STOCKHOLMS UNIVERSITET
MATEMATISKA INSTITUTIONEN
Avd. Matematik,
Lecturer: P. Alexandersson
Examiner: Sofia Tirabassi

Final exam
Mathematical Methods for Economists
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Instructions:

- You will be provided a calculator.
- Start every problem on a new page, and write at the top of the page which problem it belongs to.
- In all of your solutions, give explanations to clearly show your reasoning. Points may be deducted for unclear solutions even if the answer is correct.
- Write clearly and legibly.
- Where applicable, indicate your final answer clearly by putting A BOX around it.

There are six problems, some with multiple parts. The problems are not ordered according to difficulty.

Problem 1. (5p) Compute the degree 2 Taylor polynomial of $f(x) = x \cdot e^{1/x}$ around the point $x_0 = -1$. The answer should be expressed on the form $ax^2 + bx + c$.

Problem 2. Consider the function

$$f(x) = (2x-7)(x+4)\sum_{k=0}^{\infty}\left(\frac{x}{4}\right)^k.$$

(a) (1p) Show that $f(x)$ is defined on the interval $(-4, 4)$ but nowhere else.

(b) (2p) Find all critical points of $f(x)$.

(c) (2p) Find the minimum value of $f(x)$ on its domain, and sketch the graph. *Pay extra attention to the endpoints of the domain.*

Problem 3. Solve the following problems:

(a) (3p) Compute the integral $\displaystyle\int \frac{\ln(\sqrt{x}+1)}{\sqrt{x}}\,dx$.

(b) (2p) Compute the limit $\displaystyle\lim_{t\to\infty}\int_t^{2t}\frac{1+x}{x^2}\,dx$.

Problem 4. For every $C \in \mathbb{R}$, the equation $\sqrt{x}y^2 + x\sqrt{y} + \frac{4}{\sqrt{xy}} = C$ defines a curve in the plane.

(a) (1p) Determine C, such that the point $(4, 1)$ lies on the curve.

(b) (3p) For this value of C, find the slope of the tangent line at $(4, 1)$.

(c) (1p) Find the equation of the tangent line at $(4, 1)$.

Problem 5. Consider the function $f(x, y) = 2x^2 + 2y^2 + 3x + 4y$, and let D be the domain determined by the inequalities $x \geq 0$ and $x^2 + y^2 \leq 16$.

(a) (1p) Draw the domain D.

(b) (1p) Find all critical points of f and determine their type.

(c) (3p) Find the maximum of f on D.

Problem 6. Consider the matrices

$$A = \begin{pmatrix} 2 & k \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} x & y \\ z & w \end{pmatrix}.$$

(a) (1p) Compute $|A|$ as a function of k.

(b) (1p) For what values of k is $|A|$ non-zero?

(c) (3p) For the values of k you got in (b), solve the matrix equation $AX = B$. You should find x, y, z and w; some of these might depend on k.

Formula for geometric series

We have $1 + r + r^2 + r^3 + \cdots + r^{n-1} = \frac{1-r^n}{1-r}$, and $1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}$ whenever $-1 < r < 1$. The infinite series does not converge if $r \geq 1$ or $r \leq -1$.

Formula for Taylor polynomials

Taylor polynomial of degree k for $f(x)$ at $x = a$, is

$$f(a) + f'(a)\frac{(x-a)}{1!} + f''(a)\frac{(x-a)^2}{2!} + \cdots + f^{(k)}(a)\frac{(x-a)^k}{k!}.$$

Characterization of critical points

Let $f(x, y)$ be differentiable, and let $H = \begin{vmatrix} f''_{xx} & f''_{xy} \\ f''_{xy} & f''_{yy} \end{vmatrix}$. If, at critical point (x, y) we have

- $H > 0$ and $f''_{xx} > 0$, $f''_{yy} > 0$ then f has a local minimum at this critical point.
- $H > 0$ and $f''_{xx} < 0$, $f''_{yy} < 0$ then f has a local maximum at this critical point.
- $H < 0$ then f has neither a local maximum or minimum at this critical point.