Exam 2024-10-28 MM3001

(1) $f(x)=\sqrt{1+x^2}$ $\qquad f(0)=1$ 0.5

$$f'(x)=\frac{2x}{2\sqrt{1+x^2}}=\frac{x}{\sqrt{1+x^2}} \quad 0.5 \qquad f'(0)=0 \quad 0.5$$

$$f''(x)=\frac{\sqrt{1+x^2}-x\frac{2x}{2\sqrt{1+x^2}}}{1+x^2}=$$

$$=\frac{\frac{1+\cancel{x^2}-\cancel{x^2}}{\sqrt{1+x^2}}}{1+x^2}=\frac{1}{(1+x^2)^{3/2}} \quad 0.5 \qquad f''(0)=1 \quad 0.5$$

$$f'''(0)=2x\left(-\frac{3}{2}\right)(1+x^2)^{-5/2} \quad 0.5 \qquad f'''(0)=0 \quad 0.5$$

$$\boxed{p(x)=1+\frac{1}{2}x^2} \quad 1$$

$$f(0.1)\approx p(0.1)=1+\frac{1}{2}\,0.001=\boxed{\frac{201}{200}} \quad 0.5$$

ok but leave it like that

(2)

(a) we plug in $x=0$ $y=3$ 0.5

$$27+0=C \qquad \boxed{C=27} \quad 0.5$$

(b) $y'(x)3y(x)^2+y(x)+xy'(x)=0$ 1 $\qquad$ we let $x=0$ $y(0)=3$ 1

$$y'(0)\cdot 27+3=0 \qquad \boxed{y'(0)=\frac{-3}{27}=-\frac{1}{9}} \quad 1$$

(c) the line has equation

$$y = -\frac{1}{9}x + k$$ we know

that it goes through $(0\,3)$ so $3 = -\frac{1}{9}\,0 + k$

$= k$

the equation is

$$\boxed{y = -\frac{1}{9}x + 3}$$

(8)

(a) $f(x) = 5000\left(x + \frac{100}{x}\right)$ is defined when

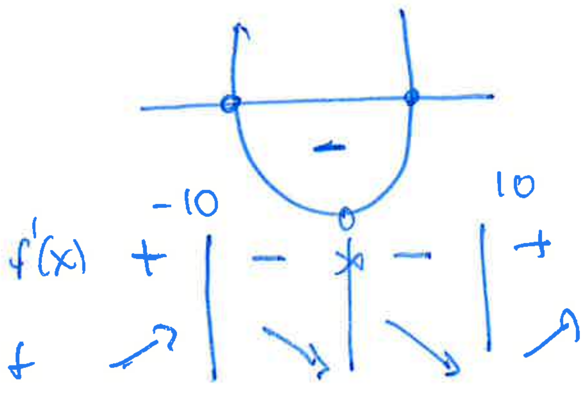
$x \in \mathbb{R} \setminus \{0\}$

(b) $f'(x) = 5000\left(1 - \frac{100}{x^2}\right) \geq 0$

$\iff \quad 1 - \frac{100}{x^2} \geq 0$

$$\frac{x^2 - 100}{x^2} \geq 0$$ since $x \neq 0$, $x^2 > 0$

$x^2 - 100 \geq 0$



$x \geq 10$

$x \leq -10$

There are two critical pt

$x = -10$ is a local max

$x = 10$ is a local min

The function is increasing when $x \geq +10$ or $x \leq -10$
and ~~decreasising~~ decreasing otherwise

(c) $f''(x) = 5000\left(-2\frac{100}{x^3}\right) = -\frac{1000000}{x^3} > 0$

when $x < 0$

The function is convex when $x > 0$ and
concave when $x < 0$

(d) $f(-11) = -5000\left(\frac{121 + 100}{11}\right)$

$= -\frac{5000}{11} 221 \approx -100454.545455$

$f(-9) = -5000\left(\frac{+81 + 100}{9}\right) = -\frac{5000}{9} 181$
≈ -100555.5555556 lowest

$f(-10) = -5000\left(\frac{200}{10}\right) = -500(200)$

$= -100\,000$ bigger

Answer the max value on $[-11, -9]$ is
$-100\,000$
while the ~~biggest~~ ~~is~~ min value is
≈ -100555.5555556

(4)

(a) $\int \frac{8x^2}{(x^3+2)^{100}} + \sqrt[8]{e^x}\, dx =$

$= \int \frac{8x^2}{(x^3+2)^{100}} dx + \int \sqrt[8]{e^x}\, dx$ 0.5

$u = x^3+2$
$du = 3x^2 dx$
$= \quad dx = \frac{du}{3x^2}$ 0.5

$= \int \frac{8\cancel{x^2}}{u^{100}} \frac{du}{3\cancel{x^2}} + \int e^{\frac{x}{8}} dx$

0.5

$= \frac{8}{3} \int u^{-100} du + 8e^{\frac{x}{8}} + C_1$

$= \frac{8}{3} \frac{1}{-99} u^{-99} + C_2 + 8e^{x/8} + C_1$

$= \frac{8}{-3(99)} (x^3+2)^{-99} + 8e^{x/8} + \boxed{C}$ 0.5

0.5

(b) $\int_0^1 x \ln(x)\, dx =$

0.5

$= \lim_{a \to 0^+} \int_a^1 x \ln(x)\, dx$

0.5

$= \lim_{a \to 0^+} \left[\frac{1}{2}x^2 \ln(x)\right]_a^1 - \int_a^1 \frac{1}{2}x^2 \frac{1}{x}\, dx$

$$= \lim_{a \to 0^+} \left(\frac{1}{2} a^2 \ln(a) - \left[\frac{1}{4} x^2 \right]_a^1 \right)$$

0.5

$$= \lim_{a \to 0^+} \frac{1}{2} a^2 \ln(a) - \frac{1}{4} a^2 + \frac{1}{4}$$

0.5

$$= \frac{1}{4} + \lim_{a \to 0^+} \frac{1}{2} a^2 \ln(a)$$

$$\frac{1}{2} a^2 \ln(a) = \frac{1}{2} \frac{\ln(a)}{a^{-2}} \xrightarrow[a \to 0]{} \text{"} \frac{-\infty}{+\infty} \text{"}$$

0.5

we can use l'Hopital

$$\lim_{a \to 0^+} \frac{1}{2} \frac{\ln(a)}{a^{-2}} = \lim_{a \to 0^+} \frac{1}{2} \frac{\frac{1}{a}}{-2a^{-3}}$$

$$= \lim_{a \to 0^+} - a^3 \frac{1}{a} = 0$$

thus the integral converges and has value $\frac{1}{4}$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ c & 1 & 1 \\ 1 & 1 & c \end{pmatrix}$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ c-1 & 0 & 0 \\ 0 & 0 & c-1 \end{vmatrix} = (-1)(c-1)^2 = 0$$

iff $c = 1$

(b) ~~A~~ A is invertible when $c \neq 1$

(c)

$$\left(\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 3 & 3 & 5 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & \boxed{1} & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 2 & 4 \end{array}\right) \quad \begin{array}{l} R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{array}{l} R_3 \quad R_3 - R_2 \\ R_4 \quad R_4 - 2R_2 \end{array}$$

I have 3 variables 2 non zero rows

$3 - 2 > 0 \Rightarrow \infty$ many solution

~~I can~~

II $y = 2 - z$

$x + y + z = 1$

$x + 2 - \cancel{z} + \cancel{z} = 1$

$x = -1$

the solutions are of the form

$(-1, 2-z, z)$ for $z \in \mathbb{R}$

$z=0 \rightsquigarrow (-1, 2, 0)$

$z=1 \rightsquigarrow (-1, 1, 1)$

(6) $f(xy) = \ln(1+x^2+y^2)$

(a) $\partial_x f(xy) = \dfrac{2x}{1+x^2+y^2} = 0$ 0.5

$(xy) = (0,0)$ 0.5

$\partial_y f(xy) = \dfrac{2y}{1+x^2+y^2} = 0$ 0.5

$$H(xy) = \begin{pmatrix} \dfrac{2(1+x^2+y^2)-4x^2}{(1+x^2+y^2)^2} & -2x\cdot 2y \dfrac{1}{(1+x^2+y^2)^2} \\ \dfrac{-4xy}{(1+x^2+y^2)^2} & \dfrac{2(1+x^2+y^2)-4y^2}{(1+x^2+y^2)^2} \end{pmatrix}$$

$$H(00) = \begin{pmatrix} \boxed{2} & 0 \\ 0 & 2 \end{pmatrix}$$

$\det H(00) = 4 > 0$ so it is a local max or min. Since $\frac{\partial^2}{(\partial x)^2} f(0,0) > 0$ we have that this is a min. 0.5

(b) On ∂D ~~A~~ we have that $x^2+y^2=2x+8$ and so if we substitute ~~f~~ in the law of f ~~x^2+y^2~~ $= 2x+8$

we get

$$g(x)=\ln(1+2x+8) \qquad x\in[-2,4]$$
$$=\ln(9+2x)$$

0.5

$$g'(x)=\frac{2}{9+2x}$$ is alway $\neq 0$ 0.5

$g(-2)=\ln(5)$ min. 0.5

$g(4)=\ln(17)$ max 0.5

(c) we have to compare

$f(0,0)$ with $\ln(5)$ and $\ln(17)$

$\parallel$

$\ln(1)=0$

this is the min

this is the max

Alternative solutions for b

I) we split the ∂D as

$$\left\{y=\sqrt{9-(x-1)^2}\right\} \qquad \left\{y=-\sqrt{9-(x-1)^2}\right\}$$

$$x\in[-2,4]$$

side 1

$g(x)=\ln\left(1+9-(x-1)^2+x^2\right)=\ln(9-2x)$ same as in the solution only $g(-2)$ and $g(4)$ no

side 2

$g(x)=\ln(9-2x)$ as before

with Lagrange multiplier

$$L(x\,y\,\lambda) = \ln(1+x^2+y^2) - \lambda(-2x+x^2+y^2-8)$$

$$\begin{cases} \partial_x L(xy\lambda) = \dfrac{2x}{1+x^2+y^2} - \lambda(-2+2x) = 0 \\ \partial_y L(xy\lambda) = \dfrac{2y}{1+x^2+y^2} - \lambda(2y) = 0 \\ -2x+x^2+y^2-8=0 \end{cases}$$

II: $$\frac{2y}{1+x^2+y^2}\left(1-\lambda(1+x^2+y^2)\right)=0$$

$y=0$ or $\lambda = \dfrac{1}{1+x^2+y^2}$

$\Downarrow$

III $x^2-2x-8=0$

$x = 1 \pm \sqrt{1+8}$

$x = -2, 4$

which gives me the points $(-2\,0)$ and $(4,0)$ that we got before

$f(-2\,0) = \ln(5)$

$f(4,0) = \ln(17)$

I
$$\frac{2x}{1+x^2+y^2} - \frac{1}{1+x^2+y^2}(-2+2x)=0$$

$\Downarrow$

$$\frac{\cancel{2x}+2-\cancel{2x}}{1+x^2+y^2}=0$$

$$\frac{2}{1+x^2+y^2}=0$$

Never