Solution to the exam 20230815

(1) $f(0)=1$

$f'(x)=2xe^{x^2}$ $\qquad f'(0)=0$

$f''(x)=2e^{x^2}+4xe^{x^2}$ $\qquad f''(0)=2$

$f'''(x)=4xe^{x^2}+8x^2e^{x^2}$ $\qquad f'''(0)=0$

we have that ~~$f(x)=e^{x^2}$~~

$$p(x)=1+x^2$$

$$p(0.1)=1+\frac{1}{100}=\frac{101}{100}=1.01$$

(2) (a)

After one year $(K-T)\left(1+\frac{P}{100}\right)$

After two years $\left((K-T)\left(1+\frac{P}{100}\right)-Y\right)\left(1+\frac{P}{100}\right)$

$$=(K-T)\left(1+\frac{P}{100}\right)^2-Y\left(1+\frac{P}{100}\right)$$

(b) $$(670000-1000)\left(1+\frac{11.5}{100}\right)^{12}-\sum_{i=1}^{12}Y\left(1+\frac{P}{100}\right)^{i}$$

$$=(669000)\left(1+\frac{11.5}{100}\right)^{12}-70000\left(1+\frac{11.5}{100}\right)\left(\sum_{i=0}^{11}\left(1+\frac{11.5}{100}\right)\right.$$

$$=(669000)\left(1+\frac{11.5}{100}\right)^{12}-700000\left(1+\frac{11.5}{100}\right)\cdot\left(\frac{\left(1+\frac{11.5}{100}\right)^{12}-1}{11.5}\right)$$

(you can leave this like that)

(c) $(K-T)\left(1+\frac{p}{100}\right)^{12} - \varphi\left(1+\frac{p}{100}\right)\left(\frac{\left(1+\frac{p}{100}\right)^{12} - \cancel{\left(1+\frac{p}{100}\right)}}{\frac{p}{100}}\right)$

(3) $f(x) = x^3 + 3x + 2$

$f'(x) = 3x^2 + 3 = 3(x^2+1) > 0$

the function is always increasing and has no critical point

$f''(x) = 6x \geq 0$ iff $x \geq 0$

So the function is concave when $x < 0$ and convex otherwise.

(4) $\int \frac{t}{t+5} + t^3 \, dt$

$= \int \frac{t}{t+5} \, dt + \int t^3 dt$

$t+5 = u$

$= \int \frac{u-5}{u} \, du + \int \frac{1}{4} t^4 + C_1$

$= \int \frac{u}{u} \, du - \int \frac{+5}{u} \, du + \frac{1}{4} t^4 + C_1$

$= u - 5\ln|u| + \frac{1}{4} t^4 + C_1$

$= (t+5) - 5\ln|t+5| + \frac{1}{4} t^4 + C$

$\int_0^1 (2x+1)^5 dx =$ $\quad$ $u = 2x+1$, $du = 2dx$

$$= \frac{1}{2}\int_1^3 (u)^5 \, du = \left[\frac{1}{2}\frac{1}{6}u^6\right]_1^3$$

$$= \left[\frac{1}{12}u^6\right]_1^3 = \frac{1}{12}3^6 - \frac{1}{12}$$

(5)

$$|A| = \begin{vmatrix} 1 & -1 & 3 \\ 0 & k+3 & k+3 \\ 0 & 1 & k \end{vmatrix}$$

$$= k(k+3) - (k+3) = (k+3)(k-1)$$

(b) $\det A = 0$ iff $k=-3$ or $k=1$
Thus A is not invertible if $k=-3, 1$

(c) For $k=0$ we know that A is invertible so the system has a unique solution

(6) (a)

$$\begin{cases} \frac{\partial f}{\partial x}(xy) = 6x^2 - 2y = 0 \\ \frac{\partial f}{\partial y}(xy) = -2x + 2y = 0 \end{cases}$$

$x = y$ so $6x^2 - 2\cancel{y}x = 0$ $x(6x-2)$
for $x = \frac{1}{3}$ $y = \frac{1}{3}$ and we get the point $(1/3\ 1/3$
we get that $x=0$ $(xy)=(00)$

$$\det H(xy) = \begin{pmatrix} 12x & -2 \\ -2 & 2 \end{pmatrix} = 24x - 4$$

$x=0$ $-4 < 0$

$x = \frac{1}{3}$ $4 > 0$

$(0;0)$ is a saddle pt

$(1/3, 1/3)$ is a local min since $12\frac{1}{3}, 2 > 0$

(2)

<u>Side 1</u> $x=0 \quad y\in[0,1]$

$f(0,y) = y^2+7 = g(y)$

$g'(y) = 2y = 0$ when $y=0$

<u>Side 2</u> $y=0 \quad x\in[0,1]$

$f(x,0) = 2x^3+7 = g(x)$

$g'(x) = 6x^2+7 > 0$

<u>Side 3</u> $y=1-x \quad x\in[0,1]$

$f(x,1-x) = 2x^3 - 2x(1-x) + (1-x)^2+7$

$= 2x^3 - 2x + 2x^2 + 1 - 2x + x^2 + 7$

$= 2x^3 + 3x^2 - 4x + 7 = g(x)$

$g'(x) = 6x^2+6x-4 = 2(3x^2+3x-2) = 0 \iff 3x^2+3x-2=0$

$x_\pm = \dfrac{-3\pm\sqrt{9+24}}{6} = \dfrac{-3\pm\sqrt{23}}{6}$ of these $\dfrac{-3-\sqrt{33}}{6} < 0$ while $\dfrac{-3+\sqrt{33}}{6} \in [0,1]$

$g(0) = 7$, $g(1) = 9$ MAX $\quad g\left(\dfrac{-3+\sqrt{33}}{6}\right) \approx 5.99$ MIN

<u>Conclusion</u>

$f(0,0)=7$

$f(0,1)=8$

$f(1,0)=9$ MAX

$f\left(\dfrac{-3+\sqrt{33}}{6}, \dfrac{3+\sqrt{33}}{6}\right) \approx 5.99$ MIN

(c) We have to compare the previous values with

$f(1/3, 1/3) = 7 - \frac{1}{27}$ <9 >6

So the min and max of the function are the same as the one on ∂D: 9 MAX, 5.99 MIN