MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Sven Raum

Tentamensskrivning i
Foundations of Analysis
7.5 hp
22nd August 2019

Please read carefully the general instructions:

- During the exam any textbook, class notes, or any other supporting material is forbidden.
- In particular, calculators are not allowed during the exam.
- In all your solutions show your reasoning, explaining carefully what you are doing. Justify your answers.
- Use natural language, not just mathematical symbols.
- Use clear and legible writing. Write preferably with a ball-pen or a pen (black or dark blue ink).
- The exam has 5 questions. If you handed in homework, you solve problem 5 and select 3 problems among questions 1-4 which you solve. State clearly which problems you chose. If you opted out of the homework, you solve problems 1-5.
- Each question is graded on a 0-20 scale.
- A score of at least 50 points will ensure a pass grade.
- The exam is returned on 2nd September at 10 o'clock in sal 16, hus 5.

GOOD LUCK!

1. Sets

   (a) Define countability and prove that the set of natural numbers $\mathbb{N}$ is countable.

   (b) Provide an example of an uncountable set. Verify that it is indeed uncountable.

   (c) Let X be a set. Show that the following two statement are equivalent:

      - X is finite or countable.
      - There is an injective map $X \to \mathbb{N}$.

   (d) Let X be a countable set and $X \to Y$ be a surjective map onto another set Y. Show that Y is countable or finite. (Even if you heard of it, you are not allowed to use the axiom of choice.)

2. Metric spaces and topology

   (a) Let (X, d) be a metric space.

      i. Define the notion of an open subset of X.

      ii. Define the notion of a closed subset of X.

   (b) Let $n \in \mathbb{N}_{\geq 1}$ and consider the function $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$d(x, y) = \sum_{i=1}^{n} |x_i - y_i|.$$

      i. Show that $(\mathbb{R}^n, d)$ is a metric space.

      ii. Show that $(0, 1)^n \subset \mathbb{R}^n$ is open with respect to the metric d.

   (c) Provide an example of a X metric space which contains a non-empty compact open subset S such that $S \neq X$.

   (d) Show that $[0, 1] \subset \mathbb{R}$ is compact, without referring to the Heine-Borel theorem.

3. Differentiation

   (a) State the implicit function theorem.

   (b) Let $O \subset \mathbb{R}^n$ be an open subset and $f : O \to \mathbb{R}$ be differentiable. Show that f is continuous.

   (c) i. Prove Rolle's theorem.

      ii. Show that the converse of Rolle's theorem does not hold, by providing an example of a differentiable function $f : [-1, 1] \to \mathbb{R}$ such that $f'(0) = 0$ holds, but f does not have a local extremum at 0.

   (d) For $k \in \mathbb{N}$ let $f_k : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function satisfying $f_k(0) = 0$. Assume that for all $k \in \mathbb{N}$ and all $x \in \mathbb{R}^n$ we have $\|f_k'(x)\| \leq \frac{1}{k^2}$.

      i. Show that the series $f(x) = \sum_{k=0}^{\infty} f_k(x)$ converges absolutely for all $x \in \mathbb{R}^n$.

      ii. Show that f is continuously differentiable.

4. Integration

   (a) Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. Define the Riemann integral of f.

   (b) Show that the function

$$\mathbb{1}_{\mathbb{Q}} : \mathbb{R} \to \mathbb{R}$$
$$\mathbb{1}_{\mathbb{Q}}(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

   is not Riemann integrable on any non-trivial compact interval.

(c) Find a monotone increasing function $\alpha : [0,1] \to \mathbb{R}$ such that for all continuous functions $f : [0,1] \to \mathbb{R}$ the following equality holds:

$$\int_0^1 f \mathrm{d}\alpha = \frac{1}{2}\left(f(0) + f(1)\right).$$

Remember to prove all claims you make.

(d) For a Riemann integrable function $f : [a,b] \to \mathbb{R}$ define $F : [a,b] \to \mathbb{R}$ by

$$F(x) = \int_a^x f(t)\mathrm{d}t.$$

i. Show that for any choice of f as above, the function F is continuous.

ii. Provide an example of a function f as above, such that F is not differentiable.

5. True / false questions

Please indicate your answers on the separate answer sheet.

(a) If $A \subset \mathbb{R}$ is countable and $B \subset \mathbb{R}$ is an arbitrary set, then $A \cap B$ is countable.

(b) If a subset E of the real numbers is bounded above, and $x = \sup E$, then $x \in E$.

(c) The set $\{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}_{\geq 1}\} = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\}$ is compact.

(d) Let X be a metric space and $A \subset X$ a proper subset with some isolated point. Then A is not open.

(e) Let X be a metric space. If $E \subset X$ a perfect subset and $F \subset E$ a subset that is closed in X. Then F is perfect.

(f) Let $(p_n)_{n\in\mathbb{N}}$ be a sequence in a metric space X. Then $(p_n)_n$ converges to $p \in X$ if and only if every neighbourhood of p contains all but finitely many members of $(p_n)_n$.

(g) Given any real sequence $(a_n)_{n\in\mathbb{N}}$, we have $\sum_{n=0}^{\infty}(a_n - a_{n+1}) = a_0$.

(h) The function $f : (1, +\infty) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ is uniformly continuous.

(i) If $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function, then f has at most countably many points of discontinuity.

(j) If $f : X \to Y$ is a continuous function between compact metric spaces and $E \subset X$ is closed, then $f(E) \subset Y$ is a closed subset.

(k) Put $X = (-\infty, 0) \cup (0, \infty)$ and let $f : X \to \mathbb{R}$ be a differentiable function satisfying $f'(x) = 0$ for all $x \in X$. Then f is constant.

(l) Let $f : \mathbb{R} \to \mathbb{R}$ be a function whose derivatives of all orders exist and are continuous. Then $f(x) = \sum_{n\in\mathbb{N}} \frac{f^{(n)}(0)}{n!}x^n$ for every real number for which the right-hand side converges.

(m) Let $f : [0,1] \to \mathbb{R}$ be a function whose absolute value $|f|$ is Riemann integrable. Then also f is Riemann integrable.

(n) Let $\alpha, \beta : [0,1] \to \mathbb{R}$ be two monotone increasing functions and assume that $\mathcal{R}(\alpha) = \mathcal{R}(\beta)$. Further assume that for all $f : [0,1] \to \mathbb{R}$ in $\mathcal{R}(\alpha)$ equality

$$\int_0^1 f \mathrm{d}\alpha = \int_0^1 f \mathrm{d}\beta$$

holds. Then $\alpha = \beta$.

(o) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in $\mathrm{C}'([0,1], \mathbb{R})$ which converges uniformly to $f \in \mathrm{C}([0,1], \mathbb{R})$. Then f is also continuously differentiable.

(p) Let X be a metric space and $E \subset X$ dense subset. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of real-valued continuous functions X such that the sequence of restrictions $(f_n|_E)_n$ converges uniformly. Then also $(f_n)_n$ converges uniformly.

(q) Let $K \subset \mathrm{C}([0,1],\mathbb{R})$ be a compact set. Then there is $B \in \mathbb{R}_{\geq 0}$ such that for all $f \in K$ and for all $x \in [0,1]$ we have $|f(x)| \leq B$.

(r) Assume that $f : \mathbb{R}^n \to \mathbb{R}$ admits all partial derivatives $\frac{\partial f}{\partial x_i}$, $i \in \{1, \ldots, n\}$. Then f is differentiable.

(s) Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable. Then there is some element $v \in \mathbb{R}^n \setminus \{0\}$ such that $\frac{\partial f}{\partial x_n} = D_v f$.

(t) Denote by $\mathrm{d} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ the Euclidean distance function on $\mathbb{R}^2$. Then inverse function theorem applies at every point in $\mathbb{R}^2$ to the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x) = \mathrm{d}(0, x)$.

ANSWER SHEET FOR QUESTION 5

EXAM CODE: ....

- Answer Question 5 by placing a clear mark (either ✓or $\times$) in the box that corresponds to your answer of the table below. Please, mark your answer ONCE only.
- In order to provide a fair exam, that does not encourage random answering, the following system is applied to calculate your points on question 5. For each correct answer you will be awarded 1 point. For each incorrect answer you will lose 1 point. If the number of incorrect answers is higher than the number of correct answers, then the total mark awarded for this question be 0.

	True	False
a.		
b.		
c.		
d.		
e.		
f.		
g.		
h.		
i.		
j.		
k.		
l.		
m.		
n.		
o.		
p.		
q.		
r.		
s.		
t.		