Solution to MM3001 Exam 17/04/24

Exercise 1

$f(x) = e^{-x^2+1}$ $\quad x_0 = 1$ $\quad f(1) = e^0 = 1$

$f'(x) = -2x e^{-x^2+1}$ $\quad f'(1) = -2$

$f''(x) = -2e^{-x^2+1} + 4x^2 e^{-x^2+1}$ $\quad f''(1) = -2+4 = 2$

$f'''(x) = +4x e^{-x^2+1} + 8x e^{-x^2+1} - 8x^3 e^{-x^2+1}$ $\quad f'''(1) = 4$

$$p(x) = 1 - 2(x-1) + \frac{2}{2}(x-1)^2 + \frac{4}{6}(x-1)^3$$

$$= 1 - 2(x-1) + (x-1)^2 + \frac{2}{3}(x-1)^3$$

$$p(1.1) = 1 - \frac{2}{10} + \frac{1}{100} + \frac{2}{3}\frac{1}{1000} = \frac{3000-600+30+2}{3000} = \frac{2432}{3000}$$

Exercise 2

(a) $$\sum_{k=4}^{90} (1.02)^k = (1.02)^4 \sum_{k=0}^{86} (1.02)^k$$

$$= (1.02)^4 \frac{1-(1.02)^{87}}{0.02}$$ it is ok to leave it like this.

(b) 5 years = 60 monthly payments

4.8 yearly interest $= \frac{4.8}{12} = 0.4$ monthly $\quad \frac{0.4}{100} = 0.004$

first payment 600 is compounded 59 times

second 600 ——— 58 times and so on

$$\sum_{k=0}^{59} 600(1.004)^k = 600 \frac{1-(1.004)^{60}}{\frac{4}{1000}} = 600 \cdot 250 \left(1-(1.004)^{60}\right)$$

$$= 150000\left(1-(1.004)^{60}\right)$$

$$= 40596,11$$

(c) Initial Capital 40596,11 $\quad$ 5.6 yearly interest

$X =$ yearly payment

After one year $\quad 40596.11(1.056) - X$

After two years $\quad 40596.1(1.056)^2 - X(1.056) - X$

At the last payment (after 10 years)

$$40596.11\,(1.056)^{10} - \sum_{k=0}^{9} x\,(1.056)^{k} = 0$$

$$40596.11\,(1.056)^{10} = x\left(\sum_{k=0}^{9}(1.056)^{k}\right) = x\,\frac{1-(1.056)^{10}}{0.056}$$

$$x = \frac{40596.11\,(1.056)^{10}}{1-(1.0056)^{10}} * 0.056$$ can be left like that

<u>Exercise 3</u>

$$f(x) = x\cdot e^{-100x}$$

(a) $$f'(x) = e^{-100x} - 100x\,e^{-100x} = \underset{\overset{\vee}{0}}{e^{-100x}}\left(1-100x\right)$$

$f'(x) \geq 0 \iff 1-100x \geq 0 \qquad x \leq \frac{1}{100}$

		$1/100$	
$f'(x)$	$+$	0	$-$
$f(x)$	$\nearrow$	$-$	$\searrow$

local max

$f(x)$ has a local max when $x = \frac{1}{100}$ it is increasing when $x < \frac{1}{100}$ and decreasing otherwise.

(c) $$f''(x) = -100\,e^{-100x}(1-100x) + e^{-100x}(-100)$$
$$= -100\,e^{-100x}(1-100x+1) = \underset{\overset{\wedge}{0}}{-100}\;\underset{\overset{\vee}{0}}{e^{-100x}}(2-100x)$$

$f''(x) > 0 \iff 2-100x < 0 \qquad 100x > 2 \qquad x > \frac{1}{50}$

The function is concave when $x < \frac{1}{50}$ and convex when $x > \frac{1}{50}$

(d) $$\lim_{x\to-\infty} x\,e^{-100x} = -\infty$$
$$\lim_{x\to+\infty} x\,e^{-100x} = \lim_{x\to+\infty}\frac{x}{e^{100x}} \overset{H}{=} \lim_{x\to+\infty}\frac{1}{100e^{x}} = 0$$

Exercise 4

(a) $$\int e^{-t+1} + 6(2t+3)^5 \, dt = \int e^{-t+1} \, dt + 6\int (2t+3)^5 \, dt$$

$u = 2t+3 \qquad dt = \frac{du}{2}$

$$= -e^{-t+1} + 3\int u^5 \, du + c_1$$

$$= e^{-t+1} + \frac{3}{6} u^6 + \underbrace{c_1 + c_2}_{C}$$

$$= e^{-t+1} + \frac{1}{2}(2t+3)^6 + C$$

(b) $$\int_1^e \ln(x^2)\, x \, dx = \int_1^e 2\ln(x)\cdot x \, dx$$

$$= 2\left[\frac{1}{2}x^2 \ln(x)\right]_1^e - \cancel{2}\int_1^e \frac{1}{\cancel{2}} x^{\cancel{2}} \frac{1}{\cancel{x}} \, dx$$

$$= e^2 \ln(e) - 1^2 \ln(1) - \left[\frac{1}{2}x^2\right]_1^e$$

$$= e^2 - \frac{1}{2}e^2 + \frac{1}{2} = \frac{1}{2}\left(e^2+1\right)$$

Exercise 5

(a) $\det\begin{pmatrix} 1 & 3c & 0 \\ 0 & 1 & -3c \\ 1 & 0 & 1 \end{pmatrix} = \det\begin{pmatrix} 1 & 3c & 0 \\ 0 & 1 & -3c \\ 0 & -3c & 1 \end{pmatrix}$

$= (1 - 9c^2) = (1-3c)(1+3c)$

(b) $\det A = 0 \iff c = \pm\frac{1}{3} \Rightarrow A$ is invertible $\iff$

$c \neq \pm\frac{1}{3}$

(c) $\left(\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 1 & 3 & 1 \\ 1 & 0 & 1 & 4 \\ 2 & 4 & -2 & 4 \end{array}\right) \overset{R_4 \leftrightarrow R_1}{\sim} \left(\begin{array}{ccc|c} 2 & 4 & -2 & 4 \\ 1 & 3 & 0 & 2 \\ 0 & 1 & 3 & 1 \\ 1 & 0 & 1 & 4 \end{array}\right)$

$R_1 \mapsto \frac{1}{2}R_1$

$\sim \left(\begin{array}{ccc|c} \boxed{1} & 2 & -1 & 2 \\ 1 & 3 & 0 & 2 \\ 0 & 1 & 3 & 1 \\ 1 & 0 & 1 & 4 \end{array}\right) \underset{\substack{R_2 \to R_2 - R_1 \\ R_4 \to R_4 - R_1}}{\sim} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & -2 & 2 & 2 \end{array}\right)$

$R_3 \to R_3 - R_2$
$R_4 \to R_4 + 2R_2$

$\sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & \boxed{2} & 1 \\ 0 & 0 & 4 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$

only one solution $\quad z = \frac{1}{2} \quad y = -\frac{1}{2} \quad x + 2y - z = 2$

$x - 1 - \frac{1}{2} = 2$

$x = 2 + \frac{3}{2} = \frac{7}{2}$

Solution $(7/2, -1/2, 1/2)$

Exercise 6 $\quad f(x,y) = xye^{-xy}$

(a) $\quad \frac{\partial}{\partial x} f(xy) = ye^{-xy} - xy^2 e^{-xy} = ye^{-xy}(1-xy) = 0$

$\frac{\partial}{\partial y} f(xy) = xe^{-xy} - x^2y\, e^{-xy} = xe^{-xy}(1-xy)$

if $y=0$ then $x \cdot e^{-xy} \cdot 1 = 0 \Rightarrow x = 0$

if $y \neq 0$ then $1-xy=0$ so $xy=1$ $\quad x = \frac{1}{y}$ and also $\frac{\partial}{\partial y} f(xy) = 0$

So the origin and $(1/y, y)$ with $y \neq 0$ are all critical poi.

$$H(xy) = \begin{pmatrix} -y^2e^{-xy}(1-xy) - y^2e^{-xy} & e^{-xy}(1-xy) - xye^{-xy}(1-xy) - xye^{-xy} \\ e^{-xy}(1-xy - xy(1-xy) - xy) & -x^2e^{-xy}(1-xy) - x^2e^{-xy} \end{pmatrix}$$

$$= \begin{pmatrix} -y^2e^{-xy}(2-xy) & e^{-xy}(1-3xy+x^2y^2) \\ e^{-xy}(1-3xy+x^2y^2) & -x^2e^{-xy}(2-xy) \end{pmatrix}$$

$$= x^2y^2e^{-2xy}(2-xy)^2 - e^{-2xy}(1-3xy+x^2y^2)^2$$

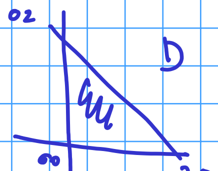
$$= e^{-2xy}\left(x^2y^2(2-xy)^2 - (1-3xy+x^2y^2)^2\right)$$

if $(xy) = (0\ 0)$ this is $e^0(0-1^2) = -1 < 0$ SADDLE

if $(xy) = (a\ 1/a)$ $\quad e^{-2}\left(1(2-1)^2 - (1-3+1)^2\right) = 0$

The Hessian does not provide us any information

(b)



Side 1 $\quad x=0 \quad y=[0,2]$

$g(y) = f(0\ y) \equiv 0$

Side 2 $\quad y=0 \quad x \in [0,2]$

$f(x\,0) = 0$

Side 3 $\quad y = 2-x \quad x \in [0,2]$

$$g(x) = f(x, 2-x) = x(2-x)\,e^{-x(2-x)} = (2x-x^2)\,e^{-(2x-x^2)}$$

$$g'(x) = (2-2x)\,e^{-2x-x^2} - (2x-x^2)(2-2x)e^{-2x-x^2}$$

$$= (2-2x)e^{-2x-x^2}\left(1-2x+x^2\right) = 0$$

$$2e^{-2x-x^2}(1-x)(1-x)^2$$ we have 0 in $x=1$

(so $y = 2-1 = 1$)

In the extremes g is 0

$g(1) = 1\cdot 1\cdot e > 0$

The minimum value of f on the boundary of D is 0 taken in $x=0$ or $y=0$

the maximum value of f on D is e

(c) we have to test f in $(a\,\frac{1}{a})$ $\quad a \neq 0 \quad 0 \le \frac{1}{a} + a \le 2$

$$0 \le \frac{1+a^2}{a} \le 2$$

$0 < a$

$1+a^2-2a \le 0 \quad a \in (0,1)$

$f(a\,\frac{1}{a}) = 1\cdot e^{-1} = f(1,1)$ max

The minimum value of f in D is 0 the maximum is e^{-1}.