Solutions 2026-04-22

1) a) $S(x) = 3\left(1 + 2e^x + (2e^x)^2 + \ldots\right)$

is a geometric series with

$r = 2e^x$. It is convergent

$\iff |r| < 1 \iff e^x < \frac{1}{2} \iff$

$\underline{x < -\ln(2)}$. We then

get $\quad S(x) = \frac{3}{1-r} = \frac{3}{1-2e^x} = 6$

$\iff e^x = \frac{1}{4} \iff \underline{x = -\ln(4)}$

b) $f(x) = e^x \ln x$

$f'(x) = e^x \cdot \frac{1}{x} + e^x \cdot \ln x$

$f''(x) = -e^x \cdot \frac{1}{x^2} + 2e^x \cdot \frac{1}{x} + e^x \cdot \ln x$

so $\quad p_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2$

$= \underline{e(x-1) + \frac{e}{2}(x-1)^2}$

2 a) Plugging $(x, y) = (1, 2)$ in the equation gives

$$1 \cdot 2^3 + a \cdot 1^5 \cdot 2 = 4 \iff$$

$$8 + 2a = 4 \iff a = -2$$

so $xy^3 - 2x^5y = 4$

Implicit derivation gives

$$y^3 + x \cdot 3y^2 \cdot y' - 10x^4y - 2x^5y' = 0$$

so at $x = 1$:

$$8 + 12y'(1) - 20 - 2y'(1) = 0$$

$$\iff y'(1) = 1{,}2 .$$

The tangent line is

$$y - y_0 = k(x - x_0) \quad \text{so}$$

$$y - 2 = 1{,}2(x - 1)$$

$$\iff y = 1{,}2x + 0{,}8$$

3) We get $f(x) = \dfrac{x}{x^2+3}$,

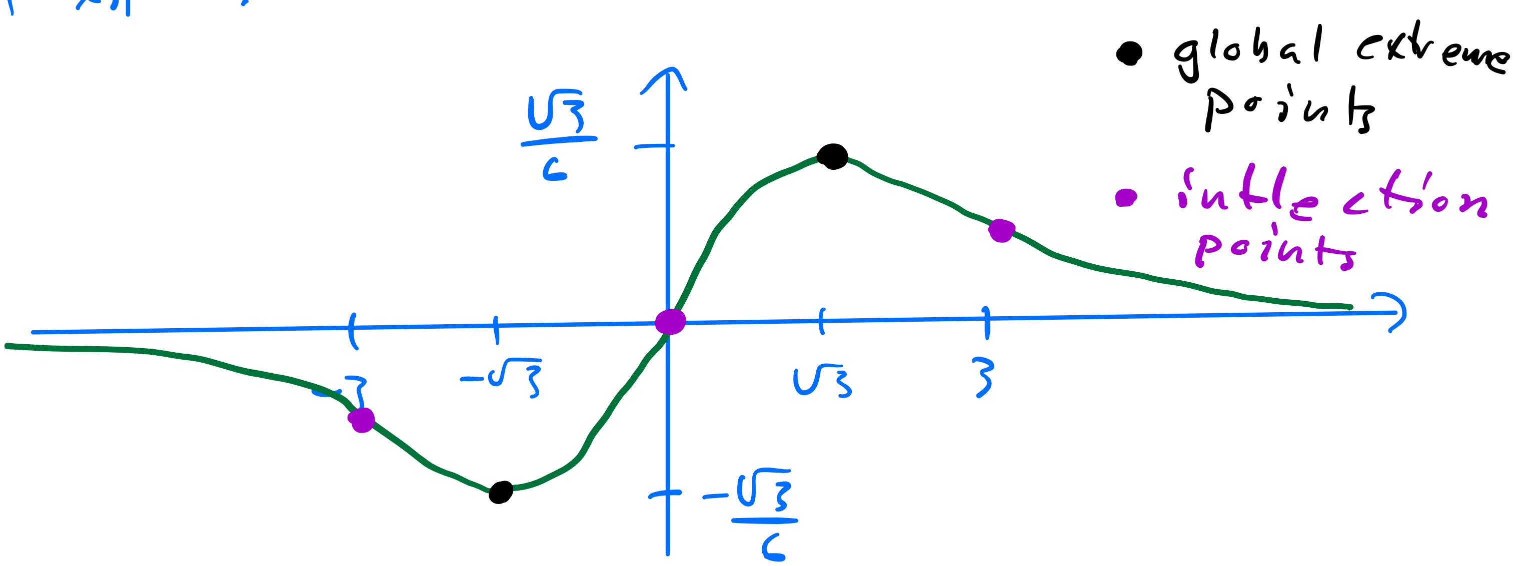
$$f'(x) = \frac{3-x^2}{(x^2+3)^2}, \quad f''(x) = \frac{2x(x-3)(x+3)}{(x^2+3)^3}$$

so $f'(x) = 0 \iff x = \pm\sqrt{3}$

and $f''(x) = 0 \iff x = 0$ or $x = \pm 3$

$$\lim_{x\to\pm\infty} \frac{x}{x^2+3} = [\text{l'Hosp.}] = \lim_{x\to\pm\infty} \frac{1}{2x} = 0$$

x	$(-\infty)$		-3		$-\sqrt{3}$		0		$\sqrt{3}$		3		(∞)
$f'(x)$		$-$	$-$	$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$	
$f(x)$	(0)	$\searrow$			$-\frac{\sqrt{3}}{6}$		$\nearrow$		$\frac{\sqrt{3}}{6}$		$\searrow$		(0)
$f''(x)$		$-$	0	$+$		$+$	0	$-$		$-$	0	$+$	
$f(x)$		$\cap$	infl.	$\cup$			infl.	$\cap$			infl.	$\cup$	



4 a) Integration by parts:

$$\int_0^1 x \cdot e^{3x} dx = \left[x \frac{e^{3x}}{3}\right]_0^1 - \int_0^1 \frac{e^{3x}}{3} dx =$$

$$= e^3 - \frac{1}{3}\left[\frac{e^{3x}}{3}\right]_0^1 = \ldots = \underline{\frac{2e^3+1}{9}}$$

b) $$\int \frac{x^2}{(1+x^3)^2} dx = \begin{bmatrix} u = 1+x^3 \\ du = 3x^2 dx \end{bmatrix} =$$

$$= \frac{1}{3}\int u^{-2} du = -\frac{1}{3u} + C =$$

$$= -\frac{1}{3(1+x^3)} + C \qquad \text{so}$$

$$\int_0^\infty \frac{x^2}{(1+x^3)^2} dx = \lim_{N\to\infty} \left[-\frac{1}{3(1+x^3)}\right]_0^N$$

$$= \lim_{N\to\infty} \left(\frac{1}{3} - \frac{1}{3(1+N^3)}\right) = \underline{\frac{1}{3}}$$

5 a) Using Gauss elimination :

$$\left(\begin{array}{cccc|c} 1 & 0 & 2 & 1 & -1 \\ -1 & 1 & -3 & 0 & 2 \\ 1 & 2 & 0 & 4 & 2 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 2 & 1 & -1 \\ 0 & 1 & -1 & 1 & 1 \\ 0 & 2 & -2 & 3 & 3 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 2 & 1 & -1 \\ 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

We set $z = t$ so $w = 1$,

$y = z - w + 1 = t$, $x = -2x - w - 1 = -2t - 2$

so $(x, y, z, t) = (-2-2t, t, t, 1)$, $t \in \mathbb{R}$

b) $$(A \mid I) = \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 3 & 0 & 1 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 0 & -2 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1/3 & 0 \end{array}\right) = (I \mid A^{-1})$$

6) We determine the stationary points :

$$\begin{cases} f'_x = y - 3x^2 = 0 & (1) \\ f'_y = x - 2y = 0 & (2) \end{cases}$$

By (2) $x = 2y$, so (1) becomes

$y - 12y^2 = 0 \iff y(1 - 12y) = 0$

so $y = 0$ or $y = \frac{1}{12}$ so

$(x, y) = (0,0)$ or $\left(\frac{1}{6}, \frac{1}{12}\right)$

The Hessian is $H = \begin{pmatrix} -6x & 1 \\ 1 & -2 \end{pmatrix}$

At $(x, y) = (0, 0)$:

$\det H = \begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix} = -1 < 1$ so

$(0,0)$ is a saddle point

At $(x, y) = \left(\frac{1}{6}, \frac{1}{12}\right)$: $H = \begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix}$

so $\det H = 1 > 0$ and $f_{xx} = -1 < 0$

so $\left(\frac{1}{6}, \frac{1}{12}\right)$ is a local max