Chaos, concentration and multiple valleys in first-passage percolation

Daniel Ahlberg* Maria Deijfen* Matteo Sfragara*

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Abstract

A decade and a half ago Chatterjee established the first rigorous connection between anomalous fluctuations and a chaotic behaviour of the ground state in certain Gaussian disordered systems. The purpose of this paper is to show that Chatterjee's work gives evidence of a more general principle, by establishing an analogous connection between fluctuations and chaos in the context of first-passage percolation. The notion of 'chaos' here refers to the sensitivity of the time-minimising path between two points when exposed to a slight perturbation. More precisely, we resample a small proportion of the edge weights, and find that a vanishing fraction of the edges on the distance-minimising path still belongs to the time-minimising path obtained after resampling. We also show that the chaotic behaviour implies the existence of a large number of almost-optimal paths that are almost disjoint from the time-minimising path, a phenomenon known as 'multiple valleys'.

Keywords: First-passage percolation, geodesics, noise sensitivity, concentration.

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1 Introduction

It has been known for quite some time that many interesting functions involving a large number of independent variables, each having little influence on the outcome as a whole, exhibit fluctuations at a lower order than what classical techniques would prescribe. This phenomenon has come to be referred to as *anomalous fluctuations* or *superconcentration*. While a robust theory for the concentration of measure has come to explain this

*Department of Mathematics, Stockholm University.
{daniel.ahlberg}{mia}{matteo.sfragara}@math.su.se

phenomenon, its potential for improvement upon classical techniques is inhibited by its generality; see e.g. [11, 17]. Understanding the mechanism that dictates the order of fluctuations remains one of the most important open problems in stochastic models for spatial growth, such as first-passage percolation.

The study of how percolative systems are affected when exposed to noise or dynamics offers a rich structure and a plethora of interesting phenomena; see e.g. [28]. A fascinating connection between sensitivity to noise and fluctuations in stochastic growth was discovered in the seminal work of Benjamini, Kalai and Schramm [8, 9]. In two later preprints, Chatterjee [13, 14] established a connection between superconcentration and a chaotic behaviour of the ground state in certain Gaussian disordered systems. (The two preprints were later combined and published as a book [17].) The purpose of this paper is to show that Chatterjee's discovery gives evidence of a much more general principle, by establishing an analogous connection between fluctuations and chaos in the context of first-passage percolation.

Since the distance-metric in first-passage percolation is known to be superconcentrated, we are able to provide the first evidence of chaos in this setting. Our results illustrate once again that various aspects of the first-passage percolation cannot be fully understood without also understanding the behaviour of distance-minimising paths of the metric.

1.1 Model and main results

First-passage percolation was introduced in the 1960s by Hammersley and Welsh [29], and has since become an archetype among stochastic models for spatial growth. The model is constructed from a graph, typically the $\mathbb{Z}^d$ nearest neighbour lattice, for some $d \geq 2$, and i.i.d. non-negative weights $\omega(e)$ assigned to the edges e of that graph. The weights may be interpreted as traversal times for some infectious phenomena, and induce a random metric T on the vertex set of the graph, defined as follows:[1] For $u, v \in \mathbb{Z}^d$ let

$$T(u,v) := \inf\left\{ \sum_{e\in\Gamma} \omega(e) : \Gamma \text{ is a nearest-neighbour path from } u \text{ to } v \right\}. \tag{1}$$

Any nearest-neighbour path π connecting u to v whose weight sum attains the infimum in (1) is referred to as a *geodesic* between u and v.

The occurrence of chaotic phenomena in models of disordered systems such as spin glasses was predicted by physicists in the 1980s [12, 23]. In the context of first-passage percolation, the analogous notion of 'chaos' refers to the sensitivity of geodesics as the first-passage metric is exposed to small perturbations. In order to address this behaviour properly, we shall be interested in a dynamical version of first-passage percolation, where edges update

[1] If weights may attain the value zero with positive probability, then T is a pseudo-metric.

their weight over time, according to independent uniform clocks. We let $\mathcal{E}$ denote the set of nearest-neighbour edges of the $\mathbb{Z}^d$ lattice, where $d \geq 2$ is fixed. Let $\{\omega(e)\}_{e\in\mathcal{E}}$ and $\{\omega'(e)\}_{e\in\mathcal{E}}$ be two independent i.i.d. families of non-negative edge weights with common distribution F, and let $\{U(e)\}_{e\in\mathcal{E}}$ be an independent collection of independent variables uniformly distributed on the interval $[0,1]$. We define a *dynamical* weight configuration $\omega_t = \{\omega_t(e)\}_{e\in\mathcal{E}}$ for $t \in [0,1]$ as

$$\omega_t(e) := \begin{cases} \omega(e) & \text{if } U(e) > t, \\ \omega'(e) & \text{if } U(e) \leq t. \end{cases} \tag{2}$$

In addition, we define the first-passage time between $u, v \in \mathbb{Z}^d$ at time $t \in [0,1]$ as

$$T_t(u,v) := \inf\left\{ \sum_{e\in\Gamma} \omega_t(e) : \Gamma \text{ is a nearest-neighbour path from } u \text{ to } v \right\}. \tag{3}$$

For fixed $t \in [0,1]$ the weight configuration ω_t consists of independent variables distributed as F, and hence $T_t(u,v)$ is equidistributed for all $t \in [0,1]$. We shall henceforth assume that F has finite second moment and does not put too much mass at zero-weight edges, although some of our results hold under weaker assumptions. More specifically, we shall assume

$$F(0) < p_c(d) \quad \text{and} \quad \int x^2\, dF(x) < \infty, \tag{4}$$

where $p_c(d)$ denotes the critical value for bond percolation on $\mathbb{Z}^d$. The former condition is mainly present in order to guarantee that the infimum in (3) is attained for some path Γ, so that a geodesic between u and v exists; see e.g. [5, Section 4.1].

When F is continuous, there exists an almost surely unique geodesic between any two points. In this case we denote by $\pi_t(u,v)$ the geodesic between u and v at time $t \in [0,1]$, i.e. the path attaining the infimum in (3). When F has atoms, there may be multiple paths attaining the infimum in (3). In this case we define $\pi_t(u,v)$ as the intersection of all geodesics between u and v, i.e.

$$\pi_t(u,v) := \big\{ e \in \mathcal{E} : e \in \Gamma \text{ for every } \Gamma \text{ that attains the infimum in (3)} \big\}. \tag{5}$$

Of course, when the geodesic between u and v is unique, then $\pi_t(u,v)$ denotes this geodesic, so the latter definition of $\pi_t(u,v)$ is indeed an extension of the former.

Chatterjee's work [17] suggests that fluctuations of $T(0,v)$ are related to the dynamical behaviour of the geodesic $\pi_t(0,v)$ through the relation[2]

$$\mathrm{Var}\big(T(0,v)\big) \asymp \int_0^1 \mathbb{E}\big[\big|\pi_0(0,v) \cap \pi_t(0,v)\big|\big]\, dt. \tag{6}$$

[2]Here, for sequences (a_v) and (b_v) of positive real numbers indexed by $\mathbb{Z}^d$, the relation $a_v \asymp b_v$ denotes that there exist constants $C_1, C_2 > 0$ such that $C_1 b_v \leq a_v \leq C_2 b_v$ for all v.

A standard coupling argument shows that the expected overlap of the two paths in the right-hand side of (6) is decreasing as a function of t (see Lemma 11). Furthermore, assuming (4), it is known that

$$c|v| \leq \mathbb{E}[|\pi_t(0,v)|] \leq \frac{1}{c}|v| \tag{7}$$

for some $c > 0$; see [5, Section 4.1]. Consequently, it is straightforward to deduce from (6) that sublinear variance growth $\mathrm{Var}(T(0,v)) = o(|v|)$ is equivalent to sublinear growth of the overlap $\mathbb{E}[|\pi_0(0,v) \cap \pi_t(0,v)|] = o(|v|)$ for $t > 0$ fixed. The latter is an expression of the chaotic nature of the first-passage metric referred to above.

A notoriously challenging problem in first-passage percolation, and more generally for models of spatial growth and disordered systems, is to determine the order of fluctuations of the random metric T. In two dimensions, first-passage percolation is believed to pertain to the KPZ-class of universality, which would suggest that $T(0,v)$ should typically fluctuate at order $|v|^{1/3}$ around its mean. In higher dimensions the fluctuations are believed to be smaller still. Classical techniques, such as an Efron-Stein estimate, give a linear upper bound on the variance of the first-passage metric. The best current bounds provide a modest improvement upon the classical techniques, and yield that for every $d \geq 2$ there exists a constant C such that, for all $v \in \mathbb{Z}^d$, we have

$$\mathrm{Var}\big(T(0,v)\big) \leq C\frac{|v|}{\log|v|}. \tag{8}$$

The bound in (8) was first obtained by Benjamini, Kalai and Schramm [9] in the special case of $\{a,b\}$-valued edge weights, for $0 < a < b < \infty$. Their result was later extended to a larger class of weight distributions by Benaïm and Rossignol [6, 7] and Damron, Hanson and Sosoe [19], and is now known to hold for weight distributions satisfying

$$F(0) < p_c(d) \quad \text{and} \quad \int x^2(\log x)_+ \, dF(x) < \infty. \tag{9}$$

Polynomial improvement on the bound in (8) is widely recognised as one of the most central open problems in the study of first-passage percolation.

In view of the variance bound (8), it will suffice to establish (6) in order to deduce the chaotic behaviour of the first-passage metric. Our first theorem concerns the integer-valued setting.

Theorem 1. *If F is supported on $\{0,1,2,\dots\}$ and satisfies (4), then (6) holds. Moreover, if also (9) holds, then there exists $C < \infty$ such that, for every $v \in \mathbb{Z}^d$ and $t > 0$, we have*

$$\mathbb{E}\big[|\pi_0(0,v) \cap \pi_t(0,v)|\big] \leq C\frac{|v|}{t\log|v|}.$$

Our second result covers the case when F is continuous, and more generally distributions that do not have any mass at the infimum of the support, i.e. $F(r) = 0$, where

$$r := \inf\{x \geq 0 : F(x) > 0\}.$$

In this case we do not establish a relation between fluctuations and geodesics that is quite as precise as (6). It is nevertheless sufficient to deduce that the first-passage metric exhibits chaos also in this setting.

Theorem 2. *If F has finite second moment and $F(r) = 0$, then, for fixed $\varepsilon > 0$,*

$$\mathrm{Var}\big(T(0,v)\big) \asymp \int_0^1 \mathbb{E}\big[\big|\pi_0(0,v) \cap \pi_t(0,v)\big|\big]\, dt \pm \varepsilon|v|. \tag{10}$$

In particular, if also (9) *holds, then*

$$\mathbb{E}\big[\big|\pi_0(0,v) \cap \pi_t(0,v)\big|\big] = o(|v|).$$

We believe that the relation (6) holds without the error term also in the setting of Theorem 2, but a more detailed analysis may be required to verify this.

Our third result shows that the chaotic behaviour of the first-passage metric implies a 'multiple valleys' property, i.e. that between any pair of distant points there are many almost disjoint paths that are almost optimal. The almost optimal paths correspond to multiple local valleys in the 'energy landscape' that reach close to the depth of the global minimum. For simplicity, we state this result only in the case when F is continuous, so that there is an almost surely unique geodesic between any pair of points.

We write $T(\Gamma) := \sum_{e \in \Gamma} \omega(e)$ for the passage time of a path Γ.

Theorem 3. *Assume that F is continuous and satisfies* (9). *For every $v \in \mathbb{Z}^d$, with probability tending to one as $|v| \to \infty$, there exists a set of paths $\{\Gamma_i\}$ from 0 to v which satisfies $|\{\Gamma_i\}| \to \infty$, $|\Gamma_i \cap \Gamma_j| = o(|v|)$, and $T(\Gamma_i) - T(0,v) = o(|v|)$.*

1.2 Method of proof

The proofs of Theorems 1 and 2 rely on writing the variance of $T := T(0,v)$ in terms of a certain correlation operator. Specifically, we define

$$Q_t(T) := \mathbb{E}\big[T_0(0,v)T_t(0,v)\big]$$

and note that

$$\mathrm{Var}(T) = Q_0(T) - Q_1(T) = -\int_0^1 Q_t'(T)\, dt,$$

assuming the derivative is well-defined. The derivative of $Q_t(T)$ can be expressed in terms of the contribution from each of the individual variables, which is typically referred to as a measure of the 'influence' of the variables. More precisely, we let $\sigma_e^x : [0,\infty)^{\mathcal{E}} \to [0,\infty)^{\mathcal{E}}$ denote the operator that replaces the value of coordinate e by x, and set

$$D_e^x T_t := T \circ \sigma_e^x(\omega_t) - \int T \circ \sigma_e^y(\omega_t)\, dF(y),$$

where the integral refers to the average over the integrand with respect to coordinate e. We define the **co-influence** of an edge e with respect to T at time t as

$$\mathrm{Inf}_e(T_0, T_t) := \int \mathbb{E}[D_e^x T_0 D_e^x T_t]\, dF(x),$$

where $T_t := T_t(0, v)$. A straightforward coupling argument shows that the co-influences are decreasing in t (see Lemma 10), and hence upper bounded by the following L^2-notion of influences

$$\mathrm{Inf}_e(T) := \mathrm{Inf}_e(T_0, T_0) = \mathbb{E}\big[(T - \mathbb{E}_e[T])^2\big], \tag{11}$$

where $\mathbb{E}_e$ refers to expectation taken over the weight at e. The derivative of $Q_t(T)$ turns out to equal the edge-sum of the co-influences, resulting in the variance formula

$$\mathrm{Var}(T) = \int_0^1 \sum_{e \in \mathcal{E}} \mathrm{Inf}_e(T_0, T_t)\, dt, \tag{12}$$

which together with (11) gives an Efron-Stein-like upper bound on the variance.

Formulas of a flavour similar to (12) have appeared in the literature before. A formula in discrete time was derived by Chatterjee in [16], and used for a similar purpose in [10]. A related formula in the Gaussian setting, in which the variables update continuously according to independent Ornstein-Uhlenbeck processes, was central in [17], and in the context of Boolean functions, a formula of this type was derived in [32].

It has been suggested that the influence of an edge (with respect to $T = T(0, v)$) is roughly proportional to the probability that the edge belongs to the geodesic between 0 and v. In the dynamical setting this corresponds to a statement of the form

$$\mathrm{Inf}_e(T_0, T_t) \asymp \mathbb{P}(e \in \pi_0 \cap \pi_t), \tag{13}$$

where π_t is short for $\pi_t(0, v)$. Much of our work will aim to establish a version of this statement, which together with (12) will give (6). Work of Ahlberg and Hoffman [2], and Dembin, Elboim and Peled [20], on the so-called 'midpoint problem', shows that when F is continuous, the influence of edges far from the end points tends to zero as $|v| \to \infty$. Our results thus help justify the claim that the influence of an edge vanishes as $|v| \to \infty$.

1.3 Related results

There has been an increasing interest in the phenomenon of chaos in recent years. The first rigorous evidence of chaos was obtained by Chatterjee [13, 14, 17] for Gaussian disordered systems, including directed Gaussian polymers, the Sherrington-Kirkpatrick model of spin glasses, and maxima of Gaussian fields. In the mathematical study of spin glasses, Chatterjee's work has been followed up in a range of papers, both within and outside of the Gaussian setting; see e.g. [3, 4, 15, 18, 22]. Chaos has also been observed in the context of eigenvalues of random matrices. Sensitivity of the eigenvector associated with the top eigenvalue of a Wigner matrix established by Bordenave, Lugosi and Zhivotovskiy [10], extending earlier work of Chatterjee [17] outside of the Gaussian setting. Moreover, in [10] the authors determine the precise order at which the transition from stability to chaos occurs, which for an $n \times n$ Wigner matrix is at $t = \Theta(n^{-1/3})$.

The literature on random matrices is extensive, and eigenvalues/-vectors thereof have a favourable structure. A significantly greater effort was required by Ganguly and Hammond [25, 26] to establish that the transition from stability to chaos in Brownian last-passage percolation occurs at $t = \Theta(n^{-1/3})$. Their work comprises the first instance of chaos in stochastic models for spatial growth. Parts of their work (the 'easy' part) have been extended to other models of last-passage percolation in a companion paper [1] of the current one, in order to illustrate that the transition from stability to chaos should more generally occur at $t = \Theta(\frac{1}{n}\mathrm{Var}(T))$, which in the exactly solvable setting is equivalent to $t = \Theta(n^{-1/3})$. In parallel to the preparation of this manuscript, Dembin and Garban [21] have given further evidence for the existence of chaos in the context of max-flow problems associated with first-passage percolation and disordered Ising ferromagnets.

Let us finally mention that a chaotic behaviour has also been observed for the set of pivotal edges in planar percolation models. In the conformally invariant setting, i.e. site percolation on the triangular lattice, the pivotal set corresponding to the crossing of an $n \times n$-box exhibits chaos, with the transition occurring at $t = \Theta(n^{-3/4})$. This can be derived as a consequence of the precise analysis of dynamical four-arm events due to Tassion and Vanneuville [32], although parts of this conclusion can also be derived from earlier work [27]; see also [24].

1.4 Future work and outline

The concept of 'noise sensitivity' was introduced in the context of Boolean functions by Benjamini, Kalai and Schramm [8], and refers to the asymptotic independence of the output of a Boolean function with respect to two highly correlated inputs. In the context of first-passage percolation, the analogous notion of noise sensitivity would refer to

$$\mathrm{Corr}(T_0, T_t) \to 0 \quad \text{as } |v| \to \infty, \tag{14}$$

for all $t > 0$. It is not known whether (14) holds in first-passage percolation, nor in other models of spatial growth. From Proposition 8 below it follows (by restricting the integral to $[t, 2t]$) that

$$\sum_{e\in\mathcal{E}} \mathrm{Inf}_e(T_0, T_{2t}) \le \frac{1}{t}\mathrm{Cov}(T_0, T_t),$$

which shows that (14) would imply that $\sum_{e\in\mathcal{E}} \mathrm{Inf}_e(T_0, T_{2t}) = o(\mathrm{Var}(T))$. This suggests, in combination with (13), that noise sensitivity of T is a stronger expression of a small perturbation than the chaotic behaviour derived here.

In this paper we establish an instance of (6), and use this relation to find evidence for chaos in the first-passage metric. It would be interesting to improve upon the bounds on the overlap between geodesics we obtain here, and conversely use (6) for the central task to improve upon existing variance bounds.

The rest of the paper is organised as follows: We first establish the formula in (12) in Sections 2 and 3. We examine the notion of co-influence in the context of first-passage percolation in Section 4, and prepare for the proofs of Theorems 1 and 2. The proof in the integer-valued setting is simpler, due to the fact that *if* changing the weight of an edge has an effect on the distance between two points, then the magnitude of the effect has to be at least one. Theorem 1 is proved in Section 5 and Theorem 2 is proved in Section 6. Finally we deduce Theorem 3 as a corollary in Section 7.

2 A dynamical covariance formula

In this section we derive a finitary version of the dynamical variance formula, stated in (12), that will be central for the remaining analysis. The corresponding formula in infinite volume will be derived in the next section. Since we shall need to consider various functions of the edge weights apart from the first-passage time, and since formulas of this kind have a general interest, we shall develop the theory in a general setting. We begin with some notation.

Let $m \ge 1$ be an integer and F (the distribution function of) some probability measure on $\mathbb{R}$. Let $\omega = (\omega(i))_{i\in[m]}$ and $\omega' = (\omega'(i))_{i\in[m]}$ be two independent random vectors with i.i.d coordinates with common distribution F. Let $U(1), U(2), \dots, U(m)$ be independent variables uniformly distributed on the interval $[0, 1]$. For $t \in [0, 1]$ and $i = 1, 2, \dots, m$, set

$$\omega_t(i) := \begin{cases} \omega(i) & \text{if } U(i) > t, \\ \omega'(i) & \text{if } U(i) \le t. \end{cases} \tag{15}$$

We shall write $\mathbb{P}$ for the associated probability measure, and $\mathbb{E}$ for its expectation.

We will derive a dynamical covariance formula for functions in $L^2(F^m)$, i.e. the collection of real-valued functions $f : \mathbb{R}^m \to \mathbb{R}$ such that

$$\mathbb{E}[f(\omega)^2] = \int f^2 \, dF^m < \infty.$$

Let $\sigma_i^x : \mathbb{R}^m \to \mathbb{R}^m$ denote the operator that replaces the value of the ith coordinate with x. Hence, $f \circ \sigma_i^x$ evaluates f with the ith coordinate fixed to be x. We introduce the following notation for the difference between f and its average over the ith coordinate: For $x \in \mathbb{R}$ let

$$D_i^x f(\omega) := f \circ \sigma_i^x(\omega) - \int f \circ \sigma_i^y(\omega) \, dF(y), \tag{16}$$

$$D_i f(\omega) := f(\omega) - \int f \circ \sigma_i^y(\omega) \, dF(y). \tag{17}$$

We define the **co-influence** of coordinate i for the function f at time t as

$$\mathrm{Inf}_i\big(f(\omega_0), f(\omega_t)\big) := \int \mathbb{E}\big[D_i^x f(\omega_0) D_i^x f(\omega_t)\big] \, dF(x). \tag{18}$$

We remark that the differences in (16)-(17) are not necessarily well-defined for every $\omega \in \mathbb{R}^m$, since the average over the ith coordinate is only guaranteed to exist almost surely. However, using Jensen's inequality, we note that

$$\mathbb{E}[(D_i f(\omega))^2] \leq \mathbb{E}\left[\left(\int f \circ \sigma_i^y(\omega) \, dF(y)\right)^2\right] \leq \mathbb{E}[f(\omega)^2] < \infty, \tag{19}$$

so that $D_i f$ is again in $L^2(F^m)$. It follows (see also Lemma 6 below) that $D_i^x f$ is in $L^2(F^m)$ for F-almost every $x \in \mathbb{R}$, and so the co-influences are well-defined for every $t \in [0, 1]$. We note further that

$$\mathrm{Inf}_i\big(f(\omega_0), f(\omega_0)\big) = \mathbb{E}[(D_i f)^2],$$

which recovers the L^2-notion of the influence of a coordinate as in (11), in analogy to its appearance in the study of Boolean functions [28, 31].

The main result of this section is a formula that relates the dynamical covariance of a function f to the (dynamic) co-influences of the respective coordinates.

Proposition 4. *For every $f \in L^2(F^m)$ the covariance $\mathrm{Cov}(f(\omega_0), f(\omega_t))$ is non-negative and non-increasing as a function of t and, for $t \in [0, 1]$, it satisfies*

$$\mathrm{Cov}\big(f(\omega_0), f(\omega_t)\big) = \int_t^1 \sum_{i=1}^m \mathrm{Inf}_i\big(f(\omega_0), f(\omega_s)\big) \, ds.$$

We shall prove the proposition in two intermediary steps, using coupling techniques similar to those in [32]. To guide the proof, we introduce the notation

$$Q_t(f) := \mathbb{E}[f(\omega_0)f(\omega_t)].$$

In this notation, we have

$$\operatorname{Cov}\big(f(\omega_0), f(\omega_t)\big) = Q_t(f) - Q_1(f) = \int_t^1 -Q_s'\,ds, \tag{20}$$

under the assumption that $Q_t(f)$ is continuously differentiable (which we verify below).

As a first step we prove the following lemma, using a standard coupling argument.

Lemma 5. *For every $f \in L^2(F^m)$ and $0 \le s \le t \le 1$ we have*

$$Q_s(f) \ge Q_t(f) \ge 0.$$

Proof. Let $0 \le s \le t \le 1$ be fixed, and let V be a random vector with i.i.d. coordinates distributed as F. We shall construct new random vectors X, Y and Z from V through a resampling procedure. Since the resampling procedure acts independently on each coordinate, the resulting vectors are again i.i.d. with marginal distribution F.

Set

$$p = 1 - \sqrt{1-s},\ q = 1 - \sqrt{1-t} \text{ and } r = \frac{q-p}{1-p}.$$

Since by assumption $0 \le s \le t \le 1$, it follows that $0 \le p \le q \le 1$, and hence that $0 \le r \le 1$.

Let W, W', W'', W''' be independent copies of V. First, let V' be obtained from V by independently replacing each coordinate of V by the corresponding coordinate from W with probability r. Next, we obtain X from V' by independently replacing each coordinate of V' with the corresponding coordinate from W' with probability p, and similarly obtain Y from V' by independently replacing each coordinate of V' with the corresponding coordinate from W'' with probability p. Finally, we obtain Z from V by independently replacing each coordinate with the corresponding coordinate of W''' with probability q.

A straightforward calculation shows that the joint distribution of the pair (X, Y) is equal to that of (ω_0, ω_s), and that the joint distribution of (X, Z) is equal to that of (ω_0, ω_t). Moreover, X and Y are conditionally independent given V', and X and Z are conditionally independent given V. Therefore, it follows that

$$\begin{aligned}
Q_s(f) &= \mathbb{E}[f(X)f(Y)] = \mathbb{E}\big[\mathbb{E}[f(X)f(Y)|V']\big] = \mathbb{E}\big[\mathbb{E}[f(X)|V']^2\big] \ge 0,\\
Q_t(f) &= \mathbb{E}[f(X)f(Z)] = \mathbb{E}\big[\mathbb{E}[f(X)f(Z)|V]\big] = \mathbb{E}\big[\mathbb{E}[f(X)|V]^2\big] \ge 0.
\end{aligned}$$

Let $\mathcal{F}$ denote the σ-algebra of information generated by V, W and the randomness used to obtain V' from V. Both V and V' are measurable with respect to $\mathcal{F}$, but $\mathcal{F}$ contains no more information regarding X. Hence, it follows from Jensen's inequality that

$$Q_t(f) = \mathbb{E}\big[\mathbb{E}[f(X)|V]^2\big] = \mathbb{E}\big[\mathbb{E}[\,\mathbb{E}[f(X)|\mathcal{F}]\,|V]^2\big] \leq \mathbb{E}\big[\mathbb{E}[f(X)|\mathcal{F}]^2\big] = Q_s(f),$$

as required. ∎

From the previous lemma we obtain an analogous statement for the co-influences.

Lemma 6. *For every $f \in L^2(F^m)$ and $i = 1, 2, \ldots, m$ we have $D_i^x f \in L^2(F^m)$ for F-almost every $x \in \mathbb{R}$. Moreover, for $0 \leq s \leq t \leq 1$ the co-influences are finite and*

$$\mathrm{Inf}_i(f(\omega_0), f(\omega_s)) \geq \mathrm{Inf}_i(f(\omega_0), f(\omega_t)) \geq 0.$$

Proof. Using Cauchy-Schwartz' inequality and (19) we obtain that

$$\mathrm{Inf}_i(f(\omega_0), f(\omega_t)) \leq \int \mathbb{E}\big[(D_i^x f(\omega))^2\big]\, dF(x) = \mathbb{E}\big[(D_i f(\omega))^2\big] < \infty.$$

In particular, the influences are well-defined and finite, and $D_i^x f$ is in $L^2(F^m)$ for F-almost every $x \in \mathbb{R}$. Moreover,

$$\mathrm{Inf}_i(f(\omega_0), f(\omega_t)) = \int \mathbb{E}\big[D_i^x f(\omega_0) D_i^x f(\omega_t)\big]\, dF(x) = \int Q_t(D_i^x f)\, dF(x).$$

By Lemma 5 we have for F-almost every $x \in \mathbb{R}$ that $Q_t(D_i^x f)$ is non-negative and non-decreasing in t. These properties are preserved under integration (over x), so the proof is complete. ∎

We next move to relate the derivative of $Q_t(f)$ with the co-influences at time t. For proof-technical reasons we shall extend the construction so that each coordinate abides to a different time variable. Let $X = (X_1, \ldots, X_m)$ and $Z = (Z_1, \ldots, Z_m)$ be two independent $\mathbb{R}^m$-valued random vectors with i.i.d. components distributed as F, and let $U_1, \ldots, U_m$ be independent uniform random variables on $[0, 1]$. Given $t_1, \ldots, t_m \in [0, 1]$, let $Y(t_1, \ldots, t_m)$ be the m-dimensional random vector, where the ith coordinate is given by

$$Y_i(t_i) = \begin{cases} X_i & \text{if } U_i > t_i, \\ Z_i & \text{if } U_i \leq t_i. \end{cases} \tag{21}$$

The vector $Y(t_1, \ldots, t_m)$ is hence obtained from X by resampling the ith coordinate with probability t_i, then replacing it with the corresponding entry from Z. When all t_i are equal, we write Y_t for $Y(t, \ldots, t)$. In particular, (Y_0, Y_t) and (ω_0, ω_t) are equal in distribution and, for f in $L^2(F^m)$, we have

$$Q_t(f) = \mathbb{E}[f(Y_0)f(Y_t)].$$

Lemma 7. *For every $f \in L^2(F^m)$ and $i = 1, 2, \ldots, m$ we have*

$$-\frac{\partial}{\partial t_i}\mathbb{E}\big[f(X)f(Y(t_1,\ldots,t_m))\big] = \int \mathbb{E}\big[D_i^x f(X) D_i^x(Y(t_1,\ldots,t_m))\big]\, dF(x),$$

and the right-hand side is continuous in each of its coordinates.

Proof. The following argument is identical for all $i \in [m]$ so we consider only $i = 1$. We then want to compute the derivative of the function

$$\phi(t_1) := \mathbb{E}\big[f(X)f(Y(t_1,t_2,\ldots,t_m))\big].$$

To this end, fix $t_1 \in (0,1)$ and let $\delta > 0$ be such that $t_1+\delta \in [0,1]$. Write $Y = Y(t_1,\ldots,t_m)$ and $Y' = Y(t_1+\delta, t_2, \ldots, t_m)$, and observe that

$$\phi(t_1+\delta) - \phi(t_1) = \mathbb{E}\big[f(X)\big(f(Y') - f(Y)\big)\big].$$

Note that the random variables Y and Y' can differ only in their first coordinate. They do so when $U_1 \in (t_1, t_1+\delta]$, in which case $Y_1 = X_1$ and $Y_1' = Z_1$. Write $Y^X = (X_1, Y_2(t_2), \ldots, Y_m(t_m))$ and $Y^Z = (Z_1, Y_2(t_2), \ldots, Y_m(t_m)$. Since U_1 takes values in $(t_1, t_1+\delta]$ with probability δ, we have

$$\phi(t_1+\delta) - \phi(t_1) = \delta\mathbb{E}\big[f(X)\big(f(Y^Z) - f(Y^X)\big)\big]. \tag{22}$$

Since Y^Z and Y^X differ only in the first coordinate, and since $f(X) - D_1 f(X)$ is determined by $(X_2,\ldots,X_m)$, it follows (by averaging over X_1 and Z_1 first) that

$$\phi(t_1+\delta) - \phi(t_1) = \delta\mathbb{E}\big[D_1 f(X)\big(f(Y^Z) - f(Y^X)\big)\big].$$

Moreover, $f(Y^Z) - F(Y^X) = D_1 f(Y^Z) - D_1 f(Y^X)$, and since $\mathbb{E}[D_1 f(X)|(X_2,\ldots,X_m)]$ is zero, we conclude that

$$\phi(t_1+\delta) - \phi(t_1) = \delta\mathbb{E}\big[D_1 f(X)\big(D_1 f(Y^Z) - D_1 f(Y^X)\big)\big] = -\delta\mathbb{E}\big[D_1 f(X) D_1 f(Y^X)\big].$$

An analogous computation yields the same conclusion for $\delta < 0$, but with the opposite sign. (For $\delta < 0$, the variable U_1 takes values in $(t_1+\delta, t_1]$ with probability $-\delta$, which gives an additional '$-$' in the right-hand side of (22).) Hence

$$-\phi'(t_1) = \mathbb{E}\big[D_1 f(X) D_1 f(Y^X)\big] = \int \mathbb{E}[D_1^x f(X) D_1^x f(Y(t_1,\ldots,t_n))]\, dF(x),$$

as required. Finally, we note that $\mathbb{E}\big[D_1 f(X) D_1 f(Y^X)\big]$ is constant in t_1, and that an argument analogous to that leading to (22) shows that it is continuous in the remaining coordinates $t_2,\ldots,t_m$. $\square$

We are finally set to prove the main result of this section.

Proof of Proposition 4. First, recall that $\mathrm{Cov}(f(\omega_0), f(\omega_t)) = Q_t(f) - Q_1(f)$, so that non-negativity and (weak) monotonicity of the covariance follows from the monotonicity of $Q_t(f)$, which was proved in Lemma 5.

Second, Lemma 7 and the chain rule show that $Q_t(f)$ is continuously differentiable on $(0, 1)$ with

$$-Q_t'(f) = \sum_{i=1}^{m} \int \mathbb{E}\big[D_i^x f(Y_0) D_i^x (Y_t)\big]\, dF(x) = \sum_{i=1}^{m} \mathrm{Inf}_i\big(f(\omega_0), f(\omega_t)\big). \tag{23}$$

Hence, by the fundamental theorem of calculus, (20) is justified, which together with (23) completes the proof. □

3 A dynamical covariance formula in infinite volume

In the previous section we derived a dynamical covariance formula for real-valued functions on $\mathbb{R}^m$. In this section we extend the covariance formula to a formula for functions of infinitely many variables, and hence establish (12). We shall again work with general functions, as we shall need to apply parts of the theory to functions other than the distance function. Since variables associated with different edges of the lattice are independent, and since there is a bijection between $\mathbb{Z}^d$ and the natural numbers, and between the set of nearest-neighbour edges of $\mathbb{Z}^d$ and the natural numbers, it will suffice to establish a formula for functions $f : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ that satisfy

$$\mathbb{E}[f(\omega)^2] = \int f^2 \, dF^{\mathbb{N}} < \infty,$$

where again F denotes some probability distribution on $\mathbb{R}$. We shall denote this class of functions by $L^2(F^{\mathbb{N}})$, and use the convention that $\mathbb{N} = \{1, 2, \ldots\}$.

Definitions of central concepts introduced in the previous section extend straightforwardly to the infinite setting. Let ω and ω' be independent $\mathbb{R}^{\mathbb{N}}$-valued random vectors with i.i.d. coordinates distributed as F. Let $U(1), U(2), \ldots$ be independent uniform random variables on $[0, 1]$, and let ω_t be the $\mathbb{R}^{\mathbb{N}}$-valued random vector defined as in (15). Note that the definitions of differences and co-influences in (16)-(18) now extend straightforwardly to function in $L^2(F^{\mathbb{N}})$.

The main result of this section is the following dynamic covariance formula.

Proposition 8. *For every $f \in L^2(F^{\mathbb{N}})$ the covariance $\mathrm{Cov}(f(\omega_0), f(\omega_t))$ is non-negative and non-increasing as a function of t and, for $t \in [0, 1]$, it satisfies*

$$\mathrm{Cov}\big(f(\omega_0), f(\omega_t)\big) = \int_t^1 \sum_{i=1}^{\infty} \mathrm{Inf}_i\big(f(\omega_0), f(\omega_s)\big)\, ds.$$

Again, since edge weights are i.i.d. and there is a bijection between the set of edges of the nearest-neighbour lattice and the natural numbers, we note that the variance formula in (12) is an immediate consequence of Proposition 8 by taking $t = 0$.

Our first lemma is an infinite volume version of Lemma 5, and is proved analogously. Since we shall need a version of the lemma where different coordinates are adjusted separately, we reuse the notation from the previous section. Let X and Z be two independent $\mathbb{R}^{\mathbb{N}}$-valued random vectors with i.i.d. components distributed as F, and let $U_1, U_2, \dots$ be independent uniform random variables on $[0, 1]$. For $t_1, t_2, \dots \in [0, 1]$, let $Y(t_1, t_2, \dots)$ be the $\mathbb{R}^{\mathbb{N}}$-valued random vector whose coordinates are defined as in (21), and write Y_t for $Y(t, t, \dots)$ for compactness. As before, (Y_0, Y_t) and (ω_0, ω_t) have the same distribution, and we let

$$Q_t(f) := \mathbb{E}[f(\omega_0)f(\omega_t)] = \mathbb{E}[f(Y_0)f(Y_t)].$$

Lemma 9. *For every $f \in L^2(F^{\mathbb{N}})$ and $0 \le s_i \le t_i \le 1$ for $i = 1, 2, \dots$, we have*

$$\mathbb{E}\big[f(X)f(Y(s_1, s_2, \dots))\big] \ge \mathbb{E}\big[f(X)f(Y(t_1, t_2, \dots))\big] \ge 0.$$

In particular, for $0 \le s \le t \le 1$, we have $Q_s(f) \ge Q_t(f) \ge 0$.

Proof. This follows from a straightforward adaptation of the proof of Lemma 5, where the probabilities p, q and r are now coordinate dependent. □

We similarly obtain the following extension of Lemma 6.

Lemma 10. *For every $f \in L^2(F^{\mathbb{N}})$ and $i = 1, 2, \dots$, we have $D_i^x f \in L^2(F^{\mathbb{N}})$ for F-almost every $x \in \mathbb{R}$. Moreover, for $0 \le s \le t \le 1$, the co-influences are finite and satisfy*

$$\mathrm{Inf}_i(f(\omega_0), f(\omega_s)) \ge \mathrm{Inf}_i(f(\omega_0), f(\omega_t)) \ge 0.$$

Proof. The proof is identical to the proof of Lemma 6. □

We are now in a position to derive the main result of this section.

Proof of Proposition 8. That the covariance is non-negative and non-increasing is immediate from Lemma 9 and the fact that $\mathrm{Cov}(f(\omega_0), f(\omega_t)) = Q_t(f) - Q_1(f)$. We shall derive the remaining formula by approximating f by the function obtained by averaging over all but the first m coordinates, and apply Proposition 4.

Let $\mathcal{F}_m$ denote the σ-algebra generated by $\{\omega(i), \omega'(i), U(i) : i = 1, 2, \dots, m\}$, and set

$$h_m(\omega_t) := \mathbb{E}[f(\omega_t)|\mathcal{F}_m].$$

Since $f \in L^2(F^{\mathbb{N}})$, the sequence $(h_m)_{m\geq 1}$ forms an L^2-bounded martingale, and by Levy's upwards theorem, as $m \to \infty$, we have that

$$h_m(\omega_t) \to f(\omega_t) \quad \text{almost surely and in } L^2.$$

It follows, in particular, that

$$\operatorname{Cov}\big(h_m(\omega_0), h_m(\omega_t)\big) \to \operatorname{Cov}\big(f(\omega_0), f(\omega_t)\big), \quad \text{as } m \to \infty. \tag{24}$$

Moreover, applying Proposition 4, we obtain that

$$\operatorname{Cov}\big(h_m(\omega_0), h_m(\omega_t)\big) = \int_t^1 \sum_{i=1}^m \operatorname{Inf}_i\big(h_m(\omega_0), h_m(\omega_s)\big)\, ds. \tag{25}$$

We next want to argue that, for every $i \in \mathbb{N}$, $m \geq i$ and $s \in [0,1]$, we have

$$\operatorname{Inf}_i\big(h_m(\omega_0), h_m(\omega_s)\big) \to \operatorname{Inf}_i\big(f(\omega_0), f(\omega_s)\big) \quad \text{as } m \to \infty, \tag{26}$$

and that the convergence is monotone in m. (For $m < i$ we have $\operatorname{Inf}_i(h_m(\omega_0), h_m(\omega_s)) = 0$, but this is not a crucial observation.) Once this has been established, since the co-influences are non-negative according to Lemma 6, it follows from the monotone convergence theorem, together with (24) and (25), that

$$\operatorname{Cov}\big(f(\omega_0), f(\omega_t)\big) = \lim_{m\to\infty} \operatorname{Cov}\big(h_m(\omega_0), h_m(\omega_t)\big) = \int_t^1 \sum_{i=1}^\infty \operatorname{Inf}_i\big(f(\omega_0), f(\omega_s)\big)\, ds,$$

as required.

It remains to establish the monotone convergence in (26). By Lemma 10, we have $D_i^x f \in L^2(F^{\mathbb{N}})$ for F-almost every $x \in \mathbb{R}$. For these values of x, Fubini's theorem shows that

$$D_i^x h_m(\omega_s) = \mathbb{E}[D_i^x f(\omega_s) | \mathcal{F}_m] \quad \text{almost surely,} \tag{27}$$

so that, by Levy's upwards theorem, as $m \to \infty$

$$D_i^x h_m(\omega_s) \to D_i^x f(\omega_s) \quad \text{almost surely and in } L^2.$$

We conclude in particular that, for F-almost every x, we have

$$\mathbb{E}[D_i^x h_m(\omega_0) D_i^x h_m(\omega_s)] \to \mathbb{E}[D_i^x f(\omega_0) D_i^x f(\omega_s)] \quad \text{as } m \to \infty. \tag{28}$$

To see that the convergence in (28) is monotone, let $Y^{(m)}$ denote the vector $Y(t_1, t_2, \ldots)$ where $t_i = s$ for $i \leq m$ and $t_i = 1$ for $i > m$. By (27) and the independence between X and $Y^{(m)}$ in coordinates $i > m$, we may rewrite the left-hand side of (28) as

$$\mathbb{E}[D_i^x h_m(\omega_0) D_i^x h_m(\omega_s)] = \mathbb{E}\big[\mathbb{E}[D_i^x f(\omega_0)|\mathcal{F}_m]\mathbb{E}[D_i^x f(\omega_s)|\mathcal{F}_m]\big] = \mathbb{E}[D_i^x f(X) D_i^x f(Y^{(m)})],$$

for those x for which $D_i^x f$ is in $L^2(F^{\mathbb{N}})$. For these values of x, the expression is non-negative and non-decreasing in m, according to Lemma 9, so the convergence in (28) is indeed monotone for F-almost every x. Consequently, it follows from (28) and the monotone convergence theorem that, as $m \to \infty$, we have

$$\mathrm{Inf}_i\big(h_m(\omega_0), h_m(\omega_s)\big) = \int \mathbb{E}[D_i^x h_m(\omega_0) D_i^x h_m(\omega_s)]\, dF(x) \to \mathrm{Inf}_i\big(f(\omega_0), f(\omega_s)\big). \tag{29}$$

Moreover, since the left-hand side of (28) is monotone in m for F-almost every x, it follows that the left-hand side in (29) is again monotone in m. Hence, the convergence in (29) is monotone, which proves (26). □

4 Influences in first-passage percolation

The dynamic covariance formula in Proposition 8 connects the variance of a function to the co-influences of its variables. We now move on to analyse the co-influences in the setting of the first-passage percolation. Our aim will be to establish the connection between fluctuations and geodesics described in (6). In this section we make some preliminary observations, and fix notation that will be used throughout.

Henceforth, F will denote (the distribution function of) some probability measure on $[0, \infty)$, used to sample the edge weights of the $\mathbb{Z}^d$ lattice. For ease of notation, we henceforth fix $v \in \mathbb{Z}^d$ and let $T := T(0, v) = T(0, v)(\omega)$ denote the distance between 0 and v in the configuration ω. It is well-known that finite second moment for F is sufficient for the first-passage time $T = T(0, v)$, for every $v \in \mathbb{Z}^d$, to have finite second moment too. In the above notation, Proposition 8 gives (12), i.e. that

$$\mathrm{Var}(T) = \int_0^1 \sum_{e \in \mathcal{E}} \mathrm{Inf}_e(T_0, T_t)\, dt, \tag{30}$$

where, as before, $T_t := T(\omega_t) = T(0, v)(\omega_t)$ and ω_t denotes the dynamical edge weight as defined in (2).

Since any two vertices of $\mathbb{Z}^d$ (assuming $d \geq 2$) are connected by (at least) two disjoint paths, it follows that the passage time $T(\sigma_e^x \omega)$ between 0 and v in the configuration $\sigma_e^x \omega$ is bounded as a function of x, for every ω and e. As a consequence, also $D_e^x T(\omega)$ takes a well-defined finite value for every x and ω and, by definition (in (16)), this function integrates to zero, i.e.

$$\int D_e^x T\, dF(x) = 0 \tag{31}$$

for every edge e and weight configuration ω.

Recall that r denotes the infimum of the support of F. We shall typically think of $x \mapsto D_e^x T$ as a function from $[r, \infty) \to \mathbb{R}$. We note that $T(\omega)$ is monotone (non-decreasing) in each of its coordinates $\omega(e)$, from which it follows that $D_e^x T$ is monotone in x, for every e and ω. In fact, either $D_e^x T$ is constant equal to zero (as a function of x), in case there is a path not using e which is optimal (for the configuration $\sigma_e^x \omega$) for all values of $x \geq r$, or $D_e^x T$ grows linearly for small values of x, and is constant for large values of x; see Figure 1.

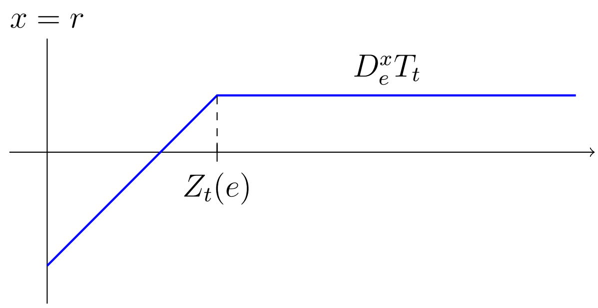

Figure 1: Plot of the function $x \mapsto D_e^x T_t$ over the interval $[r, \infty)$.

Since $D_e^x T_t$ is monotone and constant for large x, there is a point at which increasing x no longer has an effect on $D_e^x T_t$. We denote this point (depicted in Figure 1) by $Z_t(e)$, i.e.

$$Z_t(e) := \min\{x \geq r : D_e^x T_t = D_e^y T_t \text{ for all } y \geq x\}.$$

In particular, we have $r \leq Z_t(e) < \infty$ for every weight configuration ω_t. Since $D_e^x T_t$ grows linearly for $x \leq Z_t(e)$ and is constant for $x > Z_t(e)$, it must be of the form $C-(Z_t(e)-x)_+$ for some $C = C(\omega_t)$ that does not depend on x, and f_+ denotes the positive part of the function f. Since $D_e^x T_t$ integrates to zero, it follows from (31) that

$$D_e^x T_t = \int (Z_t(e) - y)_+ \, dF(y) - (Z_t(e) - x)_+. \tag{32}$$

An immediate consequence of (32) is that

$$D_e^x T_t \geq \int (Z_t(e) - y) \, dF(y) - Z_t(e) = -\mu, \tag{33}$$

where μ is the mean of F. Since $D_e^x T_t$ is constant equal to zero when $Z_t(e) = r$, we obtain

$$|D_e^x T_t| \leq (\mu + x)\mathbf{1}_{\{Z_t(e) > r\}}, \tag{34}$$

which can be used as a basis for upper bounding the co-influence of e.

Although we shall not make use of the following observation in this paper, we note that the representation in (32) allows us to interpret the co-influence as the covariance over the given coordinate, in the sense described next. Let

$$\mathcal{F}_e := \sigma\big(\{\omega(e'), \omega'(e'), U(e') : e' \neq e\}\big),$$

i.e. the sigma algebra generated by all variables except for those associated with the edge e. Since $D_e^x T_t$ is determined by $\omega_t(e')$ for $e' \neq e$, it follows that $Z_t(e)$ is $\mathcal{F}_e$-measurable. From (32) we obtain that

$$\mathrm{Inf}_e(T_0, T_t) = \mathbb{E}\big[\mathrm{Cov}\big((Z_0(e) - \tilde{\omega})_+, (Z_t(e) - \tilde{\omega})_+ \big| \mathcal{F}_e\big)\big], \tag{35}$$

where $\tilde{\omega}$ is a generic F-distributed random variable independent of everything else.

Our goal will be to relate the co-influences to the joint inclusion in the geodesic. Let $\pi_t := \pi_t(0, v)$. The connection to geodesics can be conceived from (35), as the event $\{e \in \pi_t\}$ is roughly equivalent with the weight at e not exceeding $Z_t(e)$. In turn, the joint inclusion is connected to the expected overlap through the identity

$$\mathbb{E}[|\pi_0 \cap \pi_t|] = \sum_{e \in \mathcal{E}} \mathbb{P}(e \in \pi_0 \cap \pi_t). \tag{36}$$

Before embarking on the quest to make this precise, we end this section with an observation regarding the expected overlap of the geodesics.

Lemma 11. *For every $v \in \mathbb{Z}^d$, the function $\mathbb{E}[|\pi_0 \cap \pi_t|]$ is non-increasing in t.*

Proof. Consider the function $f_e = \mathbf{1}_{\{e \in \pi\}}$. By Lemma 9, we have that

$$Q_t(f_e) = \mathbb{E}[f_e(\omega_0) f_e(\omega_t)] = \mathbb{P}(e \in \pi_0 \cap \pi_t)$$

is non-increasing in t. Hence, the result follows from (36). $\square$

5 Integer-valued edge weight distributions

Since the analysis of the co-influences is more straightforward in the integer-valued setting, we consider this setting first. The aim of this section is hence to establish (6) and prove Theorem 1. In view of Proposition 8, it will suffice to establish (13).

Proof of Theorem 1. Suppose that F is supported on $\{0, 1, 2, \ldots\}$ and has finite second moment. Then also r, the infimum of the support of F, is an integer and $F(r) = \mathbb{P}(\omega_e = r)$. Moreover, T_t and $Z_t(e)$ are supported on the integers, for all $t \in [0, 1]$ and e. The last statement implies that, if $Z_t(e) > r$ and the weight at e increases from r to some larger value, then the passage time between 0 and v must increase by at least one. It follows that, on the event $\{Z_t(e) > r\}$, we have

$$\int (Z_t(e)-x)_+ \, dF(x) \leq (Z_t(e)-r)F(r)+(Z_t(e)-r-1)(1-F(r)) = (Z_t(e)-r)-(1-F(r)).$$

Hence, by (32), we obtain that

$$D_e^r T_t = \int (Z_t(e) - x)_+ \, dF(x) - (Z_t(e) - r) \leq -(1 - F(r))\mathbf{1}_{\{Z_t(e)>r\}}, \tag{37}$$

since also $D_e^r T_t = 0$ on the event $\{Z_t(e) = r\}$.

By Lemma 9, we know that $\mathbb{E}[D_e^x T_0 D_e^x T_t] = Q_t(D_e^x T)$ is non-negative for all x, which gives the lower bound

$$\mathrm{Inf}_e(T_0, T_t) = \int \mathbb{E}[D_e^x T_0 D_e^x T_t] \, dF(x) \geq F(r)\mathbb{E}[D_e^r T_0 D_e^r T_t].$$

Together with (37), this gives that

$$\mathrm{Inf}_e(T_0, T_t) \geq F(r)(1 - F(r))^2 \, \mathbb{P}\big(Z_0(e) > r, Z_t(e) > r\big).$$

So by (30) and (36), since $\{e \in \pi_t\} \subseteq \{Z_t(e) > r\}$, we obtain

$$\mathrm{Var}(T) \geq F(r)(1 - F(r))^2 \int_0^1 \sum_{e \in \mathcal{E}} \mathbb{P}(e \in \pi_0 \cap \pi_t) \, dt = C \int_0^1 \mathbb{E}[|\pi_0 \cap \pi_t|] \, dt. \tag{38}$$

To obtain a matching upper bound, we recall (34), which gives

$$\mathrm{Inf}_e(T_0, T_t) \leq \int (\mu + x)^2 \, dF(x) \, \mathbb{P}(Z_0(e) > r, Z_t(e) > r), \tag{39}$$

which is finite since F has finite second moment. Let $A = \{Z_0(e) > r, Z_t(e) > r\}$ and $B = \{\omega_0(e) = \omega_t(e) = r\}$, and note that $A \cap B \subseteq \{e \in \pi_0 \cap \pi_t\}$. Moreover, A is $\mathcal{F}_e$-measurable, and hence independent of B. Hence,

$$\mathbb{P}(A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \leq \frac{\mathbb{P}(e \in \pi_0 \cap \pi_t)}{F(r)^2},$$

so (30) and (39) together with (36) yield that

$$\mathrm{Var}(T) \leq C' \int_0^1 \sum_{e \in \mathcal{E}} \mathbb{P}(e \in \pi_0 \cap \pi_t) \, dt = C' \int_0^1 \mathbb{E}[|\pi_0 \cap \pi_t|] \, dt,$$

for some constant C'. Combining this with (38) establishes (6).

Next suppose that (9) holds. Recall from Lemma 11 that $\mathbb{E}[|\pi_0 \cap \pi_t|]$ is monotone (non-decreasing) as a function of t. Hence, by (38), we have for every fixed $t \in (0, 1)$ that

$$\mathrm{Var}(T) \geq C \int_0^t \mathbb{E}[|\pi_0 \cap \pi_s|] \, ds \geq Ct \, \mathbb{E}[|\pi_0 \cap \pi_t|].$$

The bound on the expected overlap now follows from (8). $\square$

6 General edge weight distributions

We now proceed with the analysis in the more general setting, and prove Theorem 2. We shall bound the co-influences from below and above separately, which together amount to a version of (13), and use these bounds to prove Theorem 2 at the end.

Our bounds on the co-influences will be the result of a more detailed analysis of the function $D_e^x T_t$ depicted in Figure 2. The additional difficulty comes from adequately estimating the co-influence when $Z_t(e) > r$ is small, and the function $D_e^x T_t$ close to zero, which cannot happen for integer-valued weight distributions. In addition to $Z_t = Z_t(e)$,

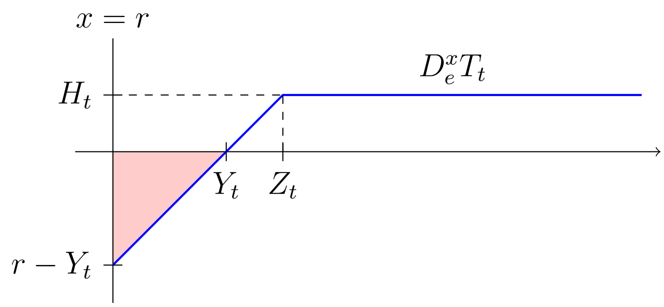

Figure 2: Illustration of the variables $Z_t = Z_t(e)$, $Y_t = Y_t(e)$ and $H_t = H_t(e)$.

we identify two other variables associated with the function $D_e^x T_t$, which are illustrated in Figure 2. Since $D_e^x T_t$ is monotone and integrates to zero, there is a first point at which it intersects the real axis, and remains non-negative after. We denote this point by $Y_t = Y_t(e)$, i.e.

$$Y_t(e) := \min\{x \geq r : D_e^x T_t \geq 0\}.$$

Of course, $r \leq Y_t(e) \leq Z_t(e)$ and $Y_t(e) = Z_t(e) = r$ if $D_e^x T_t$ is constant (and hence equal to zero). In addition, we let $H_t = H_t(e)$ denote the height of the flat segment of $D_e^x T_t$, which is attained for $x \geq Z_t(e)$, i.e.

$$H_t(e) := D_e^{Z_t(e)} T_t.$$

Recall that $Z_t(e)$ is finite, so that $H_t(e)$ is well-defined. However, we remark that $Z_t(e)$ is not necessarily in the support of F, and the value $H_t(e)$ is not necessarily attained for x in the support of F.

When it is clear from the context which edge e that is referred to, we suppress the dependence on e and simply write Y_t, Z_t and H_t for compactness.

6.1 Co-influence lower bound

We first establish the following lower bound on the co-influences.

Proposition 12. *Suppose that F has finite second moment and that $F(r) = 0$. For every $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that for all $e \in \mathcal{E}$ and $t \in [0, 1]$ we have*

$$\mathrm{Inf}_e(T_0, T_t) \geq C(\varepsilon)\big[\mathbb{P}(e \in \pi_0 \cap \pi_t) - 2\mathbb{P}\big(\{e \in \pi_0\} \cap \{\omega_0(e) \leq r + \varepsilon\}\big)\big].$$

We will derive the lower bound in three steps. We first show that it will suffice to estimate the contribution to the co-influences that comes from the shaded region below the axis in Figure 2. Second, we show that if (the contribution from) this region is small, then that is because $Z_t(e)$ is small. Finally, we deduce that an edge that has a reasonable chance of being on the geodesic will have a reasonable contribution to the co-influence.

In the first step we use the monotonicity of $D_e^x T_t$ to obtain the following lemma.

Lemma 13. *For every $e \in \mathcal{E}$ and $t \in [0, 1]$, we have*

$$\int D_e^x T_0 D_e^x T_t \, dF(x) \geq \int (D_e^x T_0)_-(D_e^x T_t)_- \, dF(x).$$

Proof. Let $e \in \mathcal{E}$ and $t \in [0, 1]$ be fixed. By symmetry, we may without loss of generality assume that $Y_0 \leq Y_t$. By (31), it holds that $D_e^x T_t$ integrates to zero, and hence, since $D_e^x T_t$ is negative below Y_t and positive above, that

$$-\int_{[r,Y_t]} D_e^x T_t \, dF(x) = \int_{[Y_t,\infty)} D_e^x T_t \, dF(x). \tag{40}$$

(At Y_t the integrand is zero, so it does not matter if we may include the point Y_t in the interval of integration or not.) Since $Y_0 \leq Y_t$ by assumption and $D_e^x T_t$ is negative below Y_t, it follows that

$$-\int_{[Y_0,Y_t]} D_e^x T_t \, dF(x) \leq \int_{[Y_t,\infty)} D_e^x T_t \, dF(x).$$

Using the fact that $D_e^x T_0$ is non-decreasing and non-negative on $[Y_0, \infty)$, we obtain that

$$-\int_{[Y_0,Y_t]} D_e^x T_0 D_e^x T_t \, dF(x) \leq \int_{[Y_t,\infty)} D_e^x T_0 D_e^x T_t \, dF(x).$$

Since both $D_e^x T_0$ and $D_e^x T_t$ are negative on $[r, Y_0)$ and $D_e^x T_0$ is non-negative on $[Y_0, \infty)$, we conclude from the above that

$$\int_{[r,\infty)} D_e^x T_0 D_e^x T_t \, dF(x) \geq \int_{[r,Y_0)} D_e^x T_0 D_e^x T_t \, dF(x) = \int_{[r,\infty)} (D_e^x T_0)_-(D_e^x T_t)_- \, dF(x),$$

as required. ∎

In the second step we show that, if $Y_t(e)$ is close to r, then $Z_t(e)$ is close to r too.

Lemma 14. *For every $e \in \mathcal{E}$, $t \in [0,1]$ and $\varepsilon > 0$ such that $F(r+\varepsilon) \leq 1/2$, we have that if $Z_t(e) \geq r + \varepsilon$, then $Y_t(e) \geq r + \varepsilon/2$.*

Proof. Let $e \in \mathcal{E}$ and $t \in [0,1]$ be fixed. Fix $\varepsilon > 0$ such that $F(r+\varepsilon) \leq 1/2$, and suppose that $Z_t - r \geq \varepsilon$. By (40), and since $-D_e^x T_t \leq Y_t - r$ on $[r, Y_t]$, we have

$$\int_{[Y_t,\infty)} D_e^x T_t \, dF(x) = -\int_{[r,Y_t]} D_e^x T_t \, dF(x) \leq (Y_t - r) F(Y_t). \tag{41}$$

Now suppose that $Y_t < r + \varepsilon/2$. We then have $D_e^{r+\varepsilon/2} T_t \geq 0$ and, since $Z_t \geq r + \varepsilon$, hence that $D_e^x T_t \geq \varepsilon/2$ on $[r+\varepsilon, \infty)$. By choice of $\varepsilon > 0$, this gives

$$\int_{[Y_t,\infty)} D_e^x T_t \, dF(x) \geq \frac{\varepsilon}{2}\big(1 - F(r+\varepsilon)\big) \geq \frac{\varepsilon}{4}. \tag{42}$$

Moreover, if $Y_t < r + \varepsilon/2$, then $F(Y_t) \leq F(r+\varepsilon) \leq 1/2$, so (41) and (42) yield that $Y_t - r \geq \varepsilon/2$. $\square$

With these lemmas at hand we prove Proposition 12.

Proof of Proposition 12. Let $e \in \mathcal{E}$ and $t \in [0,1]$ be fixed. It will suffice to consider $\varepsilon > 0$ such that $F(r+\varepsilon) \leq 1/2$, so we fix ε accordingly. By Lemma 13, the influence can be lower bounded as

$$\mathrm{Inf}_e(T_0, T_t) = \int \mathbb{E}[D_e^x T_t D_e^x T_0] \, dF(x) \geq \int \mathbb{E}[(D_e^x T_0)_- (D_e^x T_t)_-] \, dF(x).$$

Let $\Delta := \min\{Y_0(e) - r, Y_t(e) - r\}$, and note that

$$\int (D_e^x T_0)_- (D_e^x T_t)_- \, dF(x) \geq \Big(\frac{\Delta}{2}\Big)^2 F(r + \Delta/2).$$

Furthermore, by Lemma 14, we have on the event $\{Z_0(e) > r + \varepsilon\} \cap \{Z_t(e) > r + \varepsilon\}$ that $\Delta \geq \varepsilon/2$, and so it follows that

$$\mathrm{Inf}_e(T_0, T_t) \geq \Big(\frac{\varepsilon}{4}\Big)^2 F(r + \varepsilon/4) \mathbb{P}\big(\{Z_0(e) > \varepsilon\} \cap \{Z_t(e) > \varepsilon\}\big). \tag{43}$$

Now observe that

$$\begin{aligned}
\{e \in \pi_t\} &\subseteq \{Z_t(e) > \varepsilon\} \cup \big[\{e \in \pi_t\} \cap \{Z_t(e) \leq \varepsilon\}\big] \\
&\subseteq \{Z_t(e) > \varepsilon\} \cup \big[\{e \in \pi_t\} \cap \{\omega_t(e) \leq r + \varepsilon\}\big],
\end{aligned}$$

so that

$$\{Z_t(e) > \varepsilon\} \supseteq \{e \in \pi_t\} \setminus \big[\{e \in \pi_t\} \cap \{\omega_t(e) \leq r + \varepsilon\}\big].$$

This means that we can bound

$$\mathbb{P}\big(\{Z_0(e) > \varepsilon\} \cap \{Z_t(e) > \varepsilon\}\big) \geq \mathbb{P}(e \in \pi_0 \cap \pi_t) - 2\mathbb{P}\big(\{e \in \pi_0\} \cap \{\omega_0(e) \leq r + \varepsilon\}\big),$$

which together with (43) completes the proof. $\square$

6.2 Co-influence upper bound

We proceed with the upper bound on the co-influences. The argument requires only (4), and the next proposition is thus stated accordingly.

Proposition 15. *Suppose that F satisfies* (4)*. There exists $C < \infty$ such that for every $e \in \mathcal{E}$ and $t \in [0, 1)$ we have that*

$$\mathrm{Inf}_e(T_0, T_t) \le \frac{C}{1-t}\mathbb{P}(e \in \pi_0 \cap \pi_t).$$

In a first step we bound the function $D_e^x T_t$ when $Z_t(e)$ is small.

Lemma 16. *Suppose that $F(r) < 1/2$. There exists $\gamma > r$ such that, for every $e \in \mathcal{E}$ and $t \in [0, 1]$, we have that if $Z_t(e) \le \gamma$ then $H_t(e) \le F(Y_t(e))$.*

Proof. Let $e \in \mathcal{E}$ and $t \in [0, 1]$ be fixed. Fix $\gamma \in (r, r + 1/2)$ such that $F(\gamma) \le 1/2$, and suppose that $Z_t \le \gamma$. Then

$$-\int_{[r,Y_t]} D_e^x T_t \, dF(x) \le (Y_t - r)F(Y_t) \le F(Y_t)/2.$$

Furthermore, the balance equation in (40) and the assumption that $Z_t \le \gamma$ imply that

$$-\int_{[r,Y_t]} D_e^x T_t \, dF(x) \ge \int_{(Z_t,\infty)} D_e^x T_t \, dF(x) \ge H_t(1 - F(Z_t)) \ge H_t/2.$$

The two equations give $H_t \le F(Y_t)$, as required. $\square$

Proof of Proposition 15. If $F(r) > 0$, then the argument in the integer-valued setting applies. We may therefore assume that $F(r) = 0$.

Fix $e \in \mathcal{E}$ and $t \in [0, 1)$. Also fix $\gamma > r$ as in Lemma 16. We shall bound the co-influence separately depending on whether both Z_0 and Z_t exceed γ or not, by dividing the co-influence $\mathrm{Inf}_e(T_0, T_t)$ into the following sum

$$\int \mathbb{E}[D_e^x T_0 D_e^x T_t \mathbf{1}_{\{Z_0(e)>\gamma, Z_t(e)>\gamma\}}] \, dF(x) + \int \mathbb{E}[D_e^x T_0 D_e^x T_t \mathbf{1}_{\{Z_0(e)\le\gamma\}\cup\{Z_t(e)\le\gamma\}}] \, dF(x). \quad (44)$$

The former of the two terms above can be bounded in the same way as in the integer-valued setting. More precisely, appealing to (34) yields

$$\int \mathbb{E}[D_e^x T_0 D_e^x T_t \mathbf{1}_{\{Z_0(e)>\gamma, Z_t(e)>\gamma\}}] \, dF(x) \le \int (\mu + x)^2 \, dF(x) \, \mathbb{P}(Z_0(e) > \gamma, Z_t(e) > \gamma)$$

Now, with $A = \{Z_0(e) > \gamma, Z_t(e) > \gamma\}$ and $B = \{\omega_0(e) \le \gamma, \omega_t(e) \le \gamma\}$, we have $A \cap B \subseteq \{e \in \pi_0 \cap \pi_t\}$. Since A and B are independent, and since F has finite second moment, we obtain that

$$\int \mathbb{E}[D_e^x T_0 D_e^x T_t \mathbf{1}_{\{Z_0(e)>\gamma, Z_t(e)>\gamma\}}]\, dF(x) \le C \frac{\mathbb{P}(e \in \pi_0 \cap \pi_t)}{F(\gamma)^2} \tag{45}$$

for some finite constant C.

We proceed with the bound on the latter term of (44). Suppose first that $Y_0 \le Y_t$, which also implies that $Z_0 \le Z_t$. Then, on the interval $[r, Y_0]$ both $D_e^x T_0$ and $D_e^x T_t$ are negative and by (33) lower bounded by $-\mu$, whereas on $[Y_t, \infty)$ both $D_e^x T_0$ and $D_e^x T_t$ are positive and by definition and (34) upper bounded by H_0 and $\mu + x$, respectively. Since for the remaining values of x the functions have different signs, we obtain the upper bound

$$\int D_e^x T_0 D_e^x T_t \, dF(x) \le \mu^2 F(Y_0) + H_0 \int_{[Y_t,\infty)} (\mu + x)\, dF(x) \le \mu^2 F(Y_0) + 2\mu H_0.$$

By assumption $Z_0 \le Z_t$ and hence, on the event that either $Z_0 \le \gamma$ or $Z_t \le \gamma$, we have that $Z_0 \le \gamma$. On this event we have by Lemma 16 that $H_0 \le F(Y_0)$, which gives the further upper bound $(\mu^2 + 2\mu)F(Y_0)$. If instead $Z_t \le Z_0$ and either $Z_0 < \gamma$ or $Z_t < \gamma$, we obtain analogously the upper bound $(\mu^2 + 2\mu)F(Y_t)$. This shows that

$$\int D_e^x T_0 D_e^x T_t \, dF(x) \mathbf{1}_{\{Z_0(e) \le \gamma\} \cup \{Z_t(e) \le \gamma\}} \le (\mu^2 + 2\mu) F(\overline{Y}(e)).$$

where $\overline{Y}(e) = \min\{Y_0(e), Y_t(e)\}$.

Next, we note that for $y \ge 0$ and $t \in [0,1)$ we have

$$\mathbb{P}\big(\omega_0(e) \le y, \omega_t(e) \le y\big) \ge \mathbb{P}\big(\omega_0 \le y \big| U(e) > t\big) \mathbb{P}(U(e) > t) = (1-t)F(y),$$

and hence that

$$F(\overline{Y}(e)) \le \frac{1}{1-t} \mathbb{P}\big(\omega_0(e) \le \overline{Y}(e), \omega_t(e) \le \overline{Y}(e) \big| \mathcal{F}_e\big).$$

Now either $\overline{Y}(e) = 0$ or $\overline{Y}(e) < \min\{Z_0(e), Z_t(e)\}$, which gives the further upper bound

$$F(\overline{Y}(e)) \le \frac{1}{1-t} \mathbb{P}\big(\omega_0(e) < Z_0(e), \omega_t(e) < Z_t(e) \big| \mathcal{F}_e\big) \le \frac{1}{1-t} \mathbb{P}(e \in \pi_0 \cap \pi_t | \mathcal{F}_e).$$

Taking expectation results in the bound

$$\int \mathbb{E}[D_e^x T_0 D_e^x T_t \mathbf{1}_{\{Z_0(e) \le \gamma\} \cup \{Z_t(e) \le \gamma\}}]\, dF(x) \le \frac{\mu^2 + 2\mu}{1-t} \mathbb{P}(e \in \pi_0 \cap \pi_t),$$

which together with (45) completes the proof. □

6.3 Proof of Theorem 2

We finally combine the upper and lower bounds on the co-influences to prove Theorem 2. For the lower bound we borrow a lemma from [30], which gives a bound on the number of edges with low weight in a geodesic.

Lemma 17 ([30, Lemma 5.1])**.** *Suppose that $F(r) = 0$. Then there exists a constant $C < \infty$ such that for all $v \in \mathbb{Z}^d$ and $\varepsilon > 0$*

$$\mathbb{E}\big[\#\big\{e \in \pi(0,v) : \omega(e) \le r + \varepsilon\big\}\big] \le C|v|F(r+\varepsilon)^{1/d}.$$

Proof of Theorem 2. Suppose that F has finite second moment and that $F(r) = 0$. It follows from Propositions 8 and 12, together with (36), that for every $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that

$$\begin{aligned}\mathrm{Var}(T) &\ge C(\varepsilon)\int_0^1 \sum_{e\in\mathcal{E}} \big[\mathbb{P}(e \in \pi_0 \cap \pi_t) - 2\mathbb{P}\big(\{e \in \pi_0\} \cap \{\omega_0(e) \le r + \varepsilon\}\big)\big]\, dt \\ &= C(\varepsilon)\bigg(\int_0^1 \mathbb{E}[|\pi_0 \cap \pi_t|]\, dt - 2\mathbb{E}\big[\#\big\{e \in \pi_0 : \omega_0(e) \le r+\varepsilon\big\}\big]\bigg).\end{aligned} \tag{46}$$

The lower bound thus follows from Lemma 17.

It follows from Proposition 8 and the monotonicity of the co-influences, in Lemma 10, that

$$\mathrm{Var}(T) = \int_0^1 \sum_{e\in\mathcal{E}} \mathrm{Inf}_e(T_0, T_t)\, dt \le 2\int_0^{1/2} \sum_{e\in\mathcal{E}} \mathrm{Inf}_e(T_0,T_t)\, dt.$$

Furthermore, by Proposition 15, there exists a universal constant $C < \infty$ such that $\mathrm{Inf}_e(T_0,T_t) \le 2C\,\mathbb{P}(e \in \pi_0 \cap \pi_t)$ for $t \in [0, 1/2]$. Together with (36) this gives

$$\mathrm{Var}(T) \le 4C\int_0^{1/2} \mathbb{E}[|\pi_0 \cap \pi_t|]\, dt \le 4C\int_0^1 \mathbb{E}[|\pi_0 \cap \pi_t|]\, dt, \tag{47}$$

as required.

Assume next that (9) holds. Recall from Lemma 11 that $\mathbb{E}[|\pi_0 \cap \pi_t|]$ is monotone (non-increasing) as a function of t. It follows from (46) and Lemma 17 that for every $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that for all $t \in (0,1)$

$$t\,\mathbb{E}[|\pi_0 \cap \pi_t|\,] \le \int_0^t \mathbb{E}[|\pi_0 \cap \pi_s|]\, ds \le C(\varepsilon)^{-1}\mathrm{Var}(T) + 2C|v|F(r+\varepsilon)^{1/d}, \tag{48}$$

where C is a fixed constant. By (8), the variance term is known to grow sublinearly in $|v|$. Since $F(r) = 0$, we obtain a sublinear bound on the expected overlap by taking $\varepsilon = \varepsilon(|v|) \to 0$ slowly, so that $C(\varepsilon)^{-1}|v|/\log|v| = o(|v|)$. ∎

7 Multiple valleys

In this section we prove Theorem 3. Suppose that F is continuous and satisfies (9). For $i = 1, 2, \dots, k$ let $\{\omega^{(i)}(e)\}_{e\in\mathcal{E}}$ independent families of i.i.d. edge weights with common distribution F. In addition, for $i = 1, 2, \dots, k$, let $\{U^{(i)}(e)\}_{e\in\mathcal{E}}$ be independent configurations of i.i.d. uniformly distributed random variables on the interval $[0, 1]$. Starting from the usual weight configuration $\omega = \{\omega(e)\}_{e\in\mathcal{E}}$ at time $t = 0$, we define independent perturbations of this weight configuration as

$$\omega_t^{(i)}(e) = \begin{cases} \omega(e) & \text{if } U^{(i)}(e) > t, \\ \omega^{(i)}(e) & \text{if } U^{(i)}(e) \le t. \end{cases} \tag{49}$$

That is, in the resulting configuration $\omega_t^{(i)} = \{\omega_t^{(i)}(e)\}_{e\in\mathcal{E}}$ each edge has been resampled with probability t, and this is carried out independently for the k configurations.

We note that the joint distribution of the pair $(\omega_t^{(i)}, \omega_t^{(j)})$, for $i \neq j$, is equal to the joint distribution of the pair of configurations (ω_0, ω_s), where $s = 2t - t^2$. This is because, independently for each edge e, we will have $\omega_t^{(i)}(e) = \omega_t^{(j)}(e) = \omega(e)$, unless $\omega(e)$ has been replaced by an independent variable in one of two independent attempts, each having success probability t. Hence the probability of resampling equals $s = 2t - t^2$.

Write $T_t^{(i)} = T_t^{(i)}(0, v)$ for the first passage time from 0 to v in the configuration $\{\omega_t^{(i)}(e)\}_{e\in\mathcal{E}}$ and write $\pi_t^{(i)} = \pi_t^{(i)}(0, v)$ for the corresponding geodesic, which is almost surely uniquely defined, since we assume that F is continuous. Similarly, write $T = T(0, v)$ and $\pi = \pi(0, v)$ for the passage time and geodesic between 0 and v in the initial configuration $\{\omega(e)\}_{e\in\mathcal{E}}$, and note that $T_0^{(i)} = T$ and $\pi_0^{(i)} = \pi$ for all i. Finally, recall that $T(\Gamma)$ denotes the weight-sum along the path Γ, and hence that $T(\pi_t^{(i)})$ denotes the travel time of the path $\pi_t^{(i)}$, in the initial configuration $\{\omega(e)\}_{e\in\mathcal{E}}$.

We introduce a notation for the **maximum overlap** between the paths π and $\pi_t^{(i)}$, for $i = 1, 2, \dots, k$, as

$$O_k := \max\Big\{ \max_{i\neq j} \big|\pi_t^{(i)} \cap \pi_t^{(j)}\big|, \max_i \big|\pi \cap \pi_t^{(i)}\big| \Big\},$$

and the **maximum time difference** of the same paths as

$$\Delta T_k := \max_i \big[T(\pi_t^{(i)}) - T\big].$$

nopag

In the remainder of this section our goal will be to show that, as $|v| \to \infty$, we may choose $t = t(v) \to 0$ and $k = k(v) \to \infty$ in such a way that $O_k = o(|v|)$ and $\Delta T_k = o(|v|)$ with probability tending to 1. This will complete the proof of Theorem 3.

Maximum overlap

It follows from (8) and (48) that for every $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ such that for all $t > 0$ and $v \in \mathbb{Z}^d$ we have

$$\mathbb{E}\big[|\pi \cap \pi_t^{(i)}|\big] \le \frac{|v|}{C(\varepsilon)t\log|v|} + C'\frac{|v|F(r+\varepsilon)^{1/d}}{t}, \tag{50}$$

where C' is a universal constant. Denote the right-hand side in (50) by $\psi(v,\varepsilon,t)$, and note that since $s \ge t$ we obtain for $i \ne j$ that also $\mathbb{E}[|\pi_t^{(i)} \cap \pi_t^{(j)}|] \le \psi(v,\varepsilon,t)$, and hence that

$$\mathbb{E}[O_k] \le \psi(v,\varepsilon,t)(k+1)^2. \tag{51}$$

Set $t = t(\varepsilon) = F(r+\varepsilon)^{1/2d}$ so that $t(\varepsilon) \to 0$ as $\varepsilon \to 0$. Since $t(\varepsilon) > 0$ and $C(\varepsilon) > 0$ for all $\varepsilon > 0$, we may choose $\varepsilon = \varepsilon(v) \to 0$ such that $t(\varepsilon)C(\varepsilon)\sqrt{\log|v|} \ge 1$ as $|v| \to \infty$. Then

$$\psi(v) := \psi\big(v, \varepsilon(v), t(\varepsilon(v))\big) = o(|v|).$$

For $k \le k^*(v) := (|v|/\psi(v))^{1/4} - 1$ it follows from (51) that $\mathbb{E}[O_k] \le \sqrt{|v|\psi(v)}$. So, with $\alpha(v) := |v|^{3/4}\psi(v)^{1/4}$ we have $\alpha(v) = o(|v|)$ and Markov's inequality gives

$$\mathbb{P}\big(O_k > \alpha(v)\big) \le \frac{\sqrt{|v|\psi(v)}}{|v|^{3/4}\psi(v)^{1/4}} = o(1), \tag{52}$$

uniformly in $k \le k^*(v)$.

Maximum time difference

Since T and $T_t^{(i)}$ are equal in distribution, it follows that

$$\mathbb{E}\big[T(\pi_t^{(i)}) - T\big] = \mathbb{E}\big[T(\pi_t^{(i)}) - T_t^{(i)}\big] \le \mathbb{E}\Bigg[\sum_{e \in \pi_t^{(i)}} \big|\omega(e) - \omega_t^{(i)}(e)\big|\Bigg].$$

Using that

$$\big|\omega(e) - \omega_t^{(i)}(e)\big| \le \big(\omega(e) + \omega^{(i)}(e)\big)\mathbf{1}_{\{U^{(i)}(e) \le t\}},$$

that π and $\pi_t^{(i)}$ are equal in distribution, and that π is a function of ω, we find through conditioning on π, due to the independence of $\omega(e)$, $\omega^{(i)}(e)$ and $U^{(i)}(e)$, that

$$\mathbb{E}\big[T(\pi_t^{(i)}) - T\big] \le \mathbb{E}\Bigg[\sum_{e \in \pi} \big(\omega(e) + \omega^{(i)}(e)\big)\mathbf{1}_{\{U^{(i)}(e) \le t\}}\Bigg] = t\,\mathbb{E}[T] + t\,\mathbb{E}[\omega^{(i)}(e)]\mathbb{E}[|\pi|].$$

From subadditivity and by (7) we conclude that this is at most $tC|v|$ for some constant C, and hence that

$$\mathbb{E}[\Delta T_k] \le ktC|v|.$$

Now, let $t = t(v) = t(\varepsilon(v))$ and $k^*(v)$ be defined as above, so that $t(v) \to 0$ and $k^*(v) \to \infty$ as $|v| \to \infty$. Set $k(v) := \lfloor \min\{t(v)^{-1/2}, k^*(v)\} \rfloor$ and $\beta(v) := t(v)^{1/4}|v|$, so that also $k(v) \to \infty$ and $\beta(v) = o(|v|)$ as $|v| \to \infty$. Then, Markov's inequality gives

$$\mathbb{P}\big(\Delta T_k > \beta(v)\big) \le C \frac{k(v)t(v)|v|}{t(v)^{1/4}|v|} \le Ct(v)^{1/4} = o(1), \tag{53}$$

uniformly in $k \le k(v)$.

To conclude, it follows from (52) and (53) that there exists a sequence $(k(v))_{v\in\mathbb{Z}^d}$ such that $k(v) \to \infty$ as $|v| \to \infty$ and so that with high probability $O_{k(v)} = o(|v|)$ and $\Delta T_{k(v)} = o(|v|)$. This completes the proof of Theorem 3.

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