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Examination in
Mathematics for Economic and Statistical Analysis
MM1005 Fall term; 7,5 ECTS
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Instructions:
- During the exam you may not use any textbook, class notes, or any other supporting material.
- Non-graphical calculators will be provided for the exam by the department. Other calculators may not be used.
- In all solutions, justify your answers. Show clearly the computations that you do to reach the solution, answers without motivation and computations will receive zero credit.
- Use natural language, not just mathematical symbols. Write clearly and legibly.
- Mark your final answer to each question clearly by putting a box around it.
- **Do not write two exercises on the same sheet.**

Grades: Each solved problem is awarded by up to 10 points. At least 30 points guarantees grade E, 36 for D, 42 for C, 48 for B and 54 for A. Note that the problems are not ordered according to the difficulty!

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1. (a) Determine all the possible values of the parameter $b \in \mathbb{R} \setminus \{0\}$ such that

$$\lim_{x \to 0} \frac{e^{bx} + e^{-bx} - 2}{4x^2} = 1.$$

   (b) Determine the values of the parameters $a, b \in \mathbb{R} \setminus \{0\}$ such that the function

$$f(x) = (x+1)^a (1-x)^b$$

   has a local minimum at $x = 0$.

Solution

$$\lim_{x \to 0} \frac{e^{bx} + e^{-bx} - 2}{4x^2} = 1 \Leftrightarrow b = \pm 2.$$

The function

$$f(x) = (x+1)^a (1-x)^b$$

has a local minimum at $x = 0$, if and only if $a = b < 0$.

2. Calculate the integrals

$$\text{a)} \quad \int \left(\frac{2x^3 - 8x^2 + 10x - 3}{x-2} \right) \mathrm{d}x \qquad\qquad \text{b)} \quad \int_0^1 \frac{t^2 \ln(t^3+1)}{t^3+1} \mathrm{d}t$$

Solution

$$\int \left(\frac{2x^3 - 8x^2 + 10x - 3}{x-2} \right) \mathrm{d}x = \frac{2x^3}{3} - 2x^2 + 2x + \log|x-2| + C \qquad \text{where } C \in \mathbb{R}.$$

$$\int_0^1 \frac{t^2 \ln(t^3+1)}{t^3+1} \mathrm{d}t = \frac{\ln^2 2}{6}.$$

3. The expression

$$y^3 + (x-4)y - 4 = 0,$$

defines y as a smooth function of x around the point $x = 2$: $y = y(x)$.

(a) Find all the possible values of $y(2)$.

(b) Is $y(x)$ increasing or decreasing around the point $(2, y(2))$. Argument your answer.

(c) Is $y(x)$ convex or concave around the point $(2, y(2))$. Argument your answer.

Solution Substituting in the expression and solving gives

$$y(2) = 2.$$

Implicit differentiation yields

$$y'(2) = -1/5, \qquad y''(2) = -1/125.$$

We infer that y is decreasing and concave around the point $(2, y(2))$.

4. Let

$$f(x,y) = 2x^3 - 3x^2 - y^2.$$

(a) Find all stationary points for the function and determine whether they are local maximum, minimum or saddle points.

(b) Make a sketch of the set

$$S = \left\{(x,y) \in \mathbb{R}^2 : -2x + y = -1, x \geq 1, y \leq 5\right\},$$

and find the maximum and minimum value of f on S.

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Solution The function $f(x,y)$ has partial derivatives

$$f'_x(x,y) = 6x^2 - 6x \qquad \text{and} \qquad f'_y(x,y) = 2y.$$

This gives us that the only stationary points are

$$(0,0), \qquad (1,0).$$

Calculating the second partial derivatives gives

$$A = f''_{xx}(x,y) = -6 + 12x \ , \quad B = f''_{xy}(x,y) = 0 \qquad \text{and} \qquad C = f''_{yy}(x,y) = -2 \ .$$

Substituting, we obtain that the point $(0,0)$ is a local maximum point, and the points $(1,0)$ is a saddle point.

The set S is a segment given by points of the form

$$(x, 2x-1), \qquad \text{for } x \in [1,3].$$

So we need to find the optimal values of the function

$$g(t) = F(t, 2t-1), \quad \text{for } t \in [1,3].$$

Notice that

$$g'(t) = 6t^2 - 14t + 4,$$

which has roots $t = 1/3$, $t = 2$. Observe that only the second one lies within the interval $[1,3]$, and that

$$g(2) = f(2,3) = -5.$$

Moreover $g(1) = -2$ and $g(3) = 2$. So the function f attains its minimum value on S at the points $(2,3)$, and the maximum at the points $(3,5)$. These values are respectively -5 and 2.

5. Determine the minimum natural number $n \in \mathbb{N}$ such that the value of $\sum_{k=0}^{n} 3^k$ is greater or equal than $(e^{100000} - 1)/2$.

   Tip: You may use that $(\ln 3)^{-1} \approx 0.9102392266$.

Solution $n = 91023$.

6. Consider the matrix

$$A = \begin{pmatrix} 2 & 2 & 2 \\ 1 & 1 & \alpha \\ \alpha & 1 & 1 \end{pmatrix}$$

   (a) Compute the determinant of the matrix A, and determine all the values of α for which A is not invertible.

   (b) Using the Gauss elimination method, determine whether the following linear system has exactly one, none, or infinitely many solutions. In case of having any solution, provide all of them.

$$\begin{cases} x + y + z = 1 \\ x + 2y + 2z = 3 \\ x + 3y + 3z = 5 \\ y + z = 2 \end{cases}$$

Solution The determinat of A is equal to

$$2(1-\alpha)^2.$$

So the matrix A is not invertible only for $\alpha = 1$.

By Gauss elimination, we obtain that the system has an infinite number of solutions given by

$$x = -1, \qquad y = 2 - t, \qquad z = t, \qquad t \in \mathbb{R}.$$

GOOD LUCK!
