Solution to MM3001 August 20th 2024

Exercise 1

$f(x) = \ln(1+x^3)$ $\quad f(0) = \ln(1+0) = 0$

$f'(x) = 3x^2 \frac{1}{1+x^3}$ $\quad f'(0) = 3\cdot 0 \frac{1}{1+0} = 0$

$f''(x) = \frac{6x(1+x^3) - 3x^2 \cdot 3x^2}{(1+x^3)^2}$ $\quad f''(0) = 0$

$= \frac{6x - 3x^4}{(1+x^3)^2}$

$$f'''(x) = \frac{(6-12x^3)(1+x^3)^2 - 3x^2 \cdot 2(1+x^3)(6x-3x^4)}{(1+x^3)^4}$$

$f'''(0) = \frac{6\cdot 1 - 0}{(1+0)^4} = 6$

$p(x) = 0 + 0(x-0) + \frac{0}{2}(x-0)^2 + \frac{6}{3!}(x-0)^3$

$= x^3$

$f(0.1) \approx (0.1)^3 = 0.001$

Exercise 2

$f(x,y) = x^3 y + y^3 x$

(a) $C = f(1,1) = 1^3 \cdot 1 + 1 \cdot 1^3 = 2$

(b) We derive implicitly considering $y = y(x)$

$$3x^2 \cdot y(x) + x^3 y'(x) \quad + 3y(x)^2 y'(x) \cdot x + y(x)^3 \cdot 1 = 0$$

we set $x = 1$ $\quad y(1) = 1$

$$3 \cdot 1^2 \cdot 1 + \underline{1^3 \cdot y'(1)} + \underline{3 \cdot 1^2} \cdot y'(1) \cdot 1 + 1 \cdot 1 = 0$$

$$4y'(1) = -4 \qquad\qquad y'(1) = -1$$

(c) We have that

$$t: \; y = mx + k$$

with $m = -1$ and $1 = m + k$

$$\Rightarrow \; k = 2$$

So the equation is $\quad y = -x + 2$

<u>Exercise 3</u>

(a) $\quad f(x) = \ln(x^5 - 50000x)$

$$f'(x) = \frac{5x^4 - 50000}{x^5 - 50000x} = \frac{5(x^4 - 10000)}{x^5 - 50000x}$$

$$= \frac{5(x^2 - 100)(x^2 + 100)}{x^5 - 50000x} = \frac{5(x-10)(x+10)(x^2+100)}{x(x^4 - 50000)}$$

$$= \frac{5(x-10)(x+10)(x^2+100)^{>0}}{x(x - \sqrt[4]{5} \cdot 10)(x + \sqrt[4]{5} \cdot 10)(x^2 + \sqrt{5} \cdot 100)^{>0}}$$

we do a table sign to see where this function is positive or negative

Denominator		$-\sqrt[4]{5}\cdot 10$		0		$\sqrt[4]{5}\cdot 10$	
x	−	−	−	0	+	+	+
$x-\sqrt[4]{5}\cdot 10$	−	−	−	−	−	0	+
$x+\sqrt[4]{5}\cdot 10$	−	0	+	+	+	+	+
$x^5-5000x$	−	0	+	0	−	0	+

Note: the function is not def for $x \leq -\sqrt{5}\cdot 10$
and for $x \in [0, \sqrt{5}\cdot 10]$

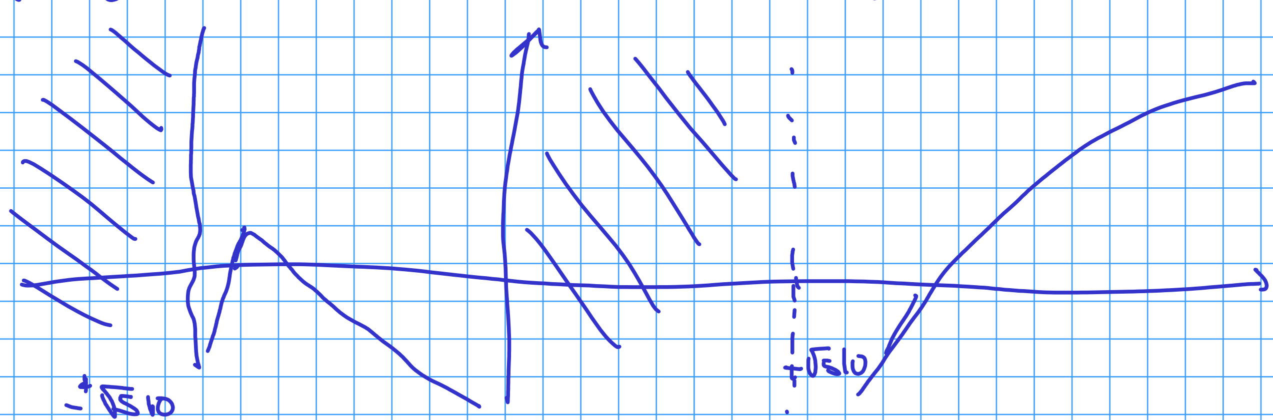
The derivative:

		$-\sqrt[4]{5}\cdot 10$		-10		0		$+10$		$\sqrt[4]{5}\cdot 10$	
$(x-10)$	−	−	−	−	−	−	−	0	+	+	+
$(x+10)$	−	−	−	0	+	+	+	+	+	+	+
denominator	−	0	+	+	+	0	−	−	−	0	+
derivative	−	?	+	0	−	?	+	0	−	?	+
function	not def	|	↗	MAX	↘	|	Not def			|	↗

There is only one critical point in $x = -10$ and it is a local MAX

(c) The function is not defined in $[9, 10]$ so there is no max or min

(d) $\lim_{x\to+\infty} \ln(x^5 - 5000x) = \ln\left(\lim_{x\to+\infty} x^5 - 5000x\right) = +\infty$



Exercise 4

(a) $\int t^3 e^{-t^2} + 4t^{-7/5} \, dt =$

$$= \int t^3 e^{-t^2} dt + 4\int t^{-7/5} dt$$

$u = -t^2$

$du = -2t \, dt$

$$= \int -u \cdot \cancel{t} \, e^{u} \frac{du}{-2\cancel{t}} + 4 \frac{1}{-\frac{7}{5}+\frac{5}{5}} t^{-7/5+1} + C_1$$

$$= \frac{1}{2}\left(e^{u} \cdot u - \int e^{u} \cdot \frac{d}{du} u \, du \right) - \cancel{4}^{2} \cdot \frac{5}{\cancel{2}} t^{-2/5} + C_1$$

$$= \frac{1}{2} e^{-t^2} (-t^2) - \frac{1}{2} e^{-t^2} - 10 t^{-2/5} + C$$

(b) $\int_0^1 \ln(x^4+1) \cdot x^3 \, dx$ $\quad u = x^4+1 \quad$ $x=0 \quad u=1$; $x=1 \quad u=2$

$du = 4x^3 dx$

$$= \frac{1}{4} \int_1^2 \ln(u) \, du$$

$$= \frac{1}{4} \left[u \cdot \ln(u) \right]_1^2 - \frac{1}{4} \int_1^2 \cancel{u} \frac{1}{\cancel{u}} du =$$

$$= \frac{1}{4} 2 \cdot \ln(2) - \frac{1}{4} \left[u \right]_1^2 =$$

$$= \frac{1}{2} \ln(2) - \frac{1}{4} \cdot 2 + \frac{1}{4} = \frac{1}{2} \ln(2) - \frac{1}{4}$$

Exercise 5

(a) $\det(A) = \begin{vmatrix} -8 & c & 2 \\ -4 & 0 & 1 \\ c & 6 & 1 \end{vmatrix} = \begin{vmatrix} 0 & c & 2 \\ \cancel{0} & \cancel{0} & \cancel{1} \\ c+4 & 6 & 1 \end{vmatrix} =$

$= -1\left(0 \cdot 6 - (c+4)\cdot c\right)$

$= c\,(c+4)$

(b) A is invertible iff $\det A \neq 0$. iff $c \neq 0, 2$

(c) We use GJ elimination

$$\left(\begin{array}{ccc|c} -8 & -4 & \boxed{2} & 2 \\ -4 & 0 & 1 & 1 \\ -4 & 6 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array}\right) \sim \qquad R_i - \frac{1}{2} R_1$$

$$\left(\begin{array}{ccc|c} -8 & -4 & 2 & 2 \\ 0 & 2 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 5 & 3 & 0 & 0 \end{array}\right) \sim \qquad R_2 \leftrightarrow R_4$$

$$\left(\begin{array}{ccc|c} -8 & -4 & 2 & 2 \\ 5 & 3 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right) \sim \qquad R_4 - \frac{1}{4} R_3$$

$$\left(\begin{array}{ccc|c} -8 & -4 & 2 & 2 \\ 5 & 3 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We have 3 variables and $3 \neq 0$ rows

$\Rightarrow$ 1 solution

$8y = 0 \quad \Rightarrow y = 0$

$5x + 3y = 0 \quad \Rightarrow x = 0$

$-8x - 4y + 2z = 2 \quad \Rightarrow z = 1$

<u>Exercise 6</u>

(a) $\frac{\partial f}{\partial x} = -4xy = 0 \qquad \frac{\partial f}{\partial y} = 2y - 2x^2 - 1 = 0$

If $x = 0$ then $2y = 1 \qquad y = \frac{1}{2}$

If $y = 0$ then $-2x^2 - 1 = 0 \qquad x^2 = -\frac{1}{2}$ no sol.

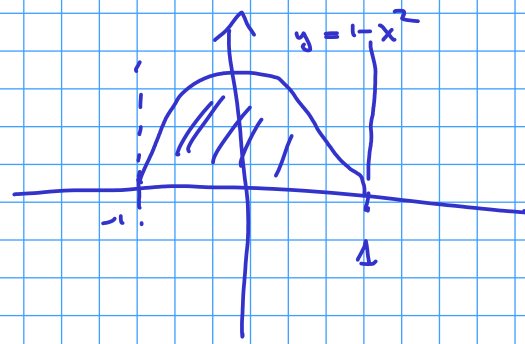
There is only 1 critical point $(0 \ \frac{1}{2})$

$$H(xy) = \det \begin{pmatrix} -4y & -4x \\ -4x & 2 \end{pmatrix} = -8y - 16x^2$$

$H(0 \ \frac{1}{2}) = -8 \frac{1}{2} = -4 < 0 \qquad$ SADDLE

(b)

D



The boundary has 2 sides $\quad y=0 \quad x\in[-1,1]$

$x^2=1-y \quad y\in[0,1]$

SIDE 1

$g(x)=f(x\ 0)=200$ for every x

SIDE 2

$$g(y)=f(1-y,y)=y^2-2(1-y)\cdot y-y+200$$

$$=y^2-2y+2y^2-y+200$$

$$=3y^2-3y+200$$

$$g'(y)=+6y-3 \qquad =0 \quad \text{for} \quad y=\frac{1}{2}$$

$$g(0)=200 \qquad g(1)=200 \qquad g\left(\frac{1}{2}\right)=3/4-\frac{3}{2}+200$$

$$=200-\frac{3}{4}=199.25$$

$\underbrace{g(0)=200 \qquad g(1)=200}_{\text{MAX}}$ $\qquad$ MIN

The min value of f on the boundary of D is

200.

the max value is 200.25

(c) the point $(0\ \frac{1}{2})\in D$ so we need to compare $f(0\ \frac{1}{2})$ with what we got in (b)

$f(0, \frac{1}{2}) = \frac{1}{4} - 0 - \frac{1}{2} + 200 = 199.75$ neither min nor max

$\Rightarrow$ In D the max value for f is 200 and the min is -199.25