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Examination in
Mathematics for Economic and Statistical Analysis
MM1005, Fall term; 7,5 ECTS
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Instructions: - During the exam you may not use any textbook, class notes, or any other supporting material.
- Non-graphical calculators will be provided for the exam by the department. Other calculators may not be used.
- In all solutions, justify your answers? communicate your reasoning. Use ordinary language where appropriate, not just mathematical symbols.
- Use natural language, not just mathematical symbols. Write clearly and legibly
- Mark your final answer to each question clearly by putting a a box around it.

Grades: Each solved problem is awarded by up to 10 points. At least 35 points would guarantee grade E, 42 for D, 49 for C, 56 for B and 63 for A. Note that the problems are not ordered according to the difficulty!

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1. Calculate the limits

$$\text{a)}\quad \lim_{x\to+\infty}\frac{\sqrt{1+9x^2}-x}{x} \qquad\qquad \text{b)}\quad \lim_{x\to 0}\frac{e^x-1}{2x+5x^2}$$

Solution

$$\lim_{x\to+\infty}\frac{\sqrt{1+9x^2}-x}{x}=2 \qquad\qquad \lim_{x\to 0}\frac{e^x-1}{2x+5x^2}=\frac{1}{2}.$$

2. Calculate the integrals

$$\text{a)}\quad \int 3\left(\frac{x^3-2x^2+x-1}{x-2}\right)\mathrm{d}x \qquad\qquad \text{b)}\quad \int_0^\infty x^3e^{-x^4}\,\mathrm{d}x$$

Solution

$$\begin{aligned}\int 3\left(\frac{x^3-2x^2+x-1}{x-2}\right)\mathrm{d}x &= \int\left(3x^2+3+\frac{3}{x-2}\right)\mathrm{d}x\\ &= x^3+3x+3\ln|x-2|+C \qquad \text{where } C\in\mathbb{R}.\end{aligned}$$

$$\int_0^\infty x^2e^{-x^4}\,\mathrm{d}x=\frac{1}{4}.$$

3. The expression

$$e^yx^2+e^xy^3-x^2+8=4x$$

defines y as a function of x. What is the equation of the tangent line to $y(x)$ at the point $x=0$?

Solution Setting $x=0$ in the expression we obtain

$$(y(0))^3+8=0\Rightarrow y(0)=-2.$$

Differentiating we obtain that

$$y'\left(e^yx^2+3y^2e^x\right)+\left(2e^yx+e^xy^3-2x\right)=4.$$

So, setting $x=0$ yields

$$12y'(0)-8=4\Rightarrow y'(0)=1.$$

Hence the equation of the line tangent to $y(x)$ at the point $x=0$ is

$$t(x)=-2+x.$$

4. Find the value of the parameters a, b such that the function given by

$$f(x) = \frac{ax^2 + bx + 1}{x - 2}$$

has a local extreme at $f(1) = 1$. Is it a local maximum or a minimum point?

Solution Differentiating

$$f'(x) = \frac{a(x-4)x - 2b - 1}{(x-2)^2}.$$

Hence we need to solve

$$\begin{cases} 1 = f(1) \\ 0 = f'(1) \end{cases} \Leftrightarrow \begin{cases} 1 = -a - b - 1 \\ 0 = -3a - 2b - 1 \end{cases} \Leftrightarrow \begin{cases} a = 3 \\ b = -5 \end{cases}$$

Therefore

$$f'(x) = \frac{3\left(x^2 - 4x + 3\right)}{(x-2)^2} = \frac{3\,(x-1)\,(x-3)}{(x-2)^2}.$$

Note that the function is not defined for $x \neq 2$. If $x < 1$ then $f'(x) > 0$ and if $1 < x < 2$, then $f'(x) < 0$. This implies that $x = 3$ is a local maximum point for f.

5. Find all stationary points for the function

$$f(x,y) = (4 - x - y^2)(x+1)$$

and determine whether they are maximum, minimum or saddle points.

Solution The function f can be written as

$$f(x,y) = (4 - x - y^2)(x+1) = 3x + 4 - x^2 - xy^2 - y^2$$

whose partial derivatives are

$$f'_x(x,y) = 3 - 2x - y^2 \qquad \text{and} \qquad f'_y(x,y) = -2xy - 2y = -2y(x+1)\ .$$

This gives the equation system

$$\begin{cases} f'_x(x,y) &= 0 \\ f'_y(x,y) &= 0 \end{cases} \Leftrightarrow \begin{cases} 3 - 2x - y^2 &= 0 \\ -2y(x+1) &= 0 \end{cases}$$

The second equation gives that $y = 0$ or $x = -1$. If we plug $y = 0$ in the first equation we obtain

$$3 - 2x = 0 = 0 \quad \Leftrightarrow \quad x = \tfrac{3}{2}\ .$$

If we use $x = -1$ in the first equation gives

$$3 + 2 - y^2 = 0 \quad \Leftrightarrow \quad y = \pm\sqrt{5}.$$

So we have three stationary points: $(\frac{3}{2}, 0)$, $(-1, \sqrt{5})$ och $(-1, -\sqrt{5})$.

The second order derivatives are

$$f''_{xx}(x,y) = -2\ ,\ \ f''_{xy}(x,y) = -2y \qquad \text{and} \qquad f''_{yy}(x,y) = -2x - 2\ .$$

If $(x,y) = \underline{(\frac{3}{2}, 0)}$ we obtain that $f''_{xx} = -2$, $f''_{xy} = 0$ and $f''_{yy} = -5$.

Since $f''_{xx} < 0$ and $f''_{xx} f''_{yy} - (f''_{xy})^2 = 10 > 0$ we have that this is a local maximum point.

If $(x,y) = \underline{(-1, \sqrt{5})}$ we have that $f''_{xx} = -2$, $f''_{xy} = -2\sqrt{5}$ and $f''_{yy} = 0$.
Since $f''_{xx} f''_{yy} - (f''_{xy})^2 = -4 \cdot 5 < 0$ we have that this point is a sadle point.

If $(x,y) = \underline{(-1, -\sqrt{5})}$ we have that $f''_{xx} = -2$, $f''_{xy} = -2\sqrt{5}$ and $f''_{yy} = 0$.
Since $f''_{xx} f''_{yy} - (f''_{xy})^2 = -4 \cdot 5 < 0$ we have that this point is a sadle point.

6. For which real numbers x is the series $S = \sum_{n\geq 1} \left(\frac{2}{3}\sqrt{x+1}\right)^n$ convergent? Find x such that $S = \frac{1}{2}$.

Solution (a) We need that $\sqrt{x+1}$ is well defined and $\sqrt{x+1} < \frac{3}{2}$, which is equivalent to

$$0 \leq x+1 < \frac{9}{4} \Leftrightarrow -1 \leq x < \frac{5}{4}.$$

(b) Note that

$$S = \frac{\frac{2}{3}\sqrt{x+1}}{1 - \frac{2}{3}\sqrt{x+1}} = \frac{2\sqrt{x+1}}{3 - 2\sqrt{x+1}} = \frac{1}{2} \Leftrightarrow \sqrt{x+1} = \frac{1}{2} \Leftrightarrow x = -\frac{3}{4}.$$

7. Write the following system in a matricial form (i.e. write the system as $A \cdot v = b$, where A is a 3×3-matrix, and v, b are two 3×1 matrices):

$$\begin{cases} x + 2y + 4z = 2 \\ \quad\; -y - 3z = 1 \\ 2x + 2y + 6z = 2 \end{cases}$$

Calculate the determinant of A and use the Gaussian elimination method to solve the system.

Solution Defining A as the matrix

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 0 & -1 & -3 \\ 2 & 2 & 6 \end{pmatrix}$$

we can write the system as

$$A. \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 8 \end{pmatrix}$$

A direct calculation gives that

$$\det A = \begin{vmatrix} 1 & 2 & 4 \\ 0 & -1 & -3 \\ 2 & 2 & 6 \end{vmatrix} = 1 \begin{vmatrix} -1 & -3 \\ 2 & 6 \end{vmatrix} - 2 \begin{vmatrix} 0 & -3 \\ 2 & 6 \end{vmatrix} + 4 \begin{vmatrix} 0 & -1 \\ 2 & 2 \end{vmatrix}$$
$$= 1(-6+6) - 2(0-(-6)) + 4(0+2) = -12+8 = -4$$

and a Gaussian elimination yields that

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}.$$

GOOD LUCK!
