STOCKHOLM UNIVERSITY
Department of Mathematics
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Examination in
Mathematics for Economic and Statistical Analysis
MM1005, Fall term; 7,5 ECTS
25 September 2023

Instructions: - During the exam you may not use any textbook, class notes, or any other supporting material.
- Non-graphical calculators will be provided for the exam by the department. Other calculators may not be used.
- In all solutions, justify your answers.
- Use natural language, not just mathematical symbols. Write clearly and legibly
- Mark your final answer to each question clearly by putting a $\boxed{\text{a box}}$ around it.
- **Do not write two exercises on the same page.**

Grades: Each solved problem is awarded by up to 10 points. At least 35 points would guarantee grade E, 42 for D, 49 for C, 56 for B and 63 for A. Note that the problems are not ordered according to the difficulty!

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1. Calculate the limits

$$\text{a)}\quad \lim_{x\to+\infty}\frac{x^2-\sqrt{1+25x^4}}{x^2+25x+1} \qquad\qquad \text{b)}\quad \lim_{x\to 0}\frac{xe^{x^2}-x-x^3}{2\ln\left[(e+x)^{x^5}\right]}$$

Solution

$$\lim_{x\to+\infty}\frac{x^2-\sqrt{1+25x^4}}{x^2+25x+1}=-4 \qquad\qquad \lim_{x\to 0}\frac{xe^{x^2}-x-x^3}{2\ln\left[(e+x)^{x^5}\right]}=-\frac{1}{4}.$$

2. Calculate the integrals

$$\text{a)}\quad \int 2\left(\frac{x^3+7x^2+16x+11}{x+1}\right)\mathrm{d}x \qquad\qquad \text{b)}\quad \int_{-\infty}^{0} x^2 5^{x^3}\,\mathrm{d}x$$

Solution

$$\begin{aligned}\int 2\left(\frac{x^3+7x^2+16x+11}{x+1}\right)\mathrm{d}x &= 2\int\left(x^2+6x+\frac{1}{x+1}+10\right)\mathrm{d}x\\ &=\frac{2}{3}x^3+6x^2+2\ln|x+1|+20x+C \qquad \text{where } C\in\mathbb{R}.\end{aligned}$$

$$\int_{-\infty}^{0} x^2 5^{x^3}\,\mathrm{d}x=\frac{1}{3\ln 5}.$$

3. The expression

$$e^{x+1}y^3+(x+1)^2e^y-x^2-6x+3=0$$

defines y as a function of x. What is the equation of the tangent line to $y(x)$ at the point $x=-1$?

Solution Substitution gives

$$y(-1)=2.$$

Implicit differentiation yields

$$y'(-1)=1.$$

Then the tangent line becomes

$$Y(x)=2+(x-1)=x+1.$$

4. Find the value of the parameters a, b such that the function given by

$$f(x) = \frac{ax^2 + x(6a+b) + 9a + 3b + 1}{x+1}$$

has a local extreme at $f(-2) = 1$. Is it a local maximum or a minimum point? Is it a global maximum/minimum point?

Solution Differentiating yields

$$f'(x) = \frac{a\left(x^2+2x-3\right) - 2b - 1}{(x+1)^2}.$$

Hence we need to solve

$$\begin{cases} 1 = f(1) \\ 0 = f'(1) \end{cases} \Leftrightarrow \begin{cases} 1 = -a - b - 1 \\ 0 = -3a - 2b - 1 \end{cases} \Leftrightarrow \begin{cases} a = 3 \\ b = -5 \end{cases}$$

In this case

$$f'(x) = \frac{3x(x+2)}{(x+1)^2}.$$

Note that the function is not defined for $x \neq -1$. If $x < -2$ then $f'(x) > 0$ and if $-2 < x < -1$ or $-1 < x < 0$ then $f'(x) < 0$. This implies that $x = -2$ is a local maximum point for f.

Besides, note that for $a = 3$ and $b = -5$

$$f(x) = 10 + 3x + \frac{3}{x+1}.$$

In particular

$$\lim_{x \to -1^+} 10 + 3x + \frac{3}{x+1} = +\infty$$

so we have that f has no global maximum.

5. Find all stationary points for the function

$$f(x,y) = -(y+1)\left(x^2+2x+y-3\right)$$

and determine whether they are maximum, minimum or saddle points.

Solution The function $f(x,y) = (-y-1)\left(x^2+2x+y-3\right)$ has partial derivatives

$$f'_x(x,y) = -2(1+x)(1+y) \qquad \text{and} \qquad f'_y(x,y) = -x^2 - 2x - 2y + 2.$$

This gives us the system of equations

$$\begin{cases} f'_x(x,y) &= 0 \\ f'_y(x,y) &= 0 \end{cases} \Leftrightarrow \begin{cases} -2(1+x)(1+y) &= 0 \\ -x^2 - 2x - 2y + 2 &= 0 \end{cases}$$

The first equation gives that $y = -1$ or $x = -1$.

Setting $x = -1$ in the scond equation gives

$$3 - 2y = 0 \quad \Leftrightarrow \quad y = \tfrac{3}{2}.$$

Setting $y = -1$ in the second equation gives

$$4 - 2x - x^2 = 0 \quad \Leftrightarrow \quad x = -1 \pm \sqrt{5}.$$

We have then three stationary points $(-1, \frac{3}{2})$, $(-1-\sqrt{5}, -1)$ och $(-1+\sqrt{5}, -1)$.

Calculating the second partial derivatives gives

$$f''_{xx}(x,y) = -2-2y\ , \ \ f''_{xy}(x,y) = -2x-2 \qquad \text{och} \qquad f''_{yy}(x,y) = -2\ .$$

Letting $(x,y) = \underline{(-1, \frac{3}{2})}$ give $A = f''_{xx} = -5$, $B = f''_{xy} = 0$ and $C = f''_{yy} = -2$.

Since $A < 0$ (even $C < 0$) and $AC - B^2 = 10 > 0$ then it is a local maximum point.

Letting $(x,y) = \underline{(-1+\sqrt{5}, -1)}$ gives $A = f''_{xx} = 0$, $B = f''_{xy} = -2\sqrt{5}$ och $C = f''_{yy} = -2$.

Since $AC - B^2 = -4 \cdot 5 < 0$ it follows that it is a saddle point.

Letting $(x,y) = \underline{(-1-\sqrt{5}, -1)}$ gives $A = f''_{xx} = 0$, $B = f''_{xy} = 2\sqrt{5}$ och $C = f''_{yy} = -2$.

Also we have that $AC - B^2 = -4 \cdot 5 < 0$ so it is also a saddle point.

6. For which real numbers x is the series $S = \sum_{n\geq 1} \left(\frac{2}{3\sqrt{x+3}} \right)^n$ convergent? Find x such that $S = \frac{1}{2}$.

Solution (a) We need that $x > -3$ for the square root to be well defined. To converge, we need that $\sqrt{x+3} > \frac{2}{3}$, which is equivalent to

$$x+3 > \frac{4}{9} \Leftrightarrow x > \frac{-23}{9}\,(> -3)\,.$$

(b) Note that for $x > \frac{-23}{9}$ we have that

$$S = \frac{2}{3\sqrt{x+3}-2} = \frac{1}{2} \Leftrightarrow x = 1.$$

7. Determine for which values of the parameter a, the system

$$\begin{cases} ax + 2y + 3z = 5 \\ 5x + 2y + az = 2 \\ 5x + 2y + 3z = a \end{cases}$$

has exactly one solution, no solutions or an infinite number of solutions.

Solution Defining A as the matrix

$$A = \begin{pmatrix} a & 2 & 3 \\ 5 & 2 & a \\ 5 & 2 & 3 \end{pmatrix}$$

we can write the system as

$$A. \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ a \end{pmatrix}$$

A direct calculation gives that

$$\det A = -30 + 16a - 2a^2 = -2(-5+a)(-3+a).$$

Hence, for $a \neq 5$ and $a \neq 3$, the system has a unique solution.

Letting $a = 5$, and doing Gaussian elimination yields that the system has an infinite number of solutions.

Letting $a = 3$, and doing Gaussian elimination, one shows that the system has no solution.

GOOD LUCK!
