MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
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Final examination in
Mathematics III Foundations of Analysis
7.5 hp
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You are not permitted to collaborate with other students or consult other individuals. Maximum total score is 20 points: 15 points and participation in the oral examination are required to pass. See course webpage for full details.
Appropriate amounts of detail are required for full marks.

1. Determine which of the following statements are true, and which are false. Explain your reasoning.
   (a) A function $f\colon \mathbb{R} \to \mathbb{R}$ that is continuous at all but countably many points, and is bounded at every point, is continuous everywhere on $\mathbb{R}$.
   (b) If a sequence of real-valued functions $\{f_n\}$ converges uniformly on $\mathbb{R}$ to a continuous function f, then all but at most finitely many of the f_n are continuous on $\mathbb{R}$.
   (c) If f is bounded on $\mathbb{R}$ and has $f'(x) = 0$ for $-1 \leq x \leq 2$ then f is constant on $[0, 1]$.
   (d) If f is continuous and the range of f contains finitely many distinct points, then f is constant.
   (e) The set of real-valued continuous functions on $[0, 1]$ equipped with the function
   $$d(f, g) = \int_0^1 |f(x) - g(x)|dx$$
   is an example of a complete metric space.

   Sketch solutions:

   (a) The statement is false. Consider, for instance, the function that is equal to 1 on $[2k, 2k + 1]$, k an integer, and equal to 0 on $[2k + 1, 2k]$. This function is bounded, is continuous except at the integers which form a countable set, but is not a continuous function overall.

   (b) The statement is false. Consider the sequence $f_n = \begin{cases} \frac{1}{n}, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$. Then f_n converges uniformly to the zero function but no f_n is a continuous function.

   (c) This statement is true. One way of seeing this is to apply the mean value theorem.

   (d) This problem intended to specify that f is a function on the real line. In that case the intermediate value theorem forces f to be constant.

   (e) This statement is false. To see this, consider the sequence of continuous functions f_n defined by setting $f_n(x) = 0$ for $0 \leq x \leq \frac{1}{2}$ and $f_n(x) = 1$ for $\frac{1}{2} + \frac{1}{n} \leq x \leq 1$ and interpolating linearly on the interval $\left[\frac{1}{2}, \frac{1}{2} + \frac{1}{n}\right]$. Then $d(f_n, f_m) \to 0$ with n, m but the limit function f is discontinuous at $x = \frac{1}{2}$.

2. A real-valued function f on the interval $[0, 1]$ is said to belong to the class $\mathcal{C}(\alpha)$, $\alpha > 0$, if there exists a constant $C > 0$ such that $|f(x) - f(y)| \leq C|x - y|^\alpha$ for any $x, y \in \mathbb{R}$.
   (a) Give an example of a uniformly continuous function on $[0, 1]$ that does not belong to any $\mathcal{C}(\alpha)$.
   (b) If f belongs to $\mathcal{C}(1)$, does this imply that f is differentiable at every point of $[0, 1]$?
   (c) Give a complete description of the functions of class $\mathcal{C}(\frac{3}{2})$ on $[0, 1]$.

Sketch solutions:

(a) The function

$$f(x) = \begin{cases} 0, & x = 0 \\ \frac{1}{\log(\frac{x}{2})}, & 0 < x \leq 1 \end{cases}$$

furnishes an example. Since $\lim_{x\to 0} f(x) = 0$, the function f is continuous, and since $[0,1]$ is compact f is uniformly continuous.

However, if $x \in [0,1]$,

$$\frac{|f(x) - f(0)|}{|x|^\alpha} = \frac{-\frac{1}{\log(x/2)}}{x^\alpha}$$

which implies that, for any $\alpha > 0$

$$\lim_{x\to 0} \frac{1}{-x^\alpha \log(x/2)} = \lim_{t\to-\infty} -\frac{2^\alpha e^{-\alpha t}}{t} = \infty.$$

This shows that there does not exist any $\alpha > 0$ such that $|f(x)| \leq C|x|^\alpha$ near 0 for some constant $C > 0$.

(b) It does not follow that f is differentiable. Consider for instance the function $f(x) = |x - \frac{1}{2}|$. This function is clearly not differentiable at $x = \frac{1}{2}$ but

$$|f(x) - f(y)| = ||x - \frac{1}{2}| - |y - \frac{1}{2}|| \leq |x - y|$$

by the reverse triangle inequality.

(c) The class $\mathcal{C}(\frac{3}{2})$ consists of the constant functions on $[0,1]$. If f is constant, then $f(x) - f(y) = 0$ for $x, y \in [0,1]$ and hence f belongs to $\mathcal{C}(\frac{3}{2})$. Conversely, suppose that $f \in \mathcal{C}(\frac{3}{2})$. Then, for $x \in (0,1)$ and h small enough, we have, for some $C > 0$,

$$\frac{|f(x+h) - f(x)|}{|h|} \leq C\frac{|h|^{\frac{3}{2}}}{|h|} = C|h|^{\frac{1}{2}},$$

and the quantity on the right tends to zero with h. This in turn implies $f'(x) = 0$ and by a theorem in Rudin, f is constant.

3. Compute the Riemann-Stieltjes integral

$$\int_0^1 f d\alpha$$

where $f(x) = x^2$ and

$$\alpha(x) = \begin{cases} 1 + x^2, & 0 \leq x \leq \frac{1}{2} \\ \frac{3}{2} + x^2, & \frac{1}{2} < x \leq 1 \end{cases}$$

Sketch solution: It is important to note that α has a jump at $x = \frac{1}{2}$. For this reason, we *cannot* apply the formula $\int_0^1 f d\alpha = \int_0^1 f\alpha' dx$ directly. However, we can decompose α as $\alpha = \alpha_1 + \alpha_2$ where $\alpha_1 = x^2$ and

$$\alpha_2(x) = \begin{cases} 1, & 0 \leq x \leq \frac{1}{2} \\ \frac{3}{2}, & \frac{1}{2} < x \leq 1 \end{cases}.$$

Then $\int f d\alpha = \int f d\alpha_1 + \int f d\alpha_2$ and we compute the integrals separately.

Since α_1 is continuously differentiable,

$$\int_0^1 f(x) d\alpha_1(x) = \int_0^1 x^2 \cdot 2x dx = \int_0^1 2x^3 dx = \left[\frac{1}{2}x^4\right]_0^1 = \frac{1}{2}.$$

Since α_2 is a step function, we obtain as in Rudin's book that

$$\int_0^1 f d\alpha_2 = \frac{1}{2} \cdot \frac{1}{2^2} = \frac{1}{8}.$$

Hence $\int_0^1 f d\alpha = \frac{5}{8}$.

4. Let f be real-valued and continuous on $[0, 1]$. Suppose that, for each $n = 0, 1, 2, \ldots$,

$$\int_0^1 f(x)x^n dx = 0.$$

Prove that $f(x) = 0$ for all $x \in [0, 1]$. *(Hint: start by looking at* f^2*.)*

Sketch solution: The assumption that $\int_0^1 fx^n dx = 0$ for all $n \in \mathbb{N}$ implies that $\int_0^1 f(x)P(x)dx = 0$ for any polynomial P with real coefficients. Using this, along with linearity of the integral, we obtain

$$\int_0^1 (f(x))^2 dx = \int_0^1 f(x)^2 dx - \int_0^1 f(x)P(x)dx = \int_0^1 f(x) \cdot (f(x) - P(x))dx \qquad (1)$$

for any polynomial P.

Now let $\epsilon > 0$ be given. Since f is a continuous function on $[0, 1]$ the Weierstrass theorem implies that there exists a polynomial P such that $\sup_{x\in[0,1]} |f(x) - P(x)| < \epsilon$. Again using properties of integrals, we deduce from (1) that

$$\int_0^1 f^2 dx \leq \epsilon \int_0^1 f dx.$$

Note that $\int_0^1 f dx$ is finite by the assumption that f is continuous. Thus, given any $\varepsilon > 0$ we obtain $\int_0^1 f^2 dx \leq \varepsilon$ by choosing $\epsilon > 0$ small enough. This in turn implies $\int_0^1 f^2 dx = 0$. We now deduce $f(x) = 0$ as desired. If this was not the case, we would have $f(x_0) \neq 0$ for some $x_0 \in [0, 1]$, and by continuity, we would have $f(x)^2 > 0$ on some interval $[a, b] \subset [0, 1]$. Since f^2 is non-negative for any real function, we arrive at a contradiction as this would imply $\int_0^1 f^2 dx \geq \int_a^b f^2 dx > 0$.