SOLUTIONS OMTENTA HT 22

$f(x) = (x+1)\ln(x^2+1) \qquad f(0) = 0$

$f'(x) = \ln(x^2+1) + \frac{2x}{x^2+1}(x+1)$

$f'(0) = 0$

$f''(x) = \frac{2x}{1+x^2} + \frac{2x}{1+x^2} + (x+1)\frac{2(x^2+1)-4x^2}{(x^2+1)^2}$

$= \frac{4x}{1+x^2} + (x+1)\left(\frac{2-2x^2}{(x^2+1)^2}\right) =$

$f''(0) = 2$

$f'''(x) = \frac{4(x^2+1)-6x^2}{(x^2+1)^2} + \frac{2-2x^2}{(x^2+1)^2}$

$+ (x+1)\left(\frac{-4x(x^2+1)^2 - 2x \cdot 2(x^2+1)(2-2x^2)}{(x^2+1)^4}\right)$

$f'''(0) = 4 + 2 + 1 \cdot 0 = 6$

$p(x) = 0 + 0x + \frac{2}{2}x^2 + \frac{6}{6}x^3$

$= x^2 + x^3$

$p(0.1) = \frac{1}{100} + \frac{1}{1000} = \frac{10+1}{1000} = \frac{11}{1000} = 0.011$

(2)

a) we set $x=0$

$$\cancel{3\cdot 0^2\cdot y} + \cancel{e^{0+y}} + \ln(y) - \cancel{e^y} = 0$$ 1

$$\ln(y)=0 \qquad \boxed{y=1}$$ 1

(b) we derive implicitly

$$6xy + 3x^2y' + (1+y')e^{x+y} + (1+y')\frac{1}{x+y} - y'e^y = 0$$ 1

we set $x=0$ and $y=1$ 0.5

$$0+0+(1+y')e + (1+y')\cdot 1 - y'e = 0$$

$$(1+y') = -e$$

$$\boxed{y' = -e-1}$$ 0.5

(c) we use the formula for a line through a point with given slope

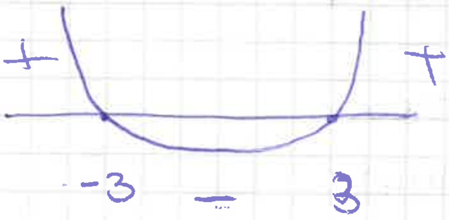
$$(y-1) = (-e-1)(x-0)$$ 0.5

$$\boxed{y = (-e-1)x+1}$$ 0.5

(3)

(a) $f'(x) = \dfrac{(2x)x - x^2 + 9}{x^2} = \dfrac{\cancel{2}\, x^2 - 9}{x^2}$ 0.5

$f'(x) = 0$ iff $x = \pm 3$ the denominator is positive $\Rightarrow$ sign = the sign of the denominator



0.5

	-3		0		3	
$f(x)$	$\nearrow$ ~~-8~~ $\searrow$		$\searrow$?	$\searrow$	$\searrow$ _ $\nearrow$	$\nearrow$
$f'(x)$	+ $\ominus$ −		− ?	−	− 0 +	+
	MAX				MIN	
	0,5				0,5	

There are two critical points $x = -3$ and $x = 3$

$x = -3$ is a local max

$x = 3$ is a local min.

0.5

(b) The function is increasing when $x \geq 3$ or $x \leq -3$ 0.5 and is decreasing otherwise.

For the concavity we compute $f''(x)$

$$f''(x) = \frac{(2x)x^2 - 2x(x^2 - 9)}{(x)^4} =$$ 0.5

$$= \frac{\cancel{2x^3} - \cancel{2x^3} + 18x}{(x)^4} = \frac{18}{x^3} > 0 \text{ iff } x > 0$$

The function is concave when $x < 0$ and convex when $x > 0$ 0.5

(c) In the interval we have only one critical point $x=3$ 0.5

The function is differentiable in the whole interval so the candidates for ~~max~~ max and min are 1, 3, and 4

$f(1) = 10$ max

$f(3) = \frac{18}{3} = 6$ min

$f(4) = \frac{25}{4} = 6.25$ 0.5

The max value is 10 & is taken at $x=1$

the min value is 6 $x=3$

(4)

(a) $\int \left(\frac{3}{\sqrt{t}} \ln(\sqrt{t}) + \frac{3}{t+1} \right) dt =$

0.5 for splitting
0.5 for 2nd inte
0.5 for C
0.5 for corett sub
0.5 for by parl
0.5 for resub

$= \int \frac{3}{\sqrt{t}} \ln(\sqrt{t})\, dt + \int \frac{3}{t+1}\, dt =$

$u = \sqrt{t}$ $\qquad v = t+1$

$= du = \frac{1}{2\sqrt{t}} dt$ $\qquad dv = dt$

$= \int 6 \ln(u)\, du + \int \frac{3}{v}\, dv$

$= 6 \left(u \ln(u) - \int u \frac{1}{u}\, du \right) + 3 \ln|v| + C_1$

$= 6 \left(u \ln(u) - u + C_2 \right) + 3 \ln|v| + C_1$

$= 6 \left(\sqrt{t} \ln(\sqrt{t}) - \sqrt{t} \right) + 3 \ln|t+1| + C$

(b) $\int_0^{+\infty} y e^{-y^2+1}\, dy =$

0.5 for the limit
0.5 for the sub
0.5 for extremes

$= \lim_{b \to +\infty} \int_0^b y e^{-y^2+1}\, dy$

$-y^2+1 = u$
$-2y\,dy = du$
$y = 0 \quad u = 1$
$y = b \quad u = -b^2+1$

$= -\frac{1}{2} \lim_{b \to +\infty} \int_1^{-b^2+1} e^{+u}\, du =$

$= -\frac{1}{2} \lim_{b \to +\infty} \left[e^u \right]_1^{-b^2+1} =$

0.5 for correct

$= -\frac{1}{2} \lim_{b \to +\infty} \left[\cancel{e^{-b^2+1}}^{\to 0} - e^1 \right] = \frac{e}{2}$

$$\det(A) = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 36 & c \\ 1 & c-2 & 6 \end{vmatrix} =$$

~~0.1~~ 1 for the method

~~0.5~~ 1 for correct answer.

$$= 36 \cdot 6 - c(c-2)$$

$$= -c^2 + 2c + 216$$

(b) $\det(A) = 0$ iff

$$c = \frac{-1 \pm \sqrt{1+216}}{-1} = \underline{+1 \pm \sqrt{217}}$$ 0.5

the matrix is invertible iff 0.5

$$c \neq 1 \pm \sqrt{217}$$

(c)

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 2 & 4 & 0 & 2 \\ 1 & 0 & 6 & 1 \end{array}\right)$$

1 pt for correct GE

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 6 & 1 \end{array}\right)$$

1 pt for correct answer

$$\sim \left(\begin{array}{ccc|c} 1 & \boxed{2} & 0 & 1 \\ 1 & 0 & \boxed{6} & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Variable = 3

of zero = zero rows 2

3-2 > 0

-> ∞ many solution

(6) $f(xy) = 5-3y+3x^2y+2(x^3+y^3)$

$$\frac{\partial f}{\partial x}(xy) = 6xy+6x^2 = 6x(y+x) = 0$$

$$\frac{\partial f}{\partial y}(xy) = -3+3x^2+6y^2 = 0 \qquad 0.5$$

from the first equation we got that

$$x=0 \quad \text{or} \quad x=-y$$

we plug in into the second equation

$x=0$ $\quad \frac{\partial f}{\partial y}(0y) = -3+6y^2 = -3(1-2y^2)$

$$= -3(1-\sqrt{2}y)(1+\sqrt{2}y)$$

we got $\quad y = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$ $\quad$ 0,5 for sol

we from the second equation we got

$x=-y$ $\quad \frac{\partial f}{\partial y}(x\ {-x}) = -3+3x^2+6x^2 = 0$

$$-3+9x^2 = 0$$

$$-3(1-3x^2)$$

$$x = \pm\frac{\sqrt{3}}{3} \qquad y = \mp\frac{\sqrt{3}}{3}$$

The critical points on

$$\left(0\ \frac{\sqrt{2}}{2}\right) \quad \left(0\ -\frac{\sqrt{2}}{2}\right) \quad \left(\frac{\sqrt{3}}{3},\ -\frac{\sqrt{3}}{3}\right)$$

$$\left(-\frac{\sqrt{3}}{3}\ \frac{\sqrt{3}}{3}\right) \qquad 0.5$$

To determine their type we consider the Hessian matrix

$$H = \begin{pmatrix} 6y + 12x & 6x \\ 6x & 12y \end{pmatrix}$$

$\partial_{xx} f = 6y + 12x$

$\partial_{yy} f = 12y$ 0.5

$\partial_{xy} f = 6x = \partial_{yx} f$

the determinant

$72 y^2 + 144xy - \cancel{6}\, 36 x^2$

we compute the value of $\frac{\partial}{\partial x} f(0, \pm\frac{\sqrt{2}}{2})$

$(0, \frac{\sqrt{2}}{2}) \rightsquigarrow 12\, 0 + 6 \frac{\sqrt{2}}{2} > 0$ MIN

$H(0 \pm \frac{\sqrt{2}}{2}) = 72 \frac{\cancel{2}}{\cancel{4}2} = 36 > 0$

$(0 - \frac{\sqrt{2}}{2}) \rightsquigarrow -12 \cdot 0 - 6 \frac{\sqrt{2}}{2} < 0$ max

MAX

$H(\frac{\sqrt{3}}{3} - \frac{\sqrt{3}}{3}) = 72 \frac{\cancel{3}^1}{\cancel{9}_3} + 144 \frac{\cancel{3}^1}{\cancel{9}_3} - 36 \frac{\cancel{3}^1}{\cancel{9}_3}$

$= 24 - 48 - 12 = -36$ SADDLE

$H(-\frac{\sqrt{3}}{3} \ \frac{\sqrt{3}}{3}) = 72 \frac{1}{3} - 144 \frac{1}{3} - 36 \frac{1}{3} = -36$

SADDLE

0.5

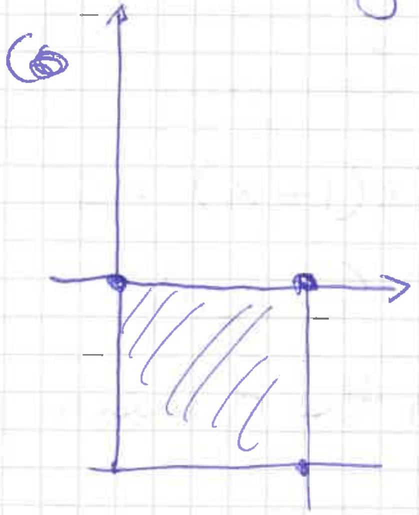
conclusion: there are 4 critical points

$\left(0 \ \frac{\sqrt{2}}{2}\right)$ is a local min

$\left(0 \ -\frac{\sqrt{2}}{2}\right)$ is a local max

$\left(\frac{\sqrt{3}}{3} \ -\frac{\sqrt{3}}{3}\right)$ & $\left(-\frac{\sqrt{3}}{3} \ \frac{\sqrt{3}}{3}\right)$ are saddle pt

(b) the boundary of the square are



0.5 for correct side

A $(x \ 0)$ $x \in [0,1]$

B $(0 \ y)$ $y \in [-1,0]$

C $(+1 \ y)$ $y \in [-1 \ 0]$

D $(x \ -1)$ $x \in [0,1]$

Candidates on the sides

1 for critical pt in the side

A: $f(x,0) = 5 + 2x^3 =: g(x)$

$\frac{d}{dx} g(x) = 6x^2 = 0$ iff $x = 0$

we got the candidate $(0 \ 0)$ which is the corner ⇝ does not count

B $f(0 \ y) = 5 - 3y + 2y^3 =: g(y)$

$\frac{d}{dy} g(y) = -3 + 6y^2$ $\quad y = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{2}}{2}$

⇝ only $-\frac{\sqrt{2}}{2}$ is ok

$\boxed{\left(0, +\frac{\sqrt{2}}{2}\right)}$

c) $f(+1,y) = 5 - \cancel{3y} + \cancel{3y} + 2 + 2y^3$

$= 7 + 2y^3 =: g(y)$

$\frac{d}{dy} g(y) = 6y^2 = 0 \quad \text{iff} \quad y = 0$

$\rightsquigarrow$ corner does not count

D) $f(x,-1) = \cancel{5+3} = 5+3 - 3x^2 + 2x^3 - 2$

$= 6 - 3x^2 + 2x^3 =: g(x)$

$\frac{d}{dx} g(x) = -6x + 6x^2 = 6x(1+x) = 0$

iff $x = 0$ (corner)

$x = -1$ not in the domain

Summary: the candidates are the corners

$(0\ 0)$	$f(0\ 0) = \boxed{5}$
$(0 -1)$	$f(0,1) = 5+3-2 = 6$
$(1\ 0)$	$f(1\ 0) = 5+2 = 7$
$(1,-1)$	$f(1-1) = 5+3-3 = \boxed{5}$
$\left(0\ -\frac{\sqrt{2}}{2}\right)$	$f\left(0\ -\frac{\sqrt{2}}{2}\right) = 5 + \frac{3\sqrt{2}}{2} + \frac{2 \cdot \cancel{2}\sqrt{2}}{\cancel{4} \cdot 2}$

0.5

$= 5 + 2\sqrt{2} > 7$

$\approx \boxed{7.8}$ MAX

c) Of all critical points the only one lying on the INTERIOR OF THE area is $\left(\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\right)$

$$f\left(\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\right) = 5 + \frac{3\sqrt{3}}{3} + \frac{3\cdot 3}{9}\frac{\sqrt{3}}{3} + 2\left(\cancel{\left(\frac{\sqrt{3}}{3}\right)^3 + \left(\frac{-\sqrt{3}}{3}\right)^3}\right)$$

$$= 5 + \frac{4}{3}\sqrt{3} = 7.30\ldots$$

The max is 7.8 taken in $\left(0\ -\frac{\sqrt{2}}{2}\right)$

the min is 5 taken in $(0,0)$ and $(1,-1)$

0.5 for correct candidate
0.5 for correct min / max