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Multi-colour competition with reinforcement

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Abstract

We study a system of interacting urns where balls of different colour/type compete for their survival, and annihilate upon contact. For competition between two types, the underlying graph (finite and connected), determining the interaction between the urns, is known to be irrelevant for the possibility of coexistence, whereas for $K \geq 3$ types the structure of the graph does affect the possibility of coexistence. We show that when the underlying graph is a cycle, competition between $K \geq 3$ types almost surely has a single survivor, thus establishing a conjecture of Griffiths, Janson, Morris and the first author. Along the way, we give a detailed description of an auto-annihilative process on the cycle, which can be perceived as an expression of the geometry of a Möbius strip in a discrete setting.

Keywords: urn model; reinforcement process; coexistence; spatial growth.

1 Introduction

Probabilistic problems phrased in terms of balls drawn from urns date back to Jacob Bernoulli's *Ars Conjectandi* in 1713. The diverse assortment of disciplines, in which central phenomena can be understood through an urn problem, have contributed to their continued importance. Pólya's urn model [16, 24], introduced around a century ago, is an archetype of a random process with reinforcement. Random processes with reinforcement are examples of processes where the entire trajectory of the process contributes to its eventual fate. In Pólya's model, a finite number of red and blue balls are initially placed in an urn. In each step of the process, a ball is drawn uniformly at random from the urn, and returned along with an identical copy of itself. The drawn colour is thus reinforced, in that balls of the same colour are more likely to be drawn in future steps. The effect of the reinforcement declines over time, resulting in the early steps of the

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