}'_{t-\tau}\mathbf{a}_{t-\tau}}{\sqrt{2tr(\boldsymbol{\Sigma}^2_{t,p}) + 4\mathbf{a}'_{t-\tau}\boldsymbol{\Sigma}_{t,p}\mathbf{a}_{t-\tau}}} \xrightarrow[p\to\infty]{d} \mathcal{N}(0,1).$$

Theorem 4.2. *Let $\{\mathbf{Y}_t\}$ be a stationary Gaussian process with $\mathbb{E}(\mathbf{Y}_t)=\boldsymbol{\mu}$ and $\mathbb{C}ov(\mathbf{Y}_{t+h},\mathbf{Y}_t)=\boldsymbol{\Gamma}(h)$.*

a) Then

$$\mathbb{E}((\mathbf{Z}_t-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}_{d;t,p}(\mathbf{Z}_t-\boldsymbol{\mu})) = tr(\boldsymbol{\Sigma}^{-1}_{d;t,p}\boldsymbol{\Sigma}_{t,p}) + \mathbf{a}'_{t-\tau}\boldsymbol{\Sigma}^{-1}_{d;t,p}\mathbf{a}_{t-\tau},$$

$$\mathbb{V}ar((\mathbf{Z}_t-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}_{d;t,p}(\mathbf{Z}_t-\boldsymbol{\mu})) = 2tr((\boldsymbol{\Sigma}^{-1}_{d;t,p}\boldsymbol{\Sigma}_{t,p})^2) + 4\mathbf{a}'_{t-\tau}\boldsymbol{\Sigma}^{-1}_{d;t,p}\boldsymbol{\Sigma}_{t,p}\boldsymbol{\Sigma}^{-1}_{d;t,p}\mathbf{a}_{t-\tau}.$$

b) Suppose that $\{\boldsymbol{\Gamma}(v)\}$ is absolutely convergent. Let p be fixed, then

$$(\mathbf{Z}_t-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}_{d;t,p}(\mathbf{Z}_t-\boldsymbol{\mu}) \xrightarrow[t\to\infty]{d} \sum_{i=1}^{p}\lambda_{dEu;l;i}\chi^2_{1,\zeta_{dEu;l;i}},$$

where $\lambda_{dEu;l;1},...,\lambda_{dEu;l;p}$ are the eigenvalues of $\boldsymbol{\Sigma}^{1/2}_{t,p}\boldsymbol{\Sigma}^{-1}_{d;t,p}\boldsymbol{\Sigma}^{1/2}_{t,p}$ and $\zeta_{dEu;l;1},...,\zeta_{dEu;l;p}$ are the components of the vector $\boldsymbol{\zeta}_{dEu;l}=\mathbf{U}'_{dEu}\boldsymbol{\Sigma}^{-1/2}_{t,p}\mathbf{a}_{t-\tau}$ with $\mathbf{U}_{dEu}$ the matrix of eigenvectors of $\boldsymbol{\Sigma}^{1/2}_{t,p}\boldsymbol{\Sigma}^{-1}_{d;t,p}\boldsymbol{\Sigma}^{1/2}_{t,p}$.

c) Suppose that $\lim_{p\to\infty}\frac{\zeta_{\tau,t,p}}{p}<\infty$. Let t be fixed, then

$$\frac{(\mathbf{Z}_t-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}_{d;t,p}(\mathbf{Z}_t-\boldsymbol{\mu}) - tr\left(\boldsymbol{\Sigma}^{-1}_{d;t,p}\boldsymbol{\Sigma}_{t,p}\right) - \mathbf{a}'_{t-\tau}\boldsymbol{\Sigma}^{-1}_{d;t,p}\mathbf{a}_{t-\tau}}{\sqrt{2tr(\boldsymbol{\Sigma}^{-1}_{d;t,p}\boldsymbol{\Sigma}_{t,p}\boldsymbol{\Sigma}^{-1}_{d;t,p}\boldsymbol{\Sigma}_{t,p}) + 4\mathbf{a}'_{t-\tau}\boldsymbol{\Sigma}^{-1}_{d;t,p}\boldsymbol{\Sigma}_{t,p}\boldsymbol{\Sigma}^{-1}_{d;t,p}\mathbf{a}_{t-\tau}}} \xrightarrow[p\to\infty]{d} \mathcal{N}(0,1).$$

The results of Theorems 4.1 and 4.2 are used to determined the designs of several control charts, that are based on the multivariate EWMA recursion of Section 3 with the control statistics computed by using the Euclidean norm and the norm employing the diagonal matrix. Depending on the usage of the exact mean and variance of the control statistics or their asymptotic couinterparts, several control schemes are obtained which are the following

$$
\begin{aligned}
T_{1,t} &= \frac{(\mathbf{Z}_t-\boldsymbol{\mu})'(\mathbf{Z}_t-\boldsymbol{\mu})-\mathbb{E}_\infty((\mathbf{Z}_t-\boldsymbol{\mu})'(\mathbf{Z}_t-\boldsymbol{\mu}))}{\sqrt{\mathbb{V}ar_\infty((\mathbf{Z}_t-\boldsymbol{\mu})'(\mathbf{Z}_t-\boldsymbol{\mu}))}},\\
T_{2,t} &= \frac{(\mathbf{Z}_t-\boldsymbol{\mu})'(\mathbf{Z}_t-\boldsymbol{\mu})-\lim_{t\to\infty}\mathbb{E}_\infty((\mathbf{Z}_t-\boldsymbol{\mu})'(\mathbf{Z}_t-\boldsymbol{\mu}))}{\sqrt{\mathbb{V}ar_\infty((\mathbf{Z}_t-\boldsymbol{\mu})'(\mathbf{Z}_t-\boldsymbol{\mu}))}},\\
T_{3,t} &= \frac{(\mathbf{Z}_t-\boldsymbol{\mu})'(\mathbf{Z}_t-\boldsymbol{\mu})-\mathbb{E}_\infty((\mathbf{Z}_t-\boldsymbol{\mu})'(\mathbf{Z}_t-\boldsymbol{\mu}))}{\sqrt{\lim_{t\to\infty}\mathbb{V}ar_\infty((\mathbf{Z}_t-\boldsymbol{\mu})'(\mathbf{Z}_t-\boldsymbol{\mu}))}},\\
T_{4,t} &= \frac{(\mathbf{Z}_t-\boldsymbol{\mu})'(\mathbf{Z}_t-\boldsymbol{\mu})-\lim_{t\to\infty}\mathbb{E}_\infty((\mathbf{Z}_t-\boldsymbol{\mu})'(\mathbf{Z}_t-\boldsymbol{\mu}))}{\sqrt{\lim_{t\to\infty}\mathbb{V}ar_\infty((\mathbf{Z}_t-\boldsymbol{\mu})'(\mathbf{Z}_t-\boldsymbol{\mu}))}},\\
T_{5,t} &= \frac{(\mathbf{Z}_t-\boldsymbol{\mu})'(\mathbf{Z}_t-\boldsymbol{\mu})-\lim_{p\to\infty}\mathbb{E}_\infty((\mathbf{Z}_t-\boldsymbol{\mu})'(\mathbf{Z}_t-\boldsymbol{\mu}))}{\sqrt{\lim_{p\to\infty}\mathbb{V}ar_\infty((\mathbf{Z}_t-\boldsymbol{\mu})'(\mathbf{Z}_t-\boldsymbol{\mu}))}},\\
T_{6,t} &= \frac{(\mathbf{Z}_t-\boldsymbol{\mu})'\boldsymbol{\Sigma}_{d;t,p}^{-1}(\mathbf{Z}_t-\boldsymbol{\mu})-\mathbb{E}_\infty((\mathbf{Z}_t-\boldsymbol{\mu})'\boldsymbol{\Sigma}_{d;t,p}^{-1}(\mathbf{Z}_t-\boldsymbol{\mu}))}{\sqrt{\mathbb{V}ar_\infty((\mathbf{Z}_t-\boldsymbol{\mu})'\boldsymbol{\Sigma}_{d;t,p}^{-1}(\mathbf{Z}_t-\boldsymbol{\mu}))}},\\
T_{7,t} &= \frac{(\mathbf{Z}_t-\boldsymbol{\mu})'\boldsymbol{\Sigma}_{d;t,p}^{-1}(\mathbf{Z}_t-\boldsymbol{\mu})-\lim_{t\to\infty}\mathbb{E}_\infty((\mathbf{Z}_t-\boldsymbol{\mu})'\boldsymbol{\Sigma}_{d;t,p}^{-1}(\mathbf{Z}_t-\boldsymbol{\mu}))}{\sqrt{\mathbb{V}ar_\infty((\mathbf{Z}_t-\boldsymbol{\mu})'\boldsymbol{\Sigma}_{d;t,p}^{-1}(\mathbf{Z}_t-\boldsymbol{\mu}))}},\\
T_{8,t} &= \frac{(\mathbf{Z}_t-\boldsymbol{\mu})'\boldsymbol{\Sigma}_{d;t,p}^{-1}(\mathbf{Z}_t-\boldsymbol{\mu})-\mathbb{E}_\infty((\mathbf{Z}_t-\boldsymbol{\mu})'\boldsymbol{\Sigma}_{d;t,p}^{-1}(\mathbf{Z}_t-\boldsymbol{\mu}))}{\sqrt{\lim_{t\to\infty}\mathbb{V}ar_\infty((\mathbf{Z}_t-\boldsymbol{\mu})'\boldsymbol{\Sigma}_{d;t,p}^{-1}(\mathbf{Z}_t-\boldsymbol{\mu}))}},\\
T_{9,t} &= \frac{(\mathbf{Z}_t-\boldsymbol{\mu})'\boldsymbol{\Sigma}_{d;t,p}^{-1}(\mathbf{Z}_t-\boldsymbol{\mu})-\lim_{t\to\infty}\mathbb{E}_\infty((\mathbf{Z}_t-\boldsymbol{\mu})'\boldsymbol{\Sigma}_{d;t,p}^{-1}(\mathbf{Z}_t-\boldsymbol{\mu}))}{\sqrt{\lim_{t\to\infty}\mathbb{V}ar_\infty((\mathbf{Z}_t-\boldsymbol{\mu})'\boldsymbol{\Sigma}_{d;t,p}^{-1}(\mathbf{Z}_t-\boldsymbol{\mu}))}},\\
T_{10,t} &= \frac{(\mathbf{Z}_t-\boldsymbol{\mu})'\boldsymbol{\Sigma}_{d;t,p}^{-1}(\mathbf{Z}_t-\boldsymbol{\mu})-\lim_{p\to\infty}\mathbb{E}_\infty((\mathbf{Z}_t-\boldsymbol{\mu})'\boldsymbol{\Sigma}_{d;t,p}^{-1}(\mathbf{Z}_t-\boldsymbol{\mu}))}{\sqrt{\lim_{p\to\infty}\mathbb{V}ar_\infty((\mathbf{Z}_t-\boldsymbol{\mu})'\boldsymbol{\Sigma}_{d;t,p}^{-1}(\mathbf{Z}_t-\boldsymbol{\mu}))}}.
\end{aligned}
$$

The control statistics $T_{1,t}$, $T_{2,t}$, $T_{3,t}$, $T_{4,t}$, and $T_{5,t}$ are more computationally efficient, since no inversion is used in their computations. On the other side, the control statistics $T_{6,t}$, $T_{7,t}$, $T_{8,t}$, $T_{9,t}$, and $T_{10,t}$ might be more sensitive to detect changes, when the components of $\mathbf{X}_t$ have different variability. To this end, we note that the control statistics $T_{5,t}$ and $T_{10,t}$ are applicable in special situations when the limits $\lim_{p\to\infty}\mathbb{E}_\infty((\mathbf{Z}_t-\boldsymbol{\mu})'(\mathbf{Z}_t-\boldsymbol{\mu}))$, $\lim_{p\to\infty}\mathbb{V}ar_\infty((\mathbf{Z}_t-\boldsymbol{\mu})'(\mathbf{Z}_t-\boldsymbol{\mu}))$, $\lim_{p\to\infty}\mathbb{E}_\infty((\mathbf{Z}_t-\boldsymbol{\mu})'\boldsymbol{\Sigma}_{d;t,p}^{-1}(\mathbf{Z}_t-\boldsymbol{\mu}))$, and $\lim_{p\to\infty}\mathbb{V}ar_\infty((\mathbf{Z}_t-\boldsymbol{\mu})'\boldsymbol{\Sigma}_{d;t,p}^{-1}(\mathbf{Z}_t-\boldsymbol{\mu}))$ can analytically be presented in terms of a finite number of model parameters. This is possible, for example, in the case of VARMA models where all parameter matrices are proportional to the identity matrix. In the simulation study of Section 5 we will consider a more general model for the target process $\{\mathbf{Y}_t\}$, and for this reason these two statistics will not be included.

The limiting values if the expression of the control statistics can be computed analytically and they are given by

$$
\begin{aligned}
\lim_{p\to\infty}\mathbb{E}_\infty((\mathbf{Z}_t-\boldsymbol{\mu})'(\mathbf{Z}_t-\boldsymbol{\mu})) &= \mathrm{tr}(\boldsymbol{\Sigma}_{l,p}),\\
\lim_{p\to\infty}\mathbb{V}ar_\infty((\mathbf{Z}_t-\boldsymbol{\mu})'(\mathbf{Z}_t-\boldsymbol{\mu})) &= 2\mathrm{tr}(\boldsymbol{\Sigma}_{l,p}^2),\\
\lim_{p\to\infty}\mathbb{E}_\infty((\mathbf{Z}_t-\boldsymbol{\mu})'\boldsymbol{\Sigma}_{d;t,p}^{-1}(\mathbf{Z}_t-\boldsymbol{\mu})) &= \mathrm{tr}(\boldsymbol{\Sigma}_{d;l,p}^{-1}\boldsymbol{\Sigma}_{l,p}),\\
\lim_{p\to\infty}\mathbb{V}ar_\infty((\mathbf{Z}_t-\boldsymbol{\mu})'\boldsymbol{\Sigma}_{d;t,p}^{-1}(\mathbf{Z}_t-\boldsymbol{\mu})) &= 2\mathrm{tr}((\boldsymbol{\Sigma}_{d;l,p}^{-1}\boldsymbol{\Sigma}_{l,p})^2),
\end{aligned}
$$

where $\boldsymbol{\Sigma}_{d;l,p}$ is obtained from $\boldsymbol{\Sigma}_{l,p}$ by setting all nondiagonal elements of $\boldsymbol{\Sigma}_{l,p}$ to zero.

5 Simulation study

The aim of this section is to investigate the performance of the control charts proposed in Section 4 and to compare them with the approaches based on the Mahalanobis distance as described in Section 3.

5.1 Results in the in-control state

To investigate the properties of the considered control schemes via simulations, one has to define both the target and the observed process. In this section, we use a VAR(1) process as a target process $\{\mathbf{Y}_t\}$, whose coefficient matrix is assumed to be proportional to the identity matrix, that is $\mathbf{\Phi} = \varphi \mathbf{I}$ with $\varphi = 0.5$. For this type of time series model, one can use the results of Lemma 3.3 to obtain the analytical expressions of $\mathbf{\Sigma}_{t,p}$ and $\mathbf{\Sigma}_{l;p}$. The covariance matrix $\mathbf{\Sigma}$ of the error process $\{\varepsilon_t\}$ is set by

$$\mathbf{\Sigma} = \mathbf{DAD},$$

where $\mathbf{D} = diag(d_1, \dots, d_p)$ is a diagonal matrix consisting of the standard deviations $d_1, \dots, d_p$ and

$$\mathbf{A} = \begin{pmatrix} 1 & \alpha & \alpha^2 & \dots & \alpha^{p-1} \\ \alpha & 1 & \alpha & \dots & \alpha^{p-2} \\ \alpha^2 & \alpha & 1 & \dots & \alpha^{p-3} \\ & \vdots & \vdots & \ddots & \vdots \\ \alpha^{p-1} & \alpha^{p-2} & \alpha^{p-3} & \dots & 1 \end{pmatrix}$$

is a correlation matrix with $\alpha = 0.5$. To make the results of the simulation study more flexible, the values of standard deviations $d_1, \dots, d_p$ are drawn randomly from the uniform distribution on the interval $[0.5, 2]$. Furthermore, it is remarkable that if the matrix $\mathbf{D}$ is proportional to the identity matrix, then the control chart $T_{1,t}$ coincides with $T_{6,t}$, $T_{2,t}$ with $T_{7,t}$, $T_{3,t}$ with $T_{8,t}$, $T_{4,t}$ with $T_{9,t}$, and $T_{5,t}$ with $T_{10,t}$. The mean of the VAR(1) process $\{\mathbf{Y}_t\}$ is set to be zero vector, i.e., $\boldsymbol{\mu} = \mathbf{0}$. To this end, we put $\mathbf{R} = r\,\mathbf{I}$ with $r \in \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1\}$ and set the dimension of the observed process and the target process p to be equal to 50.

In the definition of several control statistics in Section 4, the asymptotic mean and the asymptotic variance was used. For the target process as defined in the beginning of the section, we get that the matrices $\mathbf{\Sigma}_{t,p}$ and $\mathbf{\Sigma}_{l;p}$ are proportional following the results derived in Lemma 3.3. As such, the control charts $T_{6,t}$ and $T_{7,t}$ have the same behaviour as $T_{8,t}$ and $T_{9,t}$, respectively. Thus, we will omit presenting the results for $T_{8,t}$ and $T_{9,t}$ in this simulation study.

To study the impact of the asymptotic approximation on the performance of the control charts $T_{1,t}$, $T_{2,t}$, $T_{3,t}$, and $T_{4,t}$, we compare the exact mean and the exact variance of $\mathbf{Z}_t'\mathbf{Z}_t$, $t = 1, \dots, 30$, as derived in Theorem 4.1 with the corresponding asymptotic values. The results are obtained for $r \in \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1\}$ and are depicted in Table 1. Large differences between the exact and the asymptotic quantities occur for $r \leq 0.7$, especially when $t = 1$. For small values of r considerable differences are also present for larger values of t, up to $t = 10$. Such a poor performance of the asymptotic mean and variance to provide an accurate approximation of the exact ones might have a strong impact on the control schemes whose definitions involve the asymptotic mean and variance of $\mathbf{Z}_t'\mathbf{Z}_t$. This is especially the case with the $T_{2,t}$ chart where the exact mean and the asymptotic variance is used. This control scheme performs very poorly both in the in-control and out-of-control state and, for that reason it is excluded from the comparison study.

The in-control average run length (ARL) is used to calibrate the control charts $T_{1,t}$, $T_{3,t}$, $T_{4,t}$, $T_{6,t}$, $T_{7,t}$, $T_{Mah,t}$, and $T_{MahInf,t}$, which are used in the comparison study in Section 5.2. The control limits computed for these control schemes in the case of the considered VAR(1) process are given in Table 2 for $r \in \{0.1, 0.2, \dots, 1.0\}$. The in-control ARL is taken equal to 200. The computation of the control limits is performed by numerical computation where the ARL is found in each iteration based on Monte Carlo simulations with 10^4 independent repetitions.

	$r = 0.1$		$r = 0.2$		$r = 0.3$		$r = 0.4$		$r = 0.5$		$r = 0.6$		$r = 0.7$		$r = 0.8$		$r = 0.9$		$r = 1.0$	
t	mean	sd	mean	sd	mean	sd	mean	sd	mean	sd	mean	sd	mean	sd	mean	sd	mean	sd	mean	sd
1	0.67	0.17	2.67	0.68	6.00	1.54	10.67	2.74	16.67	4.28	24.00	6.16	32.67	8.39	42.67	10.96	54.00	13.87	66.67	17.12
2	1.81	0.46	6.51	1.67	13.14	3.37	20.91	5.37	29.17	7.49	37.44	9.62	45.41	11.66	52.91	13.59	59.94	15.39	66.67	17.12
3	3.00	0.77	9.82	2.52	18.11	4.65	26.51	6.81	34.38	8.83	41.51	10.66	48.02	12.33	54.17	13.91	60.27	15.48	66.67	17.12
4	4.09	1.05	12.28	3.15	21.06	5.41	29.11	7.48	36.20	9.30	42.55	10.93	48.48	12.45	54.31	13.95	60.29	15.48	66.67	17.12
5	5.02	1.29	13.99	3.59	22.68	5.83	30.21	7.76	36.78	9.45	42.79	10.99	48.55	12.47	54.32	13.95	60.29	15.48	66.67	17.12
6	5.81	1.49	15.14	3.89	23.54	6.05	30.66	7.88	36.96	9.49	42.84	11.00	48.57	12.47	54.32	13.95	60.29	15.48	66.67	17.12
7	6.45	1.66	15.90	4.08	23.99	6.16	30.84	7.92	37.02	9.51	42.85	11.01	48.57	12.47	54.32	13.95	60.29	15.48	66.67	17.12
8	6.98	1.79	16.39	4.21	24.21	6.22	30.91	7.94	37.03	9.51	42.86	11.01	48.57	12.47	54.32	13.95	60.29	15.48	66.67	17.12
9	7.41	1.90	16.71	4.29	24.32	6.25	30.94	7.95	37.04	9.51	42.86	11.01	48.57	12.47	54.32	13.95	60.29	15.48	66.67	17.12
10	7.76	1.99	16.92	4.34	24.38	6.26	30.95	7.95	37.04	9.51	42.86	11.01	48.57	12.47	54.32	13.95	60.29	15.48	66.67	17.12
11	8.04	2.07	17.05	4.38	24.41	6.27	30.95	7.95	37.04	9.51	42.86	11.01	48.57	12.47	54.32	13.95	60.29	15.48	66.67	17.12
12	8.27	2.12	17.13	4.40	24.42	6.27	30.95	7.95	37.04	9.51	42.86	11.01	48.57	12.47	54.32	13.95	60.29	15.48	66.67	17.12
13	8.46	2.17	17.19	4.41	24.43	6.27	30.95	7.95	37.04	9.51	42.86	11.01	48.57	12.47	54.32	13.95	60.29	15.48	66.67	17.12
14	8.61	2.21	17.22	4.42	24.43	6.27	30.95	7.95	37.04	9.51	42.86	11.01	48.57	12.47	54.32	13.95	60.29	15.48	66.67	17.12
15	8.73	2.24	17.24	4.43	24.43	6.27	30.95	7.95	37.04	9.51	42.86	11.01	48.57	12.47	54.32	13.95	60.29	15.48	66.67	17.12
16	8.83	2.27	17.26	4.43	24.43	6.27	30.95	7.95	37.04	9.51	42.86	11.01	48.57	12.47	54.32	13.95	60.29	15.48	66.67	17.12
17	8.91	2.29	17.27	4.43	24.43	6.28	30.95	7.95	37.04	9.51	42.86	11.01	48.57	12.47	54.32	13.95	60.29	15.48	66.67	17.12
18	8.97	2.30	17.27	4.44	24.43	6.28	30.95	7.95	37.04	9.51	42.86	11.01	48.57	12.47	54.32	13.95	60.29	15.48	66.67	17.12
19	9.03	2.32	17.28	4.44	24.43	6.28	30.95	7.95	37.04	9.51	42.86	11.01	48.57	12.47	54.32	13.95	60.29	15.48	66.67	17.12
20	9.07	2.33	17.28	4.44	24.43	6.28	30.95	7.95	37.04	9.51	42.86	11.01	48.57	12.47	54.32	13.95	60.29	15.48	66.67	17.12
21	9.10	2.34	17.28	4.44	24.43	6.28	30.95	7.95	37.04	9.51	42.86	11.01	48.57	12.47	54.32	13.95	60.29	15.48	66.67	17.12
22	9.13	2.35	17.28	4.44	24.43	6.28	30.95	7.95	37.04	9.51	42.86	11.01	48.57	12.47	54.32	13.95	60.29	15.48	66.67	17.12
23	9.15	2.35	17.28	4.44	24.43	6.28	30.95	7.95	37.04	9.51	42.86	11.01	48.57	12.47	54.32	13.95	60.29	15.48	66.67	17.12
24	9.17	2.36	17.28	4.44	24.43	6.28	30.95	7.95	37.04	9.51	42.86	11.01	48.57	12.47	54.32	13.95	60.29	15.48	66.67	17.12
25	9.19	2.36	17.28	4.44	24.43	6.28	30.95	7.95	37.04	9.51	42.86	11.01	48.57	12.47	54.32	13.95	60.29	15.48	66.67	17.12
26	9.20	2.36	17.28	4.44	24.43	6.28	30.95	7.95	37.04	9.51	42.86	11.01	48.57	12.47	54.32	13.95	60.29	15.48	66.67	17.12
27	9.21	2.37	17.28	4.44	24.43	6.28	30.95	7.95	37.04	9.51	42.86	11.01	48.57	12.47	54.32	13.95	60.29	15.48	66.67	17.12
28	9.22	2.37	17.28	4.44	24.43	6.28	30.95	7.95	37.04	9.51	42.86	11.01	48.57	12.47	54.32	13.95	60.29	15.48	66.67	17.12
29	9.22	2.37	17.28	4.44	24.43	6.28	30.95	7.95	37.04	9.51	42.86	11.01	48.57	12.47	54.32	13.95	60.29	15.48	66.67	17.12
30	9.23	2.37	17.28	4.44	24.43	6.28	30.95	7.95	37.04	9.51	42.86	11.01	48.57	12.47	54.32	13.95	60.29	15.48	66.67	17.12
∞	9.25	2.38	17.28	4.44	24.43	6.28	30.95	7.95	37.04	9.51	42.86	11.01	48.57	12.47	54.32	13.95	60.29	15.48	66.67	17.12

Table 1: Exact and asymptotic mean and standard deviation of $\mathbf{Z}_t'\mathbf{Z}_t$ in the in-control state for $r \in \{0.1, 0.2, \ldots, 1.0\}$ and for several values of t. The diagonal elements d_i, $i = 1, ..., p$, of $\mathbf{\Sigma}$ are all set to one.

r	$T_{1,t}$	$T_{3,t}$	$T_{4,t}$	$T_{6,t}$	$T_{7,t}$	$T_{Mah,t}$	$T_{MahInf,t}$
0.1	2.576	2.511	3.231	2.550	2.545	73.965	73.169
0.2	2.858	2.805	3.029	2.792	2.801	76.147	75.725
0.3	2.999	2.949	2.969	2.929	2.932	77.247	76.906
0.4	3.097	3.058	3.050	3.024	3.023	77.989	77.696
0.5	3.164	3.133	3.115	3.079	3.086	78.477	78.204
0.6	3.203	3.174	3.169	3.125	3.132	78.838	78.569
0.7	3.247	3.218	3.217	3.165	3.162	79.084	78.840
0.8	3.275	3.259	3.255	3.190	3.189	79.252	79.057
0.9	3.304	3.288	3.279	3.205	3.209	79.406	79.266
1	3.313	3.316	3.319	3.220	3.226	79.501	79.494

Table 2: Control limits computed for $T_{1,t}$, $T_{3,t}$, $T_{4,t}$, $T_{6,t}$, $T_{7,t}$, $T_{Mah,t}$, and $T_{MahInf,t}$ control charts in the case of VAR(1) process, $r \in \{0.1, 0.2, \ldots, 1.0\}$. The diagonal elements d_i, $i = 1, ..., p$, of $\mathbf{\Sigma}$ are drawn randomly from the uniform distribution on the interval $[0.5, 2]$.

In case $r = 1$ the MEWMA control schemes become Shewhart control charts. In this case, only the observations at a given time point are used to make a decision about the existence of a change in the model structure, previous observations are not taken into account. The control limits are the largest for $r = 1$, while they drop monotonically as r decreases. Moreover, we observe that the control limits of $T_{1,t}$, $T_{3,t}$, $T_{4,t}$, $T_{6,t}$, and $T_{7,t}$ charts are considerably smaller than the ones computed for the control schemes based on the Mahalanobis distance. This result follows directly from the observations, that $T_{Mah,t}$ and $T_{MahInf,t}$ are not normalized, while the other five control schemes are centered and normalized by the exact (asymptotic) standard deviations.

5.2 Results in the out-of-control state

In this section we study the ability of the new MEWMA control charts to detect changes in the location behaviour of high-dimensional VAR(1) processes as defined in the previous section and compare it with the two benchmark approaches that are based on the Mahalanobis distance. The changes in the mean vector are generated according to the change point model (1). Namely, the vector $\mathbf{a} = (a, \ldots, a, 0, \ldots, 0)'$ with $a \in \{-4, -3, -2, -1, 1, 2, 3, 4\}$ is added to the target process $\{\mathbf{Y}_t\}$ in the out-of-control state where the number of nonzero elements of $\mathbf{a}$ is $l \in \{12, 25, 50\}$.

The maximum expected delay (MED) is used as a performance measure in the out-of-control state. The expected delay is defined as the average delay between the time of a change and the time when the control chart detects a change under the condition that there is no false alarm before the change takes place. The MED takes the maximum value of the expected delays with respect to the possible location of the change time. In practice, for the determination of the MED only a finite number of possible time changes can be taken into account. Here, it is bounded by 20, that is $\tau \in \{1, 2, \ldots, 20\}$. To this end we note that the results obtained for negative values of a are similar to the ones obtained for the corresponding positive values of a. Therefore, only the results for $a \in \{1, 2, 3, 4\}$ are shown in Tables 3 to 6.

For each control chart and out-of-control situation the minimum value of the MED with respect to r is highlighted bold, while the smallest value across all the control schemes and the values of r are presented bold cursive. As expected, changes of small magnitude are detected faster when the small values of r are employed, for changes of moderate size one should prefer to set $r \in \{0.2, 0.3\}$, while large deviations in the mean behaviour of a VAR(1) process are best monitored when $r = 1$ or when r is close to one. These findings are in agreement with the previous results of Kramer and Schmid (1997) obtained for the multivariate EWMA control charts based on the Mahalanobis distance.

The control charts based on the distance computed with respect to the inverse of the diagonal matrix

r	$T_{1,t}$	$T_{3,t}$	$T_{4,t}$	$T_{6,t}$	$T_{7,t}$	$T_{Mah,t}$	$T_{MahInf,t}$
				$l = 12$			
0.1	**21.68**	**21.52**	**31.27**	**15.88**	***15.87***	**23.27**	**25.66**
0.2	27.57	26.45	32.07	17.86	17.94	30.92	31.32
0.3	34.71	33.29	33.77	21.24	20.99	39.16	39.24
0.4	41.34	39.57	39.25	24.32	24.32	48.12	46.50
0.5	47.37	46.36	45.13	27.59	27.86	54.60	52.70
0.6	52.49	50.58	50.91	31.26	31.23	61.65	59.19
0.7	57.75	55.80	55.88	34.36	34.44	67.81	64.48
0.8	63.28	61.58	61.29	37.25	37.30	73.46	70.57
0.9	68.16	67.24	65.72	40.82	41.00	79.05	77.94
1	72.34	72.80	73.06	44.05	44.50	84.31	84.53
				$l = 25$			
0.1	**11.80**	**11.63**	15.44	7.91	7.90	9.79	11.85
0.2	12.48	12.14	**13.74**	***7.19***	**7.21**	**9.52**	**10.48**
0.3	13.99	13.62	13.85	7.30	7.34	10.41	11.07
0.4	16.03	15.63	15.54	7.70	7.69	12.01	11.98
0.5	18.34	17.73	17.61	8.23	8.29	13.53	13.25
0.6	20.08	19.75	19.54	8.98	8.99	15.45	15.25
0.7	22.53	22.15	21.86	9.65	9.66	17.46	17.16
0.8	24.64	24.33	24.22	10.47	10.49	19.39	19.16
0.9	27.61	26.83	27.00	11.41	11.41	22.03	21.40
1	29.64	29.83	29.77	12.36	12.43	24.44	24.45
				$l = 50$			
0.1	7.12	7.02	9.29	4.70	4.71	5.76	7.59
0.2	**6.38**	6.32	7.05	3.81	3.83	4.81	5.74
0.3	6.40	**6.21**	**6.38**	3.42	3.42	4.44	5.04
0.4	6.60	6.47	6.46	3.19	3.20	**4.35**	4.70
0.5	7.04	6.94	6.95	3.06	3.08	4.40	4.50
0.6	7.49	7.45	7.41	3.01	3.01	4.52	**4.46**
0.7	8.22	8.02	8.07	2.98	***2.97***	4.79	4.69
0.8	8.84	8.81	8.78	3.00	2.98	5.03	4.96
0.9	9.76	9.60	9.53	***2.97***	3.00	5.39	5.39
1	10.60	10.62	10.68	3.03	3.04	5.86	5.84

Table 3: Maximum expected delays for $T_{1,t}$, $T_{3,t}$, $T_{4,t}$, $T_{6,t}$, $T_{7,t}$, $T_{Mah,t}$, and $T_{MahInf,t}$ control charts in the case of VAR(1) process, $r \in \{0.1, 0.2, \dots, 1.0\}$ and $a = 1$

of variances show the best performance for all considered values of a and the number of elements l whose mean values are shifted. The control schemes $T_{6,t}$ and $T_{7,t}$ are followed by the control procedures where the Mahalanobis distance is used in the construction of the control statistics. The worst performance is documented for the control charts based on the Euclidean distance, which are better than the control schemes based on the Mahalanobis distance only when $a = 1$ and $l = 12$.

6 Conclusion

Monitoring changes in the parameters of a multivariate time series is a complicated task due to large number of parameters that need to be controlled simultaneously. Although several control schemes

	$T_{1,t}$	$T_{3,t}$	$T_{4,t}$	$T_{6,t}$	$T_{7,t}$	$T_{Mah,t}$	$T_{MahInf,t}$
r				$l = 12$			
0.1	7.29	7.19	9.37	5.80	5.82	7.62	9.68
0.2	**6.54**	**6.45**	7.23	4.89	4.91	**6.90**	7.89
0.3	6.55	**6.45**	**6.63**	4.62	4.58	6.94	**7.44**
0.4	6.89	6.78	6.75	**4.50**	***4.49***	7.39	7.65
0.5	7.43	7.24	7.17	4.52	4.52	8.10	8.02
0.6	8.08	7.87	7.87	4.65	4.68	8.90	8.68
0.7	8.72	8.56	8.57	4.80	4.81	9.84	9.64
0.8	9.60	9.43	9.39	5.05	5.01	10.83	10.68
0.9	10.58	10.49	10.38	5.29	5.30	12.29	11.99
1	11.53	11.69	11.48	5.62	5.61	13.65	13.56
				$l = 25$			
0.1	4.63	4.63	6.15	3.47	3.46	4.05	5.72
0.2	3.78	3.73	4.51	2.72	2.72	3.21	4.21
0.3	3.38	3.33	3.73	2.32	2.32	2.78	3.48
0.4	3.15	3.11	3.33	2.08	2.08	2.51	3.03
0.5	3.03	3.01	3.05	1.88	1.88	2.32	2.65
0.6	2.97	2.94	2.94	1.72	1.73	2.19	2.45
0.7	**2.95**	**2.92**	**2.91**	1.59	1.58	2.07	2.28
0.8	2.96	2.94	2.93	1.46	1.46	1.97	2.03
0.9	3.01	2.96	2.97	1.37	1.38	1.88	1.88
1	3.05	3.02	3.01	***1.31***	***1.31***	**1.83**	**1.82**
				$l = 50$			
0.1	3.22	3.18	4.46	2.33	2.35	2.74	4.05
0.2	2.50	2.48	3.17	1.83	1.83	2.12	3.00
0.3	2.15	2.13	2.52	1.55	1.55	1.80	2.31
0.4	1.90	1.88	2.18	1.32	1.32	1.58	2.03
0.5	1.71	1.70	2.02	1.16	1.16	1.37	1.98
0.6	1.54	1.53	1.77	1.06	1.06	1.20	1.69
0.7	1.40	1.39	1.44	1.02	1.02	1.09	1.25
0.8	1.27	1.27	1.28	1.01	1.01	1.04	1.05
0.9	1.20	1.19	1.19	***1.00***	***1.00***	1.01	**1.01**
1	**1.13**	**1.14**	**1.14**	***1.00***	***1.00***	***1.00***	**1.01**

Table 4: Maximum expected delays for $T_{1,t}$, $T_{3,t}$, $T_{4,t}$, $T_{6,t}$, $T_{7,t}$, $T_{Mah,t}$, and $T_{MahInf,t}$ control charts in the case of VAR(1) process, $r \in \{0.1, 0.2, \ldots, 1.0\}$ and $a = 2$

exist in statistical literature and are successfully implemented in real-life applications (cf., Kramer and Schmid (1997), Bodnar and Schmid (2005), Bodnar and Schmid (2007), Bodnar and Schmid (2011)), the problem has not been investigated in detail in the high-dimensional setting, i.e., when the dimension of the target process can be very large. In a such situation, sequential surveillance of the model parameter becomes extremely challenging, especially in the case when data consist of dependent observations.

We contribute to the literature by deriving new types of control procedures based on the MEWMA recursion where in the definition of the control statistics other types of norms are used. In particular, instead of using the Mahalanobis distance, control charts based on the Euclidean distance are suggested as well as control schemes where the norm is computed by employing the inverse of the diagonal matrix consisting of the variances. The two latter approaches possess several advantages with respect to the

	$T_{1,t}$	$T_{3,t}$	$T_{4,t}$	$T_{6,t}$	$T_{7,t}$	$T_{Mah,t}$	$T_{MahInf,t}$
r				$l = 12$			
0.1	4.42	4.38	5.98	3.66	3.66	4.61	6.39
0.2	3.57	3.54	4.32	2.88	2.88	3.70	4.72
0.3	3.16	3.13	3.57	2.49	2.48	3.27	3.94
0.4	2.92	2.88	3.18	2.23	2.24	3.04	3.54
0.5	2.78	2.74	2.88	2.03	2.03	2.88	3.20
0.6	2.66	2.64	2.69	1.88	1.87	2.79	2.99
0.7	2.63	2.60	2.61	1.75	1.75	2.75	2.88
0.8	2.60	2.59	2.58	1.64	1.65	**2.73**	2.73
0.9	**2.58**	**2.58**	**2.55**	1.55	1.54	**2.73**	**2.71**
1	2.62	2.60	2.59	**1.47**	***1.46***	2.78	2.79
				$l = 25$			
0.1	3.04	3.01	4.23	2.35	2.36	2.71	4.02
0.2	2.37	2.35	3.03	1.84	1.85	2.10	2.99
0.3	2.01	2.01	2.38	1.55	1.56	1.79	2.28
0.4	1.78	1.77	2.11	1.33	1.33	1.56	2.03
0.5	1.59	1.58	1.96	1.16	1.16	1.34	1.98
0.6	1.43	1.42	1.67	1.07	1.06	1.18	1.67
0.7	1.28	1.28	1.34	1.02	1.02	1.08	1.23
0.8	1.18	1.18	1.18	1.01	1.01	1.03	1.04
0.9	1.12	1.11	1.11	***1.00***	***1.00***	1.01	1.01
1	**1.07**	**1.07**	**1.07**	***1.00***	***1.00***	***1.00***	***1.00***
				$l = 50$			
0.1	2.21	2.20	3.06	1.70	1.70	1.93	3.00
0.2	1.74	1.86	2.08	1.27	1.27	1.51	2.00
0.3	1.45	1.44	1.99	1.05	1.05	1.18	2.00
0.4	1.22	1.21	1.64	1.01	1.01	1.03	1.51
0.5	1.08	1.08	1.16	***1.00***	***1.00***	***1.00***	1.02
0.6	1.02	1.02	1.02	***1.00***	***1.00***	***1.00***	***1.00***
0.7	***1.00***	***1.00***	***1.00***	***1.00***	***1.00***	***1.00***	***1.00***
0.8	***1.00***	***1.00***	***1.00***	***1.00***	***1.00***	***1.00***	***1.00***
0.9	***1.00***	***1.00***	***1.00***	***1.00***	***1.00***	***1.00***	***1.00***
1	***1.00***	***1.00***	***1.00***	***1.00***	***1.00***	***1.00***	***1.00***

Table 5: Maximum expected delays for $T_{1,t}$, $T_{3,t}$, $T_{4,t}$, $T_{6,t}$, $T_{7,t}$, $T_{Mah,t}$, and $T_{MahInf,t}$ control charts in the case of a VAR(1) process, $r \in \{0.1, 0.2, \ldots, 1.0\}$ and $a = 3$

procedures that are based on the Mahalanobis distance. First, since no inverse of the covariance matrix must be computed, the new approaches can be easily implemented also in the high-dimensional setting. Second, avoiding the computation of the inverse covariance matrix speeds up the computation time considerably. Moreover, the new approaches, especially the ones that uses the inverse of the diagonal matrix in the computation of the control statistics, possess the best performance in the conducted simulation study independently of the model used to generate changes in the mean behaviour of a multivariate autoregressive process. When the magnitude of the change is small, we also obtain that the control schemes based on the Euclidean norm outperforms the ones based on the Mahalanobis distance.

	$T_{1,t}$	$T_{3,t}$	$T_{4,t}$	$T_{6,t}$	$T_{7,t}$	$T_{Mah,t}$	$T_{MahInf,t}$
r				$l = 12$			
0.1	3.26	3.25	4.56	2.75	2.75	3.38	4.90
0.2	2.55	2.52	3.23	2.13	2.14	2.63	3.57
0.3	2.18	2.16	2.60	1.81	1.82	2.25	2.99
0.4	1.93	1.93	2.23	1.59	1.59	1.99	2.49
0.5	1.75	1.74	2.07	1.40	1.40	1.79	2.20
0.6	1.58	1.57	1.86	1.24	1.25	1.62	2.07
0.7	1.43	1.43	1.56	1.14	1.14	1.46	1.85
0.8	1.31	1.31	1.31	1.07	1.07	1.33	1.48
0.9	1.23	1.22	1.22	1.04	1.04	1.23	1.23
1	**1.16**	**1.17**	**1.16**	***1.02***	***1.02***	**1.16**	**1.16**
				$l = 25$			
0.1	2.32	2.31	3.19	1.86	1.86	2.09	3.03
0.2	1.82	1.93	2.20	1.44	1.45	1.66	2.08
0.3	1.54	1.53	2.00	1.15	1.16	1.34	2.00
0.4	1.31	1.30	1.80	1.03	1.03	1.11	1.89
0.5	1.14	1.14	1.31	***1.00***	1.01	1.02	1.22
0.6	1.05	1.05	1.07	***1.00***	***1.00***	***1.00***	1.01
0.7	1.02	1.02	1.02	***1.00***	***1.00***	***1.00***	***1.00***
0.8	***1.00***	***1.00***	***1.00***	***1.00***	***1.00***	***1.00***	***1.00***
0.9	***1.00***	***1.00***	***1.00***	***1.00***	***1.00***	***1.00***	***1.00***
1	***1.00***	***1.00***	***1.00***	***1.00***	***1.00***	***1.00***	***1.00***
				$l = 50$			
0.1	1.76	1.97	2.30	1.33	1.33	1.58	2.01
0.2	1.33	1.33	2.00	1.02	1.02	1.09	2.00
0.3	1.08	1.07	1.29	***1.00***	***1.00***	***1.00***	1.08
0.4	1.01	1.01	1.01	***1.00***	***1.00***	***1.00***	***1.00***
0.5	***1.00***	***1.00***	***1.00***	***1.00***	***1.00***	***1.00***	***1.00***
0.6	***1.00***	***1.00***	***1.00***	***1.00***	***1.00***	***1.00***	***1.00***
0.7	***1.00***	***1.00***	***1.00***	***1.00***	***1.00***	***1.00***	***1.00***
0.8	***1.00***	***1.00***	***1.00***	***1.00***	***1.00***	***1.00***	***1.00***
0.9	***1.00***	***1.00***	***1.00***	***1.00***	***1.00***	***1.00***	***1.00***
1	***1.00***	***1.00***	***1.00***	***1.00***	***1.00***	***1.00***	***1.00***

Table 6: Maximum expected delays for $T_{1,t}$, $T_{3,t}$, $T_{4,t}$, $T_{6,t}$, $T_{7,t}$, $T_{Mah,t}$, and $T_{MahInf,t}$ control charts in the case of a VAR(1) process, $r \in \{0.1, 0.2, \ldots, 1.0\}$ and $a = 4$

7 Appendix

In this section the proofs of the theoretical results are provided.

Lemma 7.1. *Let* $\mathbf{z} \sim \mathcal{N}_p(\boldsymbol{\nu}, \boldsymbol{\Omega})$*. Let* $\boldsymbol{\Omega}$ *and* $\mathbf{B}$ *be positive definite matrices,* $\mathbf{b} \in I\!R^p$*, and let* $\mathbf{U}'\boldsymbol{\Omega}^{1/2}\mathbf{B}\boldsymbol{\Omega}^{1/2}\mathbf{U} = \boldsymbol{\Lambda} = diag(\lambda_1, ..., \lambda_p)$*. Further let* $\boldsymbol{\delta} = (\delta_1, ..., \delta_p)' = \mathbf{U}'\boldsymbol{\Omega}^{-1/2}(\boldsymbol{\nu} - \mathbf{b})$*. Suppose that*

$$\frac{\max_{1 \le i \le p} \lambda_i^2 (1 + 2\delta_i^2)}{\sum_{v=1}^{p} \lambda_v^2 (1 + 2\delta_v^2)} \xrightarrow[p \to \infty]{} 0 \tag{10}$$

then

$$\frac{(\mathbf{z} - \mathbf{b})'\mathbf{B}(\mathbf{z} - \mathbf{b}) - tr(\mathbf{B}\boldsymbol{\Omega}) - (\mathbf{b} - \boldsymbol{\nu})'\mathbf{B}(\mathbf{b} - \boldsymbol{\nu})}{\sqrt{2tr(\mathbf{B}\boldsymbol{\Omega}\mathbf{B}\boldsymbol{\Omega}) + 4(\boldsymbol{\nu} - \mathbf{b})'\mathbf{B}\boldsymbol{\Omega}\mathbf{B}(\boldsymbol{\nu} - \mathbf{b})}} \xrightarrow[p \to \infty]{d} \mathcal{N}(0, 1).$$

Proof. From $\mathbf{z} \sim \mathcal{N}_p(\boldsymbol{\nu}, \boldsymbol{\Omega})$, we get that

$$\mathbf{u} = (u_1, ..., u_p)' = \mathbf{U}'\boldsymbol{\Omega}^{-1/2}(\mathbf{z} - \mathbf{b}) \sim \mathcal{N}_p(\boldsymbol{\delta}, \mathbf{I})$$

and, hence,

$$(\mathbf{z} - \mathbf{b})'\mathbf{B}(\mathbf{z} - \mathbf{b}) = \mathbf{u}'\boldsymbol{\Lambda}\mathbf{u} = \sum_{i=1}^{p} \lambda_i u_i^2, \tag{11}$$

where $u_i^2 \sim \chi^2_{1,\delta_i^2}$. Consequently,

$$\mathbb{E}(u_i^2 - 1 - \delta_i^2) = 0, \mathbb{V}ar(u_i^2) = 2(1 + 2\delta_i^2),$$

$$\mathbb{E}((u_i^2 - 1 - \delta_i^2)^4) = 12(1 + 2\delta_i^2)^2 + 48(1 + 4\delta_i^2).$$

Now we can apply Theorem 5 in Dette and Dörnemann (2020) because

i) $u_i^2, i = 1, ..., p$ are independent random variables for all $p \in I\!N$,

ii) $\mathbb{E}(u_i^2 - 1 - \delta_i^2) = 0$ for all $i \in \{1, ..., p\}, p \in \mathbb{N}$,

iii) $\mathbb{E}((u_i^2 - 1 - \delta_i^2)^4) \leq C(2(1 + 2\delta_i^2))^2$ with $C = 15$.

Moreover, we get with

$$g_p(i) = \frac{\lambda_i}{\sqrt{2\sum_{v=1}^{p} \lambda_v^2(1 + 2\delta_v^2)}}$$

that

iv) $\max_{1 \leq i \leq p} g_p(i)^2 \; \mathbb{V}ar(u_i^2) \xrightarrow[p\to\infty]{} 0$ by assumption (10).

Furthermore, it holds that

$$\begin{aligned}\sum_{v=1}^{p} \lambda_v^2(1 + 2\delta_v^2) &= \operatorname{tr}(\mathbf{B}\boldsymbol{\Omega}\mathbf{B}\boldsymbol{\Omega}) + 2(\boldsymbol{\nu} - \mathbf{b})'\boldsymbol{\Omega}^{-1/2}\mathbf{U}\boldsymbol{\Lambda}^2\mathbf{U}'\boldsymbol{\Omega}^{-1/2}(\boldsymbol{\nu} - \mathbf{b}) \\ &= \operatorname{tr}(\mathbf{B}\boldsymbol{\Omega}\mathbf{B}\boldsymbol{\Omega}) + 2(\boldsymbol{\nu} - \mathbf{b})'\mathbf{B}\boldsymbol{\Omega}\mathbf{B}(\boldsymbol{\nu} - \mathbf{b}),\end{aligned}$$

which completes the proof of the lemma □

Next, we prove Lemma 3.3:

Proof of Lemma 3.3. It follows from Reinsel (1993) that

$$\boldsymbol{\Gamma}(h) = \boldsymbol{\Phi}^h\boldsymbol{\Gamma}(0) = \varphi^h\boldsymbol{\Gamma}(0)$$

for $h \geq 0$ and since $\boldsymbol{\Gamma}(0) = \frac{1}{1-\varphi^2}\boldsymbol{\Sigma}$ and $\boldsymbol{\Sigma}$ is a symmetrical matrix, we obtain

$$\boldsymbol{\Gamma}(-h) = \boldsymbol{\Gamma}(h)' = \varphi^h\boldsymbol{\Gamma}(0)' = \varphi^h\boldsymbol{\Gamma}(0) = \boldsymbol{\Gamma}(h).$$

Thus

$$\begin{aligned}\boldsymbol{\Sigma}_{t,p} &= \mathbf{R}\sum_{i,j=0}^{t-1}(\mathbf{I} - \mathbf{R})^i\boldsymbol{\Gamma}(j - i)(\mathbf{I} - \mathbf{R})^j\mathbf{R} \\ &= r\mathbf{I} \cdot \sum_{i,j=0}^{t-1}(1 - r)^i\mathbf{I}\varphi^{|j-i|}\mathbf{I}\boldsymbol{\Gamma}(0)(1 - r)^j\mathbf{I} \cdot r\mathbf{I} \\ &= r^2\boldsymbol{\Gamma}(0)\sum_{i,j=0}^{t-1}(1 - r)^{i+j}\varphi^{|j-i|}.\end{aligned}$$

Assume now that $1-r \neq \varphi$ and $\varphi \neq 0$. Then

$$\begin{aligned}
\mathbf{\Sigma}_{t,p} &= r^2\mathbf{\Gamma}(0)\sum_{i=0}^{t-1}\left(\sum_{j=0}^{i-1}(1-r)^{i+j}\varphi^{i-j}+\sum_{j=i}^{t-1}(1-r)^{i+j}\varphi^{j-i}\right)\\
&= r^2\mathbf{\Gamma}(0)\sum_{i=0}^{t-1}(1-r)^i\left(\varphi^i\sum_{j=0}^{i-1}\left(\frac{1-r}{\varphi}\right)^j+\varphi^{-i}\sum_{j=i}^{t-1}((1-r)\varphi)^j\right)\\
&= r^2\mathbf{\Gamma}(0)\sum_{i=0}^{t-1}\left((1-r)^i\varphi^i\frac{1-\left(\frac{1-r}{\varphi}\right)^i}{1-\frac{1-r}{\varphi}}+(1-r)^{2i}\frac{1-((1-r)\varphi)^{t-i}}{1-(1-r)\varphi}\right)\\
&= r^2\mathbf{\Gamma}(0)\left[\frac{1}{1-\frac{1-r}{\varphi}}\left(\frac{1-(1-r)^t\varphi^t}{1-(1-r)\varphi}-\frac{1-(1-r)^{2t}}{1-(1-r)^2}\right)\right.\\
&+ \left.\frac{1}{1-(1-r)\varphi}\left(\frac{1-(1-r)^{2t}}{1-(1-r)^2}-\sum_{i=0}^{t-1}(1-r)^{2i}(1-r)^{t-i}\varphi^{t-i}\right)\right]\\
&= r^2\mathbf{\Gamma}(0)\left[\frac{\varphi}{\varphi-(1-r)}\left(\frac{1-(1-r)^t\varphi^t}{1-(1-r)\varphi}-\frac{1-(1-r)^{2t}}{1-(1-r)^2}\right)\right.\\
&+ \left.\frac{1}{1-(1-r)\varphi}\left(\frac{1-(1-r)^{2t}}{1-(1-r)^2}-(1-r)^t\varphi\frac{\varphi^t-(1-r)^t}{\varphi-(1-r)}\right)\right].
\end{aligned}$$

Consequently,

$$\begin{aligned}
\mathbf{\Sigma}_{t,p} &= \frac{r^2\mathbf{\Gamma}(0)}{(\varphi-(1-r))(1-(1-r)\varphi)(1-(1-r)^2)}\left[\varphi\left(1-(1-r)^t\varphi^t\right)\left(1-(1-r)^2\right)\right.\\
&- \varphi\left(1-(1-r)^{2t}\right)\left(1-(1-r)\varphi\right)+\left(1-(1-r)^{2t}\right)\left(\varphi-(1-r)\right)\\
&- \left.(1-r)^t\varphi\left(\varphi^t-(1-r)^t\right)\left(1-(1-r)^2\right)\right]\\
&= \frac{r^2\mathbf{\Gamma}(0)}{(\varphi-(1-r))(1-(1-r)\varphi)(1-(1-r)^2)}\left[\left((1-r)^t\varphi\left(\varphi^t-(1-r)^t\right)\right.\right.\\
&- \left.\varphi\left(1-(1-r)^t\varphi^t\right)\right)(1-r)^2+\left(\varphi^2\left(1-(1-r)^{2t}\right)\right.\\
&- \left.\left(1-(1-r)^{2t}\right)\right)(1-r)+\varphi\left(1-(1-r)^t\varphi^t\right)+\varphi\left(1-(1-r)^{2t}\right)\\
&- \left.(1-r)^t\varphi\left(\varphi^t-(1-r)^t\right)-\varphi\left(1-(1-r)^{2t}\right)\right]\\
&= \frac{r^2\mathbf{\Gamma}(0)}{(\varphi-(1-r))(1-(1-r)\varphi)(1-(1-r)^2)}\left[\varphi\left((1-r)^t\left(2\varphi^t-(1-r)^t\right)-\right.\right.\\
&- \left.\left.1\right)(1-r)^2+\left(\varphi^2-1\right)\left(1-(1-r)^{2t}\right)(1-r)+\varphi\left(1-2(1-r)^t\varphi^t+(1-r)^{2t}\right)\right]
\end{aligned}$$

and thus

$$\begin{aligned}
\mathbf{\Sigma}_{t,p} &= \mathbf{\Gamma}(0)\,\frac{r}{2-r}\,\frac{1}{1-(1-r)\varphi}\,\left(1+\varphi(1-r)+\varphi\frac{\varphi^t-(1-r)^t}{\varphi-(1-r)}\left(\varphi^t-(1-r)^t+(1-r)^{t+2}\right)\right.\\
&- \left.\frac{\varphi^{2t+1}-(1-r)^{2t+1}}{\varphi-(1-r)}\;+\;\varphi^2(1-r)^{t+2}\,\frac{\varphi^{t-1}-(1-r)^{t-1}}{\varphi-(1-r)}\right).
\end{aligned}$$

If $\varphi=0$ then $1-r\neq 0=\varphi$ and we get

$$\mathbf{\Sigma}_{t,p}=r^2\frac{1-(1-r)^{2t}}{1-(1-r)^2}\,\mathbf{\Gamma}(0)=(1-(1-r)^{2t})\frac{r}{2-r}\,\mathbf{\Gamma}(0)$$

which is also obtained by setting $\varphi=0$ in (7).

Results for $\mathbf{\Sigma}_{l;p}$ follow from the respective formulas for $\mathbf{\Sigma}_{t,p}$ by letting t tends to ∞.

Now let $1 - r = \varphi$. Then

$$
\begin{aligned}
\mathbf{\Sigma}_{t,p} &= (1-\varphi)^2 \mathbf{\Gamma}(0) \sum_{i,j=0}^{t-1} \varphi^{i+j} \varphi^{|j-i|} \\
&= (1-\varphi)^2 \mathbf{\Gamma}(0) \sum_{i=0}^{t-1} \left(\sum_{j=0}^{i-1} \varphi^{2i} + \sum_{j=i}^{t-1} \varphi^{2j} \right) \\
&= (1-\varphi)^2 \mathbf{\Gamma}(0) \sum_{i=0}^{t-1} \left(i\varphi^{2i} + \varphi^{2i} \frac{1-\varphi^{2(t-i)}}{1-\varphi^2} \right) \\
&= (1-\varphi)^2 \mathbf{\Gamma}(0) \left[\frac{\varphi}{2} \left(\sum_{i=0}^{t-1} \varphi^{2i} \right)' + \sum_{i=0}^{t-1} \frac{\varphi^{2i} - \varphi^{2t}}{1-\varphi^2} \right] \\
&= (1-\varphi)^2 \mathbf{\Gamma}(0) \left[\frac{\varphi}{2} \left(\frac{1-\varphi^{2t}}{1-\varphi^2} \right)' + \frac{1-\varphi^{2t}}{(1-\varphi^2)^2} - t \frac{\varphi^{2t}}{1-\varphi^2} \right] \\
&= (1-\varphi)^2 \mathbf{\Gamma}(0) \left[\frac{-2t\varphi^{2t-1}(1-\varphi^2) - (1-\varphi^{2t})(-2\varphi)}{(1-\varphi^2)^2} \cdot \frac{\varphi}{2} + \frac{1-\varphi^{2t}}{(1-\varphi^2)^2} \right. \\
&- \left. t \frac{\varphi^{2t}}{1-\varphi^2} \right] = \frac{(1-\varphi)^2 \mathbf{\Gamma}(0)}{2\left(1-\varphi^2\right)^2} \left[-2t\varphi^{2t}(1-\varphi^2) \right. \\
&+ \left. 2\varphi^2 \left(1-\varphi^{2t}\right) + 2\left(1-\varphi^{2t}\right) - 2t\left(1-\varphi^2\right)\varphi^{2t} \right] \\
&= \frac{\mathbf{\Gamma}(0)}{(1+\varphi)^2} \left[\left(1+\varphi^2\right)\left(1-\varphi^{2t}\right) - 2t\varphi^{2t}(1-\varphi^2) \right].
\end{aligned}
$$

Note that this result is also obtained from (7) by letting φ converge to $1 - r$. □

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