MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
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Tentamensskrivning i
Mathematics Methods for Economist MM3001
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Instructions:

- During the exam you MAY NOT use textbooks, class notes, or any other supporting material.
- Use of calculators is permitted for performing calculations. The use of graphic or programmable features is NOT permitted.
- Start every problem on a new page, and write at the top of the page which problem it belongs to. (But in multiple part problems it is not necessary to start every part on a new page)
- In all of your solutions, give explanations to clearly show your reasoning. Points may be deducted for unclear solutions even if the answer is correct.
- Write clearly and legibly.
- Where applicable, indicate your final answer clearly by putting A BOX around it.

Note: There are six problems, some with multiple parts. The problems are not ordered according to difficulty

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(1) (5pt) Compute the degree 2 Taylor polynomial of the function $f(x) = e^{-\sqrt{x+3}}$ around the point $x_0 = 1$.
Soulution: We begin by computing the first and second derivative of $f(x)$ in $x = 1$:

$$f'(x) = \frac{-1}{2}(x+3)^{-\frac{1}{2}}e^{-\sqrt{x+3}}.$$

We plug in $x = 1$ and get $f'(1) = \frac{-1}{4}e^{-2}$. For the second derivative we have

$$f''(x) = \frac{1}{4}(x+3)^{-\frac{3}{2}}e^{-\sqrt{x+3}} + \frac{1}{4}(x+3)^{-1}e^{-\sqrt{x+3}}.$$

In $x = 1$ we get $f''(1) = \frac{3}{32}e^{-2}$. Now we have all the ingredients to compute the Taylor polynomial:

$$p_2(x) = e^{-2} - \frac{1}{4}e^{-2}(x-1) + \frac{3}{64}e^{-2}(x-1)^2.$$

(2) Consider the function $f(x) = \frac{x^2+x+3}{x+1}$.
(a) (2pt) Assume that the function is defined on the interval $[0, 2]$, compute the maximum and minimum value of the function and give the points in which these values are taken.
(b) (2pt) Assume now that the function is defined on $[0, +\infty)$. Does the function have a maximum value? Explain why.
(c) (1pt) Compute $\lim_{x\to+\infty} \frac{f(x)}{6x}$.
Solution: We need to find eventual values in the interval $(0, 2)$ where $f'(x)$ vanishes and we need to compare the value of f at those points with $f(0)$

and $f(2)$. We start by computing the first derivative of f:

$$\begin{aligned} f'(x) &= \frac{(2x+1)(x+1) - (x^2+x+3)\cdot 1}{(x+1)^2} \\ &= \frac{2x^2+2x+x+1-x^2-x-3}{(x+1)^2} \\ &= \frac{x^2+2x-2}{(x+1)^2}. \end{aligned}$$

The polynomial at the numerator has roots $x_1 = -1+\sqrt{3}$ and $x_2 = -1-\sqrt{3}$. Of these the second is negative and lies outside the given interval while the other is in $[0,2]$. We see that $f(x_1) = \frac{7\sqrt{3}-9}{3} \simeq 1.0415$. We compare this value with $f(0) = 3 = f(2)$ and deduce that the maximum of the function in $[0,2]$ is 2 while the minimum is $f(\sqrt{3}-1)$.
For the second point we observe that $f'(x) > 0$ when $x > 2$, thus the function is increasing. In addition we can see that $\lim_{x\to+\infty} f(x) = +\infty$. We deduce that f grows with no limits if $x > 2$ and thus the function has no max on $[0,+\infty]$.
For the last point:

$$\begin{aligned} \lim_{x\to+\infty} \frac{f(x)}{6x} &= \lim_{x\to+\infty} \frac{x^2+x+3}{6x^2+6x} \\ &= \lim_{x\to+\infty} \frac{x^2(1+\frac{1}{x}+\frac{3}{x^2})}{x^2(6+6\frac{1}{x})} = \frac{1}{6} \end{aligned}$$

(3) Consider the series $2 + \frac{4}{x} + \frac{8}{x^2} + \frac{16}{x^3} + \cdots$. Determine
 (a) (2pt) for which values of x the series converges;
 (b) (3pt) for which values of x the value of the series is 1.
Solution: We need to express the series as a geometric series. We observe that

$$\begin{aligned} 2 + \frac{4}{x} + \frac{8}{x^2} + \frac{16}{x^3} + \cdots &= 2(1 + \frac{2}{x} + \frac{4}{x^2} + \frac{8}{x^3} + \cdots \\ &= 2(1 + \frac{2}{x} + (\frac{2}{x})^2 + (\frac{2}{x})^3 + \cdots \end{aligned}$$

Thus the series converges as long as $|\frac{2}{x}| < 1$ which is equivalent to the condition $|x| > 2$ which is satisfied in the union $(-\infty,-2) \cup (2,\infty)$.
For the second part, by using the formula for the value of geometric series we have that

$$2 + \frac{4}{x} + \frac{8}{x^2} + \frac{16}{x^3} + \cdots = 2\frac{1}{1-\frac{2}{x}}.$$

Thus we set

$$2\frac{1}{1-\frac{2}{x}} = 1$$

which is equivalent to $2 = 1 - \frac{2}{x}$. This has only one solution $x = -2$.

(4) Compute the following integrals:
 (a) (3pt) $\int \left(x\ln(x^2+1) + 3\sqrt{x^5}\right) dx$,
 (b) (2pt) $\int_0^{\ln(2)} \frac{5e^{2x}}{3e^{2x}-2} dx$.

Solution: We begin with the first

$$\int \left(x\ln(x^2+1)+3\sqrt{x^5}\right)dx = \int x\ln(x^2+1)dx + \int 3\sqrt{x^5}dx$$
$$= \frac{1}{2}\int \ln(u)du + \frac{6}{7}x^{\frac{7}{2}} + C_1$$

Where in the first summand I used the substitution $u = x^2+1$, so $du = 2xdx$. Going forward we get

$$\int \left(x\ln(x^2+1)+3\sqrt{x^5}\right)dx = \frac{1}{2}(\ln(u)u-u)+C_2+\frac{6}{7}x^{\frac{7}{2}}+C_1$$
$$= \frac{1}{2}(\ln(x^2+1)(x^2+1)-(x^2+1))+\frac{6}{7}x^{\frac{7}{2}}+C.$$

For the second integral we set $u = 3e^{2x}-2$. Thus $du = 6e^{2x}$. We need to recompute the extremes of integration. In order to do that we have that if $x=0$ then $u=1$, and if $x=\ln(2)$, the $u=10$. Thus we get

$$\int_0^{\ln(2)} \frac{5e^{2x}}{3e^{2x}-2}dx = \frac{5}{6}\int_0^{\ln(2)}\frac{6e^{2x}}{3e^{2x}-2}dx$$
$$= \frac{5}{6}\int_1^{10}\frac{du}{u}$$
$$= \frac{5}{6}[\ln(u)]_1^{10} = \frac{5}{6}\ln(10).$$

(5) Consider the matrix

$$A = \begin{pmatrix} 5 & 4 & 1 \\ k-7 & k-4 & -2 \\ 5 & k-2 & 1 \end{pmatrix}$$

(a) (2 pt) Compute the determinant of A, $|A|$ as a function of k.
(b) (1 pt) Find all the values of k for which A is not invertible.
(c) (2 pt) Use Gauss–Jordan elimination to solve the linear system

$$\begin{cases} 5x & +4y & +1z & = & 1 \\ -7x & -4y & -2z & = & -3 \\ 5x & -2y & +z & = & 1 \end{cases}$$

Solution: We perform elementary operation onnthe rows of A to make the computations of the determinant easier:

$$\begin{vmatrix} 5 & 4 & 1 \\ k-7 & k-4 & -2 \\ 5 & k-2 & 1 \end{vmatrix} = \begin{vmatrix} 5 & 4 & 1 \\ k-7 & k-4 & -2 \\ 0 & k-6 & 0 \end{vmatrix}$$
$$= (-1)(k-6)(-10-(k-7)) = (k-6)(k+3)$$

We see immediately that $\det(A)=0$ if and only if $k=6$ or $k=-3$, so we deduce that A is invertible if $k \neq 6, 3$.
To solve the system we perform elementary row operations to the matrix

$$\left(\begin{array}{ccc|c} 5 & 4 & 1 & 1 \\ -7 & -4 & -2 & -3 \\ 5 & -2 & 1 & 1 \end{array}\right)$$

We subtract the firs row to the third row:

$$\left(\begin{array}{ccc|c} 5 & 4 & 1 & 1 \\ -7 & -4 & -2 & -3 \\ 5 & -2 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 5 & 4 & 1 & 1 \\ -7 & -4 & -2 & -3 \\ 0 & -6 & 0 & 0 \end{array}\right)$$

We add twice the first row to the second row

$$\left(\begin{array}{ccc|c} 5 & 4 & 1 & 1 \\ -7 & -4 & -2 & -3 \\ 0 & -6 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 5 & 4 & 1 & 1 \\ 3 & 4 & 0 & -1 \\ 0 & -6 & 0 & 0 \end{array}\right)$$

From the last row we deduce that $y = 0$. We plug in this in the second row and find that $3x = -1$. Therefore $x = \frac{-1}{3}$. We plug both the solutions we found for y and x in the first row, and get that $5 \cdot \frac{-1}{3} + z = 1$. We deduce that $x = \frac{8}{3}$.

(6) We define the area D to be the square in $\mathbb{R}^2$ with vertices:

$$(-1,-1), \quad (-1,1), \quad (1,-1), \quad (1,1).$$

Let $f(x,y) = xe^{3y} - e^x$.

(a) (1pt) Draw a sketch of D.

(b) (4pt) Find the maximum and minimum value of $f(x,y)$ in D.

Solution: We first compute the partial derivatives of f and set them to zero to find critical points for f in the interior of D. We get

$$\frac{\partial}{\partial x} f(x,y) = e^{3y} - e^x = 0$$

$$\frac{\partial}{\partial y} f(x,y) = 3xe^{3y} = 0.$$

From the second equation we find $x = 0$. If we plug that in the first equation we get $e^{3y} = 1$ which correspond to $3y = \ln(1) = 0$. We deduce that $y = 0$. The point $(0,0)$ lies in the interior of D so it is a candidate for a max or a min point for f.

Now we look for other candidates along the boundary. The four corners are candidates, and we need to find other possible candidates around the four edges. Note that the four edges have equaions $x = \pm 1$ and $y = \pm 1$.

Edge $x = 1$ We restrict the function on the side $(1,y)$. We have that $f(1,y) = e^{3y} - e$ is just a function of y of which we can compute the derivative. We have that $\frac{d}{dy} f(1,y) = 3e^{3y}$ which is never 0. Thus we get no candidate on this side. *Edge* $x = -1$ We restrict the function on the side $(-1,y)$. We have that $f(-1,y) = -e^{3y} - e^{-1}$ is just a function of y of which we can compute the derivative. We have that $\frac{d}{dy} f(-1,y) = -3e^{3y}$ which is never 0. Thus we get no candidate on this side. *Edge* $y = 1$ We restrict the function on the side $(x,1)$. We have that $f(x,1) = xe^3 - e^x$ is just a function of x of which we can compute the derivative. We have that $\frac{d}{dx} f(x,1) = e^3 - e^x$ which is 0 if $x = 3$. The point $(3,1)$ is not in D we get no candidate on this side. *Edge* $y = -1$ We restrict the function on the side $(x,-1)$. We have that $f(x,-1) = xe^{-3} - e^x$ is just a function of x of which we can compute the derivative. We have that $\frac{d}{dx} f(x,1) = e^{-3} - e^x$ which is 0 if $x = -3$. The point $(-3,-1)$ is not in D we get no candidate on this side.

<u>*Conclusion:*</u> From the reasoning above we get 5 candidates for max and min points of f, $(0,0)$, $(-1,-1)$, $(1,-1)$, $(-1,1)$ and $(1,1)$. We compute the values that f takes at these points:

$$f(0,0) = -1, \; f(-1,-1) \sim -0.4177 \; f(1,-1) \sim -2.6685$$

$$f(-1,1) \sim -20.4534 \; f(1,1) = e^3 - e.$$

Of these the biggest is surely $f(1,1)$ - it is the only one which is positive- and the smallest is $f(-1,1)$, so these are respectively the maximum and the minimum value of f in D. These value are taken at $(1,1)$ and $(-1,1)$.

FORMULAS

- Taylor polynomial of degree n of $f(x)$ around the point $x_0 = a$:

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

GOOD LUCK!!!

Texten på svenska

(1) (5pt) Bestäm Taylor polynomet av grad 2 till funktionen $f(x) = e^{-\sqrt{x+3}}$ kring punkten $x_0 = 2$.

(2) Betrakta funktionen $f(x) = \frac{x^2+x+3}{x+1}$.
(a) (2pt) Antar att f defineras på $[0, 2]$: bestäm max och min värden av f samt ge var dessa värden anta.
(b) (2pt) Antar nu att f defineras på $[0, +\infty)$. Har funktionen nu en max värde? Förklara din mening.
(c) (1pt) Bestäm $\lim_{x\to+\infty} \frac{f(x)}{6x}$.

(3) Bestäm följande integraler:
(a) (3pt) $\displaystyle\int \left(x \ln(x^2 + 1) + 3\sqrt{x^5}\right) dx$,
(b) (2pt) $\displaystyle\int_0^1 \frac{5e^{3x}}{1 - 3e^{3x}} dx$.

(4) Betrakta serien $2 + \frac{4}{x} + \frac{8}{x^2} + \frac{16}{x^3} + \cdots$. Bestäm
(a) (2pt) for vilka x convergerar serien;
(b) (3pt) för vilka x är seriens värden 1.

(5) Låt

$$A = \begin{pmatrix} 5 & 4 & 1 \\ k-7 & k-4 & -2 \\ 5 & k-2 & 1 \end{pmatrix}$$

(a) (2 pt) Räkna determinanten till A, $|A|$ som en funktion av k.
(b) (1 pt) Hitta alla värden k sådana att A inte är invertibar.
(c) (2 pt) Använd Gausselimination för att lösa det linjär systemet nedanför

$$\begin{cases} 5x & +4y & +1z & = & 1 \\ -7x & -4y & -2z & = & -3 \\ 5x & -2y & +z & = & 1 \end{cases}$$

(6) Vi definearar området D i $\mathbb{R}^2$ som kvadraten med hörnen

$$(-1, -1), \quad (-1, 1), \quad (1, -1), \quad (1, 1).$$

Låt $f(x, y) = xe^{3y} - e^x$.
(a) (1pt) Skissa området D.
(b) (4pt) Bestäm max och min värde av $f(x, y)$ i D.