MATEMATISKA INSTITUTIONEN STOCKHOLMS UNIVERSITET Avd. Matematik Examinator: A.A. Sola	Final examination in Mathematics III Foundations of Analysis 7.5 hp January 7th 2021

You are not permitted to collaborate with other students or consult other individuals. Maximum total score is 20 points: 15 points and participation in the oral examination are required to pass. See course webpage for full details.
Appropriate amounts of detail are required for full marks.

1. Determine which of the following statements are true, and which are false. Explain your reasoning.
   (a) A function $f\colon \mathbb{R} \to \mathbb{R}$ that is continuous at all but countable many points, and is bounded at every point, is continuous everywhere on $\mathbb{R}$.
   (b) If a sequence of real-valued functions $\{f_n\}$ converges uniformly on $\mathbb{R}$ to a continuous function f, then all but at most finitely many of the f_n are continuous on $\mathbb{R}$.
   (c) If f is bounded on $\mathbb{R}$ and has $f'(x) = 0$ for $-1 \leq x \leq 2$ then f is constant on $[0, 1]$.
   (d) If f is continuous and the range of f contains finitely many distinct points, then f is constant.
   (e) The set of real-valued continuous functions on $[0, 1]$ equipped with the function
   $$d(f, g) = \int_0^1 |f(x) - g(x)|dx$$
   is an example of a complete metric space.
2. A real-valued function f on the interval $[0, 1]$ is said to belong to the class $\mathcal{C}(\alpha)$, $\alpha > 0$, if there exists a constant $C > 0$ such that $|f(x) - f(y)| \leq C|x - y|^\alpha$ for any $x, y \in \mathbb{R}$.
   (a) Give an example of a uniformly continuous function on $[0, 1]$ that does not belong to any $\mathcal{C}(\alpha)$.
   (b) If f belongs to $\mathcal{C}(1)$, does this imply that f is differentiable at every point of $[0, 1]$?
   (c) Give a complete description of the functions of class $\mathcal{C}(\frac{3}{2})$ on $[0, 1]$.
3. Compute the Riemann-Stieltjes integral
   $$\int_0^1 f d\alpha$$
   where $f(x) = x^2$ and
   $$\alpha(x) = \begin{cases} 1 + x^2, & 0 \leq x \leq \frac{1}{2} \\ \frac{3}{2} + x^2, & \frac{1}{2} < x \leq 1 \end{cases}$$
4. Let f be real-valued and continuous on $[0, 1]$. Suppose that, for each $n = 0, 1, 2, \ldots$,
   $$\int_0^1 f(x)x^n dx = 0.$$
   Prove that $f(x) = 0$ for all $x \in [0, 1]$. *(Hint: start by looking at f^2.)*