Introduction to Real Analysis

Lecture 3: Compact and Connected sets

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Questions?

Lecture Plan

- Compact spaces (Rudin 2.31-2.42)
- Connected sets (Rudin 2.45-2.47)

Section 1
Compact sets continued

Open covering

Given $E \subseteq X$, an open cover of E is a collection $\{U_\alpha\}_{\alpha \in A}$ of open sets of X such that

$$E \subseteq \bigcup_{\alpha} U_\alpha$$

Definition: Compact set

Given $K \subseteq X$, we say that it is compact if, for every open covering $\{U_\alpha\}_{\alpha \in A}$ there are finitely many $\alpha_1, \ldots, \alpha_n$ such that

$$E \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$$

Compact and closed

Goal: → prepare the ground for limits in compact sets
→ describe compact subsets in $\mathbb{R}^n$

Theorem

Compact sets are closed.

$\mathbb{R}^n \rightsquigarrow$ Euclidean distance

$d(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$ $\quad n=1 \quad |x-y| = d(x,y)$

Compact and closed 2

$C \subseteq K \subseteq X$ metric space

Theorem 2

Let K be a compact set and $C \subseteq K$ a closed set (relatively to X). Then C is compact.

Corollary

If C is closed and K is compact then $C \cap K$ is compact.

Theorem 1 K is closed

$\underset{\text{cmpt}}{K} \supseteq K \cap C$ is closed

Theorem 2 $\Rightarrow$ $K \cap C$ is compact

Proof of the theorem 2

We use the def of compact: we take $\{V_\alpha\}$ an open cover of C

C is closed $\quad C^c = X \setminus C$ the complementary is OPEN

$\{V_\alpha\} \cup \{C^c\}$ open covering of X
open covering of K

$$K \subseteq X \subseteq \bigcup_\alpha V_\alpha \cup C^c$$

K is compact $\exists\ \alpha_1 \dots \alpha_n$ such that

$$K \subseteq \bigcup_1^n V_{\alpha_i} \cup C^c \quad \text{finite subcovering}$$

$$C = K \cap C \subseteq \bigcup_1^n (V_{\alpha_i} \cap C) \cup \underbrace{(C^c \cap C)}_{= \emptyset} = \bigcup_1^n (V_{\alpha_i} \cap C) \subseteq \bigcup_1^n V_{\alpha_i}$$

$\{V_{\alpha_i}\}_{i=1\dots n}$ finite subcovering ☺

Intersection of compact

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Theorem

Let $\{K_\alpha\}$ a collection of compact sets such as any finite intersection is non-empty. Then,

$$\bigcap_\alpha K_\alpha \neq \emptyset$$

Corollary

If $\{K_n\}$ is a sequence of compact sets such that $K_n \supset K_{n+1}$ then we have that

$$\bigcap_n K_n \neq \emptyset$$

The intersection of finitely many K_n is $K_{\text{smallest index}}$ non empty $\Rightarrow$ use Thm.

Proof: (Theorem)

By contradiction $\bigcap_{\alpha} K_\alpha = \phi$

$\Leftrightarrow$ (de Morgan) $\underline{\bigcup_{\alpha} K_\alpha^c} = X$

K_α cmpt $\Rightarrow K_\alpha$ closed

$\Rightarrow K_\alpha^c$ open

$K_{\bar{\alpha}} \in \{K_\alpha\}$ $\qquad K_{\bar{\alpha}} \subseteq X = \bigcup_{\alpha} K_\alpha^c$

open cover for $K_{\bar{\alpha}}$

($K_{\bar{\alpha}}$ cmpt)

$\exists \ \alpha_1 \dots \alpha_n$ such that $K_{\bar{\alpha}} \subseteq \bigcup_{1}^{n} K_{\alpha_i}^c$

$$= \left(\bigcap_{i} K_{\alpha_i}\right)^c$$

$K_{\bar{\alpha}} \cap K_{\alpha_1} \cap \quad \cap K_{\alpha_n} = \phi$ contradiction

we found finitely many K_α's that have ϕ-intersectio

Limits in compact sets

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Theorem

Any infinite subset of a compact set K has a limit point in K.

Proof: By contradiction no point of K is a limit point of E

$E \subseteq K$ $\quad (|E| = +\infty)$

$q \in K$ $\quad \exists$ some $r_q > 0$ $\quad V_q := N_{r_q}(q)$

$\hookrightarrow$ OPEN

$(V_q \cap E) \setminus \{q\} = \phi$

Consider $\quad V_q \cap E$ $\quad \begin{cases} \phi \\ \{q\} \end{cases}$

$K \subseteq \bigcup_{q \in K} V_q$ $\quad$ compact $\quad \exists\, q_1 \dots q_n$ $\quad$ such that

$K \subseteq \bigcup_{i=1}^{n} V_{q_i}$

$$E = E \cap K \subseteq \dot{E} \cap \bigcup_{1}^{n} V_{q_i}$$

$$= \left(\bigcup_{1}^{n} V_{q_i} \cap \dot{E} \right) \subseteq \{q_1 \ldots q_n\}$$

$$\phi \qquad \{q_i\}$$

$\Rightarrow$ $|E| \leq n$ contradiction.

Compact subsets of $\mathbb{R}^n$

Let (X, d) be $\mathbb{R}^k$ with the Euclidean distance. We want to prove the following theorem

Theorem: Heine–Borel

For $\boxed{E \subseteq \mathbb{R}^k}$ the following are equivalent

1. E is closed and bounded (totally bounded)
2. E is compact
3. every infinte subset of E has a limit point in E.

general

Bounded $\iff$ $\exists$ $p \in X$ and $r > 0$ such that

$E \subseteq N_r(p)$

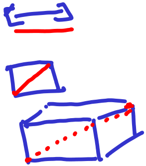
k-cells

1-cell $[a,b]$ 2-cell $[a,b]\times[c,d]$

A (closed) k-cell I is a subset of $\mathbb{R}^k$ which is a product of (closed) intervals. That is there are $(a_1, \dots, a_k)$ and $(b_1, \dots, b_k)$, with $b_i \geq a_i$ such that

$$I = \{(x_1, \dots, x_k) \in \mathbb{R}^k \mid a_i \leq x_i \leq b_i\}$$

If we set



length of diagonal

$$\delta(I) := \left(\sum_{i=1}^{k} (b_i - a_i)^2\right)^{\frac{1}{2}}$$

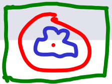
we have that

$$d(x, y) < \delta(I)$$

for all x and y in I.

Proposition

We have that $E \subseteq \mathbb{R}^k$ is bounded if, and only if, it is contained in a k-cell.



$2 \cdot r \rightsquigarrow$ you consider a k-cell of $\delta(I) > 2r$ with center in the set

Two Lemmas

Idea: We want to show that k-cells are compact ⇒ we show the theorem.

Lemma

The intersection of a sequence of nested interval in $\mathbb{R}$ is not empty.

Lemma

The intersection of a sequence of nested k-cells in $\mathbb{R}^k$ is not empty.

$K_n \supseteq K_{n+1} \supseteq \dots \; K_{\kappa} \supseteq \dots$

$\bigcap K_n \neq \emptyset$

Lemmas ⇒ the same is true for cells

The main technical step

Theorem

Every k-cell is a compact subset of $\mathbb{R}^k$.

Skip the proof now:

Proof of Heine-Borel thm (Char of cpt in $\mathbb{R}^k$)

① ⇒ ② E closed and bounded

$E \subseteq I$ a k-cell

closed cmpct $\Rightarrow$ E is cpt

② ⇒ ③ holds in general

③ ⇒ ① 3 - Every ∞-subset of E has a limit pt in E

1 - E closed & bounded

if E is not bounded for every $n \in \mathbb{Z}^+$

$\exists\, x_n \in E$ such that $d(\underline{0}, x_n) \geq n$

Otherwise $E \subseteq N_n(0)$ bounded

$\{x_n\}$ has no limit pt in $\mathbb{R}^k$ $\Rightarrow$ $\{x_n\}$ has no limit pt in E

$3 \Rightarrow$ bounded

Suppose that E is not closed

$\Rightarrow \quad E \subsetneq \bar{E} = E \cup E' \quad \Rightarrow \exists \; p \in E' \; p \notin E$

$\phi \neq \left(N_{\frac{1}{n}}(p) \cap E \right) \setminus \{p\}$ choose

$\underset{q_n}{\cup}$

$\{q_n\}$ this is not finite or $q_n = \bar{q}$ for $n >> 0$
impossible if $\frac{1}{n} < d(p, \bar{q})$

By construction $p \in \{q_n\}'$

There is no other limit point for $\{q_n\}$ in $\mathbb{R}^n$
(observe that we constructed a ∞ subset of E with only a limit point p such that $p \notin E$ $\Rightarrow$ 3 does not hold and $3 \Rightarrow$ closed $\Rightarrow$ $3 \Rightarrow$ closed & bounded)

let $q \in \mathbb{R}^n$

Δ

$$d(q,p) \le d(q,q_n) + d(q_n,p)$$

$$d(q_n, q) \geq d(q,p) - \underset{\substack{\wedge \\ \frac{1}{n} < \frac{1}{2} d(q,p)}}{d(q_n,p)} \geq \frac{1}{2} d(p,q) = \delta$$

$$\left| N_\delta(q) \cap \{q_n\} \right| < \infty$$

$q_1 \; \dots \; q_n$ $\quad \frac{1}{n} > \frac{1}{2} d(q,p)$

$\Rightarrow$ q is not a limit point for $\{q_n\}$

$\boxed{q \in E'}$ $\Longrightarrow$ $N_r(q) \cap E$ contain ∞ many points

$\Downarrow$

Weierstrass Theorem

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Theorem

Every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.

E bounded $\Rightarrow$ $E \subseteq \underset{k\text{-cell}}{I}$ $\Rightarrow$ I cmpt

$|E| = +\infty$

E has a limit pt in $I \subseteq \mathbb{R}^k$

Questions?

Section 2
Connected sets

Connected sets

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Definiton (usual connected set)

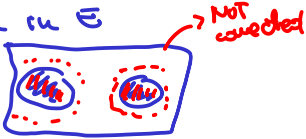
A subset E of X is said to be connected if it cannot be written as union as a union of two disjoint, nonempty sets open with respect to the restricted metric

$E \neq U_1 \cup U_2$

$U_i \subseteq E$ open in E

$U_1 \cap U_2 = \phi$

$U_1, U_2 \neq \phi$



Connected sets

Definiton (usual connected set)

A subset E of X is said to be connected if it cannot be written as union as a union of two disjoint, nonempty sets open with respect to the restricted metric

In other words, if there are two open sets U_1 and U_2 in X such that

- $E \subseteq U_1 \cup U_2$
- $E \cap U_1 \cap U_2 = \emptyset$

Then $E \cap U_1 = \emptyset$ or $E \cap U_2 = \emptyset$

Connected sets - A la Rudin

Two subsets A and B of a metric space X are said to be separated if

$$\overline{A} \cap B = \emptyset \quad \text{and} \overline{B} \cap A = \emptyset$$

Connected sets - A la Rudin

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Definiton

A subset E of X is said to be connected if it cannot be written as union of two nonempty separtaed subsets.

Idea

A, B (RUDIN)

$U_1 = \overline{A}^c$ $U_2 = \overline{B}^c$

Connected sets - A la Rudin

Two subsets A and B of a metric space X are said to be separated if

$$\overline{A} \cap B = \emptyset \quad \text{and} \overline{B} \cap A = \emptyset$$

Definiton

A subset E of X is said to be connected if it cannot be written as union of two nonempty separtaed subsets.

The two definitions are equivalent. This is a nice exercise to do :).

Connected subset of the real line

$|E| \geq 2$ $|E| = 1$ $E = \{x\}$ conned

A subset $E \subseteq \mathbb{R}$ is connected if, and only if, for all $x < y$ in E we have that $[x, y] \subseteq E$ $E \supseteq (\inf E, \sup E)$

$E = \{\inf E, \sup E\}$ $\{ = (, [$

Proof E connected $E \subseteq \mathbb{R}$

$x, y \in E$ $x < y$ $z \in (x, y)$

Suppose that $z \notin E$ union of two $\neq \emptyset$ open disjoint subset $\Rightarrow$ not conneded

$E = ((-\infty, z) \cap E) \cup ((z, +\infty) \cap E)$

$A = (-\infty, z)$ $B = (z, +\infty)$ separated

E not conected $E = A \cup B$ non empty + separated

$x \in A$
$y \in B$
up to swap them can assume $x < y$

$A \cap [x, y]$ bounded above (by y)

$B \cap [x,y]$ bounded below

$z = \sup(A \cap [x,y])$

$x \leq z \leq y$

$z \in \overline{A} \Rightarrow z \notin B$ (separated) $x \leq z < y$

if $z \notin A$ then $z \in (x,y)$ $z \notin E \Rightarrow$ contradict

assume that $z \in A \Rightarrow A \cap \overline{B} = \emptyset$ $z \notin \overline{B}$

let $\varepsilon > 0$

$\underbrace{((z-\varepsilon, z+\varepsilon) \cap B)}_{= \emptyset} \subseteq \{z\}$ but $z \notin B$

I take $z_1 > z$ $z_1 - z < \varepsilon$ $z_1 > \sup A \Rightarrow z_1 \notin A$

$z_1 \notin B \Rightarrow z_1 < y \Rightarrow \boxed{z_1 \in E}$

Questions?

find the hole

$\sup A \notin E$

↳ hole?

maybe is to close to A

$(\in \bar{A}) \Rightarrow$ move a litle bit and find the right hole

Thank you for your attention!