STOCKHOLM UNIVERSITY
Department of Mathematics
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Examination in
Mathematics for Economic and Statistical Analysis
MM1005, Fall term; 7,5 ECTS
Tuesday 28 September, 2021

Instructions: - During the exam you may not use any textbook, class notes, or any other supporting material.
- Nongraphical calculators will be provided for the exam by the department. Other calculators may not be used.
- In all solutions, justify your answers? communicate your reasoning. Use ordinary language where appropriate, not just mathematical symbols.
- Use natural language, not just mathematical symbols. Write clearly and legibly
- Mark your final answer to each question clearly by putting a a box around it.

Grades: Each solved problem is awarded by up to 10 points. At least 30 points guarantee grade grade E, 36 for D, 42 for C, 48 for B and 54 for A. Note that the problems are not ordered according to the difficulty!

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1. Calculate the limits

$$\text{a)} \quad \lim_{x\to+\infty} \frac{(-4x^2+8x+5)^{2021}}{(2x+3)^{4042}} \qquad \text{b)} \quad \lim_{x\to 0} \frac{x^2+2x^3}{1-e^{x^2}}$$

Solution

$$\lim_{x\to+\infty} \frac{(-4x^2+8x+5)^{2021}}{(2x+3)^{4042}} = -1 \qquad \lim_{x\to 0} \frac{x^2+2x^3}{1-e^{x^2}} = -1.$$

2. Calculate the integrals

$$\text{a)} \quad \int (e^x(x+1)+2x)\mathrm{d}x \qquad \text{b)} \quad \int_0^{e-1} \frac{4\ln(t+1)}{t+1}\mathrm{d}t$$

Solution

$$\int (e^x(x+1)+2x)\mathrm{d}x = xe^x + x^2 + C, \quad C\in\mathbb{R}.$$

$$\int_0^{e-1} \frac{4\ln(t+1)}{t+1}\mathrm{d}t = 2(\ln(t+1))^2|_0^{e-1} = 2.$$

3. The expression

$$y^3 - xy^2 + y = 1,$$

defines y as a function of x: $y = y(x)$.

(a) Find all the possible values of $y(1)$.

(b) Find the equation of the tangent line to $y(x)$ at the point $P = (1, y(1))$.

Solution Setting $y(1)$ one sees that the equations is satisfied, if and only if

$$(y(1)-1)(y(1)^2+1) = 0 \Leftrightarrow y(1) = 1.$$

Implicit differentiation gives

$$(3y^2-2xy+1)y' - y^2 = 0 \Rightarrow y'(1) = 1/(3-2+1) = 1/2.$$

So the equation of the tangent line at the point P is

$$Y = 1 + \frac{1}{2}(x-1).$$

4. Elon Tusk's fortune doubles every year and the value of his fortune was estimated at \$10 000 000 in 2010. How much was his fortune at the end of 2014? At the end of which year will his fortune surpass $\$10^{13}$.

Solution1 At the end of 2014, his fortune is

$$F_0 2^4$$

At the end of year $2010+L$ his fortune would be

$$F_0 2^L \geq 10^{13} \Leftrightarrow L \geq 6\frac{\ln 10}{\ln 2} \approx 19.93.$$

So, his fortune will surpasss 10^{13} when $L=20$. That is, at the end of 2030.

Solution2 Some students interpreted the statement in a different way, that could be re-stated as follows:

> Elon Tusk's **profit** doubles every year and the value of his fortune was estimated at \$10 000 000 in 2010. How much was his fortune at the end of 2014? At the end of which year will his fortune surpass $\$10^{13}$.

The interpretation of the word 'fortune'was a subtlety, we gave credits for solutions of this statement.

The solution to this problem is given below. At the end of 2014 his fortune is

$$F_0(2^0+2^1+2^2+2^3+2^4) = F_0(2^5-1) = 31*10^7,$$

where $F_0 = 10^7$.

At the end of the year $2010+(L-1)$ his fortune is $F_0(2^L-1)$. We are looking for L such that

$$F_0(2^L-1) \geq 10^{13} \Leftrightarrow 2^L \geq 10^6+1 \Leftrightarrow L \geq \frac{\ln(10^6+1)}{\ln 2} \approx 19.9315.$$

So

$$L=20 \Rightarrow \text{year } 2010+19=2029.$$

5. Let $f(x,y) = x^2-4x+y^2-6y+13$.

   (a) Find all stationary points for f and determine whether they are local maximum, minimum, or saddle points.

   (b) Find the maximum and the minimum value of the function f with constraints $x \geq 0$, $y \geq 0$ and $x+y \leq 6$.

Solution The partial derivatives of f are

$$f_x(x,y) = 2x-4 \qquad f_y(x,y) = 2y-6.$$

So the only stationary point is

$$x = (2,3),$$

and the Hessian is

$$H(x,y) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

By convexity of the domain, we know that $f(2,3) = 4-8+9-18+13 = 0$ is a global minimum point.

The boundary of the domain consist of three part. On the first one we have

$$f(0,y) = y^2-6y+13, \quad y \in [0,6],$$

which has critical point at $y=3$. Evaluating at $y=3$ and in extremes of the interval gives us

$$f(0,3) = 9-18+13 = 4, \quad f(0,0) = 13, \quad f(0,6) = 13.$$

On the second part of the boundary we have

$$f(x,0) = x^2 - 4x + 13, \quad x \in [0,6],$$

which has a critical point at $x = 2$. Evaluating gives us

$$f(2,0) = 13 \quad f(6,0) = 36 - 24 + 13 = 25.$$

On the diagonal part of the domain, we obtain

$$\begin{aligned} h(x) = f(x, 6-x) &= x^2 - 4x + 36 - 12x + x^2 - 36 + 6x + 13 \\ &= 2x^2 - 10x + 13 \quad x \in [0,6]. \end{aligned}$$

which has a stationary point at $x = 5/7$. Evaluating yields

$$f(5/2, 7/2) = 5^2/2 - 5^2 + 13 = 13 - 25/2 = 1/2.$$

Then, we obtain that the maximum and minimum values of f with the given constrains are respectively

$$25 \qquad \text{and} \qquad 0.$$

6. Write the following system in a matricial form (i.e. write the system as $A \cdot v = b$, where A is a 3×3-matrix, and v, b are two 3×1 matrices):

$$\begin{cases} -2x - 2y - 2z &= 0 \\ 6x + 2y - 2z &= 0 \\ 6x - 4z &= 8 \end{cases}$$

Calculate the determinant of A and use the Gaussian elimination method to solve the system.

Solution Defining A as the matrix

$$A = \begin{pmatrix} -2 & -2 & -2 \\ 6 & 2 & -2 \\ 6 & 0 & -4 \end{pmatrix}$$

we can write the system as

$$A. \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 8 \end{pmatrix}$$

A direct calculation gives that

$$\det A = 16.$$

and a Gaussian elimination yields that

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -8 \\ 4 \end{pmatrix}.$$

GOOD LUCK!
