Mm5023 Lecture 5

Generating functions II

Plan:

- Exponential generating functions
- Summation operator
- Examples

Exponential generating series & functions

Motivation: to solve problems in which the order matters

Def Given a sequence $(a_n)_{n \in \mathbb{N}}$ its exponential generating series is the (formal) power series

$$\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$$

$a_n \rightsquigarrow \sum_{n=0}^{\infty} a_n x^n$

Mclaurin exp of

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

If $\boxed{\sum_{n\geq 0} \frac{a_n}{n!} x^n}$ has a positive radius of absolute convergence then we say that

$$\sum_{n\geq 0} \frac{a_n}{n!} x^n$$ is the generating function

of $(a_n)_{n\geq 0}$.

Examples

Exp

- $a_n = 1 \rightsquigarrow$ Generating series is $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$

$\Longrightarrow e^x$ generating function

$a_n = 2^n$

$$\sum_{n=0}^{\infty} \frac{2^n}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} (2x)^n \Longrightarrow e^{2x}$$

this is the generating functions of

$a_n = 2^n$

$$a_n = \begin{cases} 1 & \text{for } n = 0, 1 \\ 0 & \text{for } n \neq 0, 1 \end{cases}$$

$$\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n = \frac{a_0}{0!} x^0 + \frac{a_1}{1!} x^1 = a_0 + a_1 x^1$$

$$= 1 + x$$

the generating function

Fix n $\quad a_k = P(n,k)$

↳ Permutations of k objects / n letter

$$P(n,k) = \binom{n}{k} k!$$

$$\sum_{k=0}^{\infty} \frac{a_k}{k!} x^k = \sum_{k=0}^{\infty} \frac{\binom{n}{k} \cancel{k!}}{\cancel{k!}} x^k = \sum_{k=0}^{\infty} \underbrace{\binom{n}{k}}_{= 0 \text{ if } k > n} x^k$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^k$$

$$= (1+x)^k \qquad \text{generating function.}$$

48 flags of 4 colors (12 in each color)

Blue Purple Yellow Black.

12 flag poles.

You send different messages by placing flags of different color in the flag poles

ORDER MATTERS!

How many messages.

1	2			12	
.	.	.	.	.	$\Rightarrow 4^{12}$
↓	↓			↓	
4 choices	4 choices			4 choices	

How many signal with even number of blue and an odd number of black.

Count using generating functions

→ Find the exponential generating function of the problem.

→ Compute the coefficient of deg 12.

→ multiply it by 12!

purple & yellow

$$\longrightarrow \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \quad + \frac{x^{12}}{12!}\right)$$

blue

$$\longrightarrow \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \frac{x^{10}}{10!} + \frac{x^{12}}{12!}\right)$$

black

$$\longrightarrow \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \frac{x^{11}}{11!}\right)$$

the number we are looking for is the coefficient of degree 12 of

$$\underbrace{\left(\sum_{k=0}^{12} \frac{x^k}{k!}\right)^2}_{P \,\&\, Y} \underbrace{\left(\sum_{k=0}^{6} \frac{x^{2k}}{(2k)!}\right)}_{B} \underbrace{\left(\sum_{k=0}^{5} \frac{x^{2k+1}}{(2k+1)!}\right)}_{\text{Black.}}$$

This is the same as the coefficient of deg 12 of

$$\underbrace{\left(\sum_{k=0}^{\infty} \frac{x^k}{k!}\right)^2}_{P \,\&\, Y} \underbrace{\left(\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}\right)}_{B} \underbrace{\left(\sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}\right)}_{\text{Black.}}$$

$$\Rightarrow (e^x)^2 \cdot \underbrace{\left(\frac{e^x + e^{-x}}{2}\right)}_{\cosh(x)} \underbrace{\left(\frac{e^x - e^{-x}}{2}\right)}_{\sinh(x)}$$

$$= e^{2x}\,\frac{e^{2x}-e^{-2x}}{4} = \frac{1}{4}\left(e^{4x}-1\right) =$$

Go back to power series

$$= \frac{1}{4}\left(\sum_{k=0}^{\infty}\frac{(4x)^k}{k!} - 1\right) = \frac{1}{4}\sum_{k=1}^{\infty}\frac{4^k x^k}{k!} \qquad a_{12}$$

$$= \sum_{k=1}^{\infty}\frac{4^{k-1}}{k!}x^k = \sum_{k=0}^{\infty}\frac{a_k}{k!}x^k$$

coefficient of deg 12 $\qquad \dfrac{4^{11}}{12!}$

$$\Rightarrow a_n = \frac{4^{11}}{12!}\cdot 12! = 4^{11}$$

3) How many signals have at least 3 purple or no purple at all.

yellow black blue $\longrightarrow$ e^x

purple: 0 3 4 5 6 7 8 9 10 11 12

$$1 + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \frac{x^9}{9!} + \frac{x^{10}}{10!} + \dots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \frac{x^{\boxed{12}}}{12!} - x - \frac{x^2}{2!}$$

$$\rightsquigarrow \quad e^x - x - \frac{x^2}{2}$$

Generating function of the problem is

$$\underbrace{(e^x)^3}_{Y\,B\,Bl} \cdot \left(e^x - x - \frac{x^2}{2}\right) = e^{3x}\left(e^x - x - \frac{x^2}{2}\right)$$

I want the coefficient of degree 12.

$$e^{4x} - xe^{3x} - \frac{x^2}{2} \cdot e^{3x}$$

$$= \sum_{k=0}^{\infty} \frac{4^k x^k}{k!} - \sum_{k=0}^{\infty} \frac{3^k}{k!} x^{k+1} - \sum_{k=0}^{\infty} \frac{3^k}{2 \cdot k!} x^{k+2}$$

$$\text{deg } 12 = \frac{4^{12}}{12!} - \frac{3^{11}}{11!} - \frac{3^{10}}{2 \cdot 10!} \qquad = \sum_{k=0}^{\infty} \frac{a_k}{k!} x^k$$

$$= \frac{4^{12} - 3^{11} \cdot 12 - 3^{10} \cdot 11 \cdot 6}{12!} = \frac{a_{12}}{12!}$$

$$a_{12} = 4^{12} - 3 \cdot 12 - 3 \cdot 11 \cdot 6$$

$$= 10,754,218$$

SUMMARY

Generating function	Exponential Gen. function
1) Find the gf of your problem.	1) Find the EGF of the problem
2) Describe it as a power series.	2) Describe it as a power series
3) Find the right coeffs	3) Find the right coeffs
	multiply it by deg!
ANSWER (no order)	ANSWER (order)

PS ⟹ FUNCTION ⟹ PS (find the coefficient you are looking for)

The summation operation (usual gf)

Problem Given the generating function of $(a_n)_{n\in\mathbb{N}}$, what is the generating function of

$$\sum_{k=0}^{n} a_k =: b_n \qquad ?$$

Proof

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\left(f(x) \cdot \frac{1}{1-x} \right) = \left(\sum_{n=0}^{\infty} a_n x^n \right) \cdot \left(\sum_{n=0}^{\infty} x^n \right)$$

$$= \sum_{n=0}^{\infty} c_n x^n$$

$$\sum a_n x^n \cdot \sum b_n x^n$$

$$\shortparallel$$

$$\sum c_n x^n$$

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

$$c_n = \sum_{k=0}^{n} a_k \cdot 1$$

$$= \sum_{k=0}^{n} a_k$$ $\hookrightarrow$ sum of the first n terms of (a_n)

#

Prop If $f(x)$ is the generating function of $(a_k)_{k\in\mathbb{N}}$ then $\frac{f(x)}{1-x}$ is the generating function of $\left(\sum_{n=0}^{k} a_n\right)_{k\in\mathbb{N}}$

we call $\boxed{\frac{1}{1-x}}$ the SUMMATION OPERATOR.

Examples:

$a_k = 1 \quad \forall \quad k$ first $n+1$ terms

$$\sum_{k=0}^{n} a_k = n+1$$

$$a_k \rightsquigarrow f(x) = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

the generating function of $a_n = n+1$ $\left(\frac{1}{1-x}\right)^2$

- $a_k = \begin{cases} 1 & \text{if } k = 1, 2 \\ 0 & \text{otherwise.} \end{cases}$

$a_k = x + x^2$.

$$\sum_{k=0}^{n} a_k = b_n$$

$b_0 = 0$

$b_1 = 1$

$b_2 = 2$

$b_n = 2 \quad \forall \quad n \geq 2.$

the generating function of b_n

$$\frac{x+x^2}{1-x}$$

$$C_n = \sum_{k=0}^{n} b_k$$

$C_0 = b_0 = 0$

$C_1 = b_1 = 1.$

$C_2 = b_1 + b_2 = 3.$

$C_3 = b_1 + b_2 + b_3 = 5.$

$C_4 = 7$

$C_5 = 9$

$$C_k = \begin{cases} 0 & \text{if } k=0 \\ 2(k-1)+1 & k>0 \end{cases}$$

$C_k \longrightarrow$ k th odd numb

$= 2(k-1)+1$

the generating function of odd number

$(2(k-1)+1)_{k=1 \ldots \ldots}$

$$\frac{x+x^2}{(1-x)^2}$$

What if we want to know a formula for computing the sum of the first n odd numbers (d_n)

$$d_n = \sum_{k=0}^{n} c_n$$

$$= \sum_{k=1}^{n} c_n + \underset{c_0}{\underset{\shortparallel}{0}}$$

$$\frac{x+x^2}{(1-x)^3}$$ generating function.

$$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{n} x^k$$

$$\frac{x+x^2}{(1-x)^3} = \sum_{k=0}^{\infty} \binom{3+k-1}{k} x^{k+1} + \sum_{k=0}^{\infty} \binom{3+k-1}{k} x^{k+2}$$

coeff of k

$$\Big(\quad 1-1 \Big) + \binom{3+k-2-1}{3}$$

$$= \binom{k+1}{\ } + \binom{k}{k-3}$$

$$= \frac{(k+1)!}{(k-2)!\,(3)!} + \frac{k!}{(k-3)!\,3!} = \frac{(k+1)(k)(k-1)}{3!} + \frac{k(k-1)(k-2)}{3!}$$

$$\frac{k(k-1)(k+1+k-2)}{3!}$$

$$= \frac{k((k-1))(2k-1)}{3!}.$$

$$\left(\frac{1}{1-x}\right)^n$$ the coefficient of deg k

$$\left(1+x+x^2+\cdots\right)^n$$

I distribute k candies to n children

$$\cdot \;|\; \circ \;|\; \circ \;|\; \;|\; |$$

$n-1$

# of strings of length $k+n-1$
with $n-1$ ones

$$\binom{k+n-1}{n-1}$$

want the coefficient of deg k of

$$\frac{x+x^2}{(1-x)^3}$$

$n=3$

$k=$ whatever.

$$(x+x^2)\cdot\sum_{k=0}^{\infty}\binom{k+3-1}{3-1}x^k$$

$$=\sum_{k=0}^{\infty}\binom{k+2}{2}x^{k+1}+\sum_{k=0}^{n}\binom{k+2}{2}x^{k+2}$$

deg k

$$\binom{k+1}{2}+\binom{k}{2}=\frac{1}{2}k(k+1)+\frac{1}{2}k(k-1)$$

$$=\frac{1}{2}k(k+1+k-1)=\frac{1}{2}k(2k)$$

k^2.

<u>Ex</u> $a_n = \begin{cases} 2 & n=0 \\ 0 & \text{otherwise} \end{cases}$

$b_n = 2, 4, 6 \ldots$

Sum of the first n even number.

Generating functions of the catalan #'s

$\boxed{C_0 \quad C_1 = 1}$?

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}$$

! Convergence matters.

$$(*) \quad \sum_{n=0}^{\infty} C_{n+1} x^{n+1} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} C_k C_{n-k} \right) x^{n+1} = (f(x))^2 \cdot x$$

suppose that $\sum_{k=0}^{\infty} C_k x^k$ converges.

then it converges to some function $f(x)$ which is the generating function of C_n

$$f(x) - \underbrace{C_0 x^0}_{1} = x\,(f(x))^2$$

$$\boxed{x\,(f(x))^2 - f(x) + 1 = 0}$$

$$f(x) = \frac{1 \pm \sqrt{1+4x}}{2x}$$

two possibilities. how do I choose?

$$\lim_{x \to 0} f(x) \in \mathbb{R}$$

f(x) is a given by a convergent power series $\Rightarrow$ so it converges near 0

$$\lim_{x \to 0} \left| \frac{1+\sqrt{1+4x}}{2x} \right| = \text{"} \left| \frac{1}{0} \right| \text{"} = +\infty$$

$$\lim_{x \to 0} \frac{1-\sqrt{1+4x}}{2x} \in \mathbb{R}$$

the generating function for Catalan

$$f(x) = \frac{1-\sqrt{1+4x}}{2x}$$

Partition of 10

- such that every $\boxed{\text{summand is at most 3}}$

$$p(n) \rightsquigarrow \prod_{k=0}^{\infty} \frac{1}{1-x^k} \qquad \boxed{\text{with at most 3 summands}}$$

m_k : # of summand with value k

1 → $(1 + x^{m_1} + x^{m_2} + \dots) \rightsquigarrow \frac{1}{1-x}$

2 $(1 + x^{2m_1} + x^{2m_2} - - - \rightsquigarrow \frac{1}{1-x^2}$

k $\frac{1}{1-\cancel{k}^k}$

the generating function of this partition

$$\prod_{k=1}^{3} \frac{1}{1-x^k}$$

coefficient of deg 10.

$$(1+x+x^2+x^4+x^5+x^6+x^7+x^8+x^9+x^{10})$$

$$(1+x^2+x^4+x^6+x^8+x^{10})$$

$$(1+x^3+x^6+x^9)$$

$$\frac{1}{1-x}\left(1+x^3+x^6+x^9+x^2+x^5+x^6+x^4+x^7+x^{10} + x^6+x^9+x^8+x^{10}+\text{higher deg}\right)$$

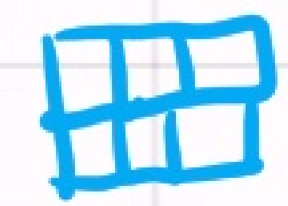
summ.

$$\frac{1}{1-x}\left(1+x^2+x^3+x^4+x^5+3x^6+x^7+2x^9+x^8+2x^{10}+\text{hig}\right)$$

$$= 14x^{10} + \text{other terms}$$

5.3 Examples

1) Find the number of partitions of $\boxed{10}$ in

a) in parts not exceeding 3 → 

b) in at most 3 summands ~~parts~~ $= 14$

c) in odd parts

d) in even parts

Thursday we start recursion.
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