STOCKHOLM UNIVERSITY
Department of Mathematics
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Examination in
Mathematics for Economic and Statistical Analysis
MM1005 Fall term; 7,5 ECTS
1 October 2024

Instructions:
- During the exam you may not use any textbook, class notes, or any other supporting material.
- Non-graphical calculators will be provided for the exam by the department. Other calculators may not be used.
- In all solutions, justify your answers.
- Use natural language, not just mathematical symbols. Write clearly and legibly
- Mark your final answer to each question clearly by putting a box around it.
- **Do not write two exercises on the same sheet.**

Grades: Each solved problem is awarded by up to 10 points. At least 35 points would guarantee grade E, 42 for D, 49 for C, 56 for B and 63 for A. Note that the problems are not ordered according to the difficulty!

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1. Calculate the limits

a) $\lim_{x\to 0} \frac{e^{-x^3}-e^{x^3}}{\ln\left[(e^2+x)^{x^3}\right]}$ b) $\lim_{x\to+\infty}\left(x^4-\sqrt{9x^8-2}\right)$

Solution

$$\lim_{x\to 0} \frac{e^{-x^3}-e^{x^3}}{\ln\left[(e^2+x)^{x^3}\right]} = -1, \qquad \lim_{x\to+\infty}\left(x^4-\sqrt{9x^8-2}\right) = -\infty$$

2. Calculate the integrals

a) $\int \left(\frac{x^3+4x^2-12x+2}{x-2}\right)\mathrm{d}x$ b) $\int_0^{+\infty}(x^2-2)2^{-x^3+6x}\mathrm{d}x$

Solution

$$\int \left(\frac{x^3+4x^2-12x+2}{x-2}\right)\mathrm{d}x = \frac{x^3}{3}+3x^2+2\log|x-2|+C \qquad \text{where } C\in\mathbb{R}.$$

$$\int_0^{+\infty}(x^2-2)2^{-x^3+6x}\mathrm{d}x = \frac{1}{\ln 8}.$$

3. The expression

$$y^3+3x^2y-3x-1=0$$

defines y as a function of x. What is the MacLaurin polynomial of degree 2 for $y(x)$?

Solution Substitution gives

$$y(0)=1.$$

Implicit differentiation yields

$$y'(0)=1, \qquad y''(0)=-4.$$

Thus

$$P_2(x)=1+x-2x^2.$$

4. Find the value of the parameters a, b, c such that the function given by

$$f(x) = \frac{25a+5b+c+(10a+b)x+ax^2}{x+4}$$

has a local extreme at the point $(-3,3)$ when $a+b+c=1$. Is this local extreme a local maximum or a local minimum? Argument your answer.

Solution Differentiating yields

$$f'(x) = \frac{ax^2+8ax+15a-b-c}{(x+4)^2}.$$

Hence we need to solve

$$\begin{cases} 3 = f(-3) \\ 0 = f'(-3) \\ a+b+c=1 \end{cases} \Leftrightarrow \begin{cases} 3 = 4a+2b+c \\ 0 = -b-c \\ 1 = a+b+c \end{cases} \Leftrightarrow \begin{cases} a = 1 \\ b = 1 \\ c = -1 \end{cases}$$

In this case

$$f'(x) = \frac{x^2+8x+15}{(x+4)^2} = \frac{(x+5)(x+3)}{(x+4)^2}.$$

Note that the function is not defined for $x=-4$. If $x<-5$ or $x>-3$ then $f'(x)>0$ and if $-5<x<-4$ or $-4<x<-3$ then $f'(x)<0$. This implies that $x=-2$ is a local minimum point for f.

5. Let

$$f(x,y) := x^2+xy-4x+2y^2+5y+3.$$

(a) Find all stationary points for the function and determine whether they are local maximum, minimum or saddle points.

(b) Find the maximum and minimum value of f on the set

$$S = \left\{(x,y) \in \mathbb{R}^2 : x+y=1, x \geq 0, y \geq 0\right\}.$$

Solution The function $f(x,y) = x^2+xy-4x+2y^2+5y+3$ has partial derivatives

$$f'_x(x,y) = 2x+y-4 \qquad \text{and} \qquad f'_y(x,y) = x+4y+5.$$

This gives us the system of equations

$$\begin{cases} f'_x(x,y) &= 0 \\ f'_y(x,y) &= 0 \end{cases} \Leftrightarrow \begin{cases} 2x+y-4 &= 0 \\ x+4y+5 &= 0 \end{cases}$$

which has only the solution $x=3$, $y=-2$. We have then one stationary point $(3,-2)$.

Calculating the second partial derivatives gives

$$A = f''_{xx}(x,y) = 2\ , \quad B = f''_{xy}(x,y) = 1 \qquad \text{and} \qquad C = f''_{yy}(x,y) = 4\ .$$

Since for all $(x,y) \in \mathbb{R}^2$

$$AC - B^2 = 8-1 = 7 > 0$$

and both $A>0$ and $C>0$, can we deduce that it is a <u>global minimum point</u>.

Parametrising

$$S = \{(x,y) : x=t, y=1-t, \qquad t \in [0,1]\}.$$

So we need to find the optimal values of the function

$$g(t) = F(t, 1-t), \quad \text{for } t \in [0,1].$$

Notice that

$$g'(t) = 2t + (1-t) - t - 4 - 4(1-t) - 5 = 4t - 12.$$

So $t = 3$ is the only stationary point for g, but this lies outside of the interval $[0,1]$.

Looking at the signg of $g'(t)$, we deduce that the function g is decreasing on the interval $[0,1]$.

So the maximum value is attained at $t = 0$ and the minimum at $t = 1$. This values are respectively

$$10 = F(0,1) = g(0), \qquad g(1) = F(1,0) = 0.$$

6. For which real numbers x is the series $S = \sum_{n\geq 1} \left(\frac{\sqrt{x+1}}{4} \right)^n$ convergent? Find x such that $S = 1$.

Solution (a) We need $x \geq -1$ for the square root to be well defined. To converge, we need that $\sqrt{x+1} < 4$, which is equivalent to

$$x+1 < 16 \Leftrightarrow x < 15.$$

(b) Note that for $0 \leq x < 4$ we have that

$$S = \frac{\sqrt{x+1}}{4 - \sqrt{x+1}} = 1 \Leftrightarrow x = 3.$$

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7. Let

$$M = \begin{pmatrix} 1 & 1 \\ 6 & 5 \end{pmatrix}$$

Find a matrix $N = (n_{i,j})$ with 3 rows and 2 columns such that $N \cdot M$ satisfies

$$N \cdot M = \begin{pmatrix} 0 & -1 \\ -1 & 0 \\ 0 & 1 \end{pmatrix} - 2 \begin{pmatrix} 0 & -1 \\ 0 & 0 \\ 1 & -1 \end{pmatrix}.$$

Solution We need to solve

$$N \cdot M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ -2 & 3 \end{pmatrix} \Leftrightarrow N = \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ -2 & 3 \end{pmatrix} M^{-1}.$$

On the other hand

$$M^{-1} = \begin{pmatrix} -5 & 1 \\ 6 & -1 \end{pmatrix}.$$

Hence

$$N = \begin{pmatrix} 6 & -1 \\ 5 & -1 \\ 28 & -5 \end{pmatrix}.$$

GOOD LUCK!
