MM3001 6-12-24

1) $f(x) = \ln(1+x)$ $f(0)=0$ 0.5

$f'(x) = \frac{1}{1+x}$ 0.5 $f'(0)=1$ 0.5

$f''(x) = -\frac{1}{(1+x)^2}$ 0.5 $f''(0)=-1$ 0.5

$f'''(x) = \frac{2}{(1+x)^3}$ 0.5 $f'''(0)=2$ 0.5

$$p(x) = 0 + (x-0) - \frac{1}{2}(x-0)^2 + \frac{1}{6}(x-0)^3$$

$$= x - \frac{1}{2}x^2 + \frac{1}{3}x^3$$ 1

$$p(0.1) = \frac{1}{10} - \frac{1}{200} + \frac{1}{3000}$$ 0.5

2) a) $$\sum_{k=10}^{20} 3(0.5)^k = 3(0.5)^{10} \sum_{k=0}^{10} (0.5)^k$$

$$= 3(0.5)^{10} \frac{1-(0.5)^{11}}{1-0.5}$$

b) first payment immediately

after first payments

debt. $=$ $D - a$

after 2nd payment

debt $=$ $(D-a)\left(1+\frac{p}{100}\right) - a$

$$= D\left(1+\frac{p}{100}\right) - a\left(1+\frac{p}{100}\right) - a$$

we see a geometric series being formed

$$D\left(1+\frac{p}{100}\right)^{119} - a\sum_{k=0}^{119}\left(1+\frac{p}{100}\right)^{k} = 0$$

If the first payment after a month

$$D\left(1+\frac{p}{100}\right)^{120} - \sum_{k=0}^{119}\left(1+\frac{p}{100}\right)^{k}$$

Using the formula for geometric series.

First payment immediately

$$D\left(1+\frac{p}{100}\right)^{119} - a\,\frac{\left(1+\frac{p}{100}\right)^{120}-1}{\left(1+\frac{p}{100}\right)-1} = 0$$

$$\boxed{D = \frac{a}{\left(1+\frac{p}{100}\right)^{119}} \frac{100}{p}\left(\left(1+\frac{p}{100}\right)^{120} - 1\right)}$$

if the first payment is afte after a month

$$\boxed{D = \frac{a}{\left(1+\frac{p}{100}\right)^{120}} \frac{100}{p}\left(\left(1+\frac{p}{100}\right)^{120} - 1\right)}$$

c) We modify the formula above to suit our purpose

let d the downpayment

we have that

$$(D-d)\left(1+\frac{p}{100}\right)^{119+12}$$

$$= a\left(\left(1+\frac{p}{100}\right)^{120} - 1\right)\frac{100}{p}$$

so that

$$a = \frac{(D-d)\left(1+P/100\right)^{131}}{\left(1+\frac{P}{100}\right)^{120}-1}\;\frac{P}{100}$$

We plug in the values and get

$P = \frac{3.6}{12} = 0.3$

$$\boxed{a = \frac{(100\,000)\left(1+\frac{3}{1000}\right)^{131}}{\left(1+\frac{3}{1000}\right)^{120}-1}\;\frac{3}{1000}}$$

can be left so

3) $f(x) = \frac{1}{5000}\, e^{-x^4+200x^2}$

a) $f'(x) = \underbrace{\frac{1}{5000}}_{>0}\left(-4x^3+400x\right)\underbrace{e^{-x^4+200x}}_{>0}$

0.5

we want to know when

$f'(x) \geq 0 \quad \Leftrightarrow$

$-4x^3+400x \geq 0 \quad \Leftrightarrow$

$-4x\,(x^2-100) \geq 0$

$\Leftrightarrow \quad 4x(x-10)(x+10) \leq 0$ 0.5

		-10		0		10	
x	$-$	$-$	$-$	0	$+$	$+$	$+$
$(x-10)$	$-$	$-$	$-$	$-$	$-$	0	$+$
$(x+10)$	$-$	0	$+$	$+$	$+$	$+$	$+$
$-f'(x)$	$-$	0	$+$	0	$-$	0	$+$
$f'(x)$	$+$	0	$-$	0	$+$	0	$-$
$f(x)$	$\nearrow$	$-$	$\searrow$	$-$	$\nearrow$	$-$	$\searrow$
		local max		local min			local max

0.5

0.5 correct conclusion

A: the function has 3 critical points $x = -10, 0, +10$

± 10 are local max

0 is a local min.

$f(x) \nearrow$ if $x \leq 10$, $0 \leq x \leq 10$

and decreases otherwise

b) on $[-1, 11]$ there are 2 critical points, 0 and 10

$f(0) = \frac{1}{5000}$ MIN.

$$f(-1) = \frac{1}{5000} e^{-1+200} = \frac{1}{5000} e^{199}$$

$$f(10) = \frac{1}{5000} e^{-10000+20000} = \frac{1}{5000} e^{+10000} \quad \text{MAX}$$

$$f(11) = \frac{1}{5000} e^{-11^4+200(11)^2} = \frac{1}{5000} e^{+9554}$$

the min value is $\frac{1}{5000}$

the max value is $\frac{1}{5000} e^{10000}$

c) $$\lim_{x \to -\infty} \frac{1}{5000} e^{-x^4+200x^2}$$

$$= " \frac{1}{5000} e^{-\infty} " \quad (x^4 \text{ wins to } x^2)$$

$$= 0$$

0.5

$$\lim_{x \to +\infty} f(x) = 0$$ for the same reason

1

on $[+9, +\infty)$ the function has a max value

$f(10)$ $\frac{1}{5000} e^{10000}$ 0.5

but not min value since it will get closer & closer to 0 without ever reaching it

4) a) $\int \ln(x) + e^{x^3} x^2 \, dx$ 0.5

$= \int \ln(x)\,dx + \int e^{x^3} x^2 \, dx$ $x^3 = u$, $3x^2 dx = du$

0.5

$= x\ln(x) - \int x \frac{1}{x} dx + \frac{1}{3}\int e^u du =$

$= x\ln(x) - x + C_1 + \frac{1}{3} e^u + C_2$

0.5 0.5

$= x\ln(x) - x \; \frac{1}{3} e^{x^3} + C =$

0.5

b) $\int_0^1 x\sqrt{x^2+1}\, dx$

the function $x\sqrt{x^2+1}$ is defined and continuous on $[0,1]$ so this is a regular integral

$\int_0^1 x\sqrt{x^2+1}\,dx =$

$u = x^2+1$

$\frac{du}{dx} = 2x$

$du = 2\,dx$

$x=0 \quad u=1$

$x=1 \quad u=2$

$\int_1^2 \frac{1}{2}\sqrt{u}\,du$ 1

$= \left[\frac{1}{2}\,\frac{1}{1+\frac{1}{2}}\,u^{1+\frac{1}{2}}\right]_1^2$ 0.5

$= \left[\frac{1}{\cancel{2}}\,\frac{\cancel{2}}{3}\,u^{3/2}\right]_1^2$

$= \frac{1}{3}2^{3/2} - \frac{1}{3}1^{3/2}$ 1

5) a) $A \cdot A$ is not possible

$A \cdot A^T$ is possible

$A\,B^T$ is possible

AB is not possible

b) $B \cdot A$ is not possible 1

$$A^T \cdot A = \begin{pmatrix} 1 \\ 2 \end{pmatrix} (1, 2)$$

$$= \begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}$$ 1

$$\det(A^T \cdot A) = 8 - 8 = 0$$

c) $$\left(\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 4 & -1 & 5 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & \boxed{1} & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 3 & -2 & 4 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & -2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -5 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We have one solution!

$$z = \frac{2}{5}$$

$$y + \frac{2}{5} = 2 \qquad y = \frac{8}{5}$$

$$x + \frac{8}{5} + \frac{2}{5} = 1$$

$$x + \cancel{\frac{10}{5}}^{\,2} = 1 \qquad x = -1$$

there is only one solution

$(-1, 8/5, 2/5)$

6) a) $f(x,y) = \sqrt{x^2+y^2} - y^2 - 1$

$\frac{\partial}{\partial x} f(x,y) = 2x \frac{1}{\sqrt{x^2+y^2}} \cancel{\;}^{0} = 0$ $\quad x = 0$

$\frac{\partial}{\partial y} f(x,y) = 2y \frac{1}{\sqrt{x^2+y^2}} - 2y$

$= 2y \left(\frac{1}{\sqrt{x^2+y^2}} - 1 \right) = 0$

note that the partial derivatives are not defined in $(0,0)$

So we have $\frac{1}{\sqrt{x^2+y^2}} = 1$ $\quad x = 0$

$\frac{1}{|y|} = 1$ $\quad y = \pm 1$

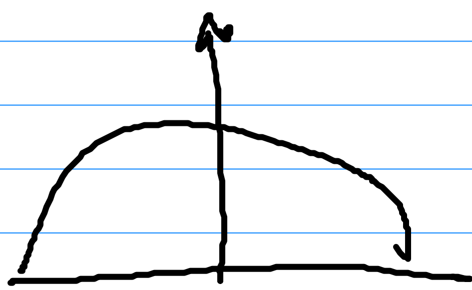
We have 2 critical points $(0,1)$ $(0,-1)$

$$H(x,y) = \begin{pmatrix} \dfrac{2\sqrt{x^2+y^2} + 4x\frac{1}{\sqrt{x^2+y^2}}}{x^2+y^2} & -\frac{1}{2}4xy\,(x^2+y^2)^{-3/2} \\ -2xy(x^2+y^2)^{-3/2} & 2\left(\frac{1}{\sqrt{x^2+y^2}} - 1\right) - 2y^2(x^2+y^2)^{-3/2} \end{pmatrix}$$

$$H(0,1) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

$$H(0-1) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

both points are saddle



there are two sides

$y=0 \quad x \in [-3,3]$

$\sqrt{x^2+y^2} = 3 \quad y \in [0,3]$

Side 1 $\quad f(x,0) = \sqrt{x^2} - 1 = |x| - 1$

we have a min when $x = 0$

& max when $x = \pm 3$

$f(0,0) = -1$

$f(\pm 3, 0) = 2$

Side 2

$f_{| x^2+y^2=9}(x,y) = 3 - y^2 - 1 = g(y)$

we have a max when

$y = 0$ & $x = \pm 3$.

(already taken care of)

when $y = 3$ $x = 0$

$f(0,3) = 3 - 9 - 1 = -7$

the min value of f on ∂D is

-7

the max value is 2

We have to compare this with the value in the critical pts

$f(0, \pm 1) = 1-1-1 = -1$

which is neither bigger than 2 nor smaller than -7

$\Rightarrow$ the max & min on D are the same as those in ∂D

(Which we know since the critical points are SADDLE)

Note: Since $\partial_x f(x,y)$ and $\partial_y f(x,y)$ are not defined in $(0,0)$ one should also have compared the values in (b) with $f(0,0) = \sqrt{0} - 0 - 1 = -1$

(You do not lose points for missing this)