MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
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Final examination in
Mathematics III Foundations of Analysis
7.5 hp
February 17th 2021

You are not permitted to collaborate with other students or consult other individuals. Maximum total score is 20 points: 15 points and participation in the oral examination are required to pass. See course webpage for full details.
Appropriate amounts of detail are required for full marks.

1. Determine which of the following statements are true, and which are false. Explain your reasoning.
   (a) If f is a differentiable real-valued function on the interval $[a, b]$ whose derivative is non-zero in (a, b) then f does not attain a largest value on $[a, b]$.
   (b) If a sequence of real-valued functions $\{f_n\}$ converges uniformly on $\mathbb{R}$ to a differentiable function then there exists an integer N such that, for each $n \geq N$, the function f_n is differentiable.
   (c) If a set K is compact subset of the real line, then either K contains some closed interval or else K is a countable union of points.
   (d) If the real power series $\sum_{k=1}^{\infty} a_k x^k$ converges at some point $x_0 \in \mathbb{R}$ then there exists some neighborhood containing x_0 on which the series converges.
   (e) The set of real-valued differentiable functions on $[0, 1]$ equipped with the function
   $$d(f, g) = \int_0^1 |f'(x) - g'(x)|^2 dx$$
   is an example of a complete metric space.

   Sketch solutions:

   (a) This statement is false. Consider the function $f(x) = x$ on $[a, b]$. Then $f'(x) \neq 0$ for every $x \in (a, b)$ but attains a largest value at b.
   (b) This statement is false. Consider the sequence of functions f_n with $f_n(x) = 0$ for irrational x and $f_n(x) = 1/n$ for $x \in \mathbb{Q}$. Then $f_n \to 0$ uniformly but f_n is not a differentiable function for any $n \in \mathbb{N}$.
   (c) This statement is false as can be seen by taking K to be the middle-third Cantor set, which is uncountable and contains no interval.
   (d) This statement is false. Consider the power series $\sum_{k=0}^{\infty} (k!)^{k!} x^k$ which converges at $x = 0$ only.
   (e) This is not a complete metric space: in fact, d is not even a metric since $d(f, g) = 0$ whenever f and g are constant functions.

2. A subset $E \subset \mathbb{R}^2$ is said to be *complement-connected* if its complement E^c is connected.
   (a) Give an example of a set in $\mathbb{R}^2$ that is connected but not complement-connected.
   (b) If $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a bounded continuous function and E is a complement-connected set, need $f(E)$ be complement-connected?

   Sketch solutions:

   (a) One possible example of a connected set that is not complement-connected is $E = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \setminus \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < \frac{1}{2}\}$.

(b) It need not be the case that $f(E)$ is complement connected. One way of seeing this is to identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, and to observe that the unit disk, which is complement-connected, can be mapped to the vertical strip $\{z = x + iy \in \mathbb{C} \colon 1 \leq x \leq 2\}$, which in turn is mapped to an annulus using the complex exponential e^{iz}. Taking real and imaginary parts gives the desired map.

3. Compute the Riemann-Stieltjes integral

$$\int_0^{\frac{\pi}{2}} f d\alpha$$

where $f(x) = x^2$ and

$$\alpha(x) = \begin{cases} \sin x, & 0 \leq x \leq \frac{\pi}{4} \\ 1 + \sin x, & \frac{\pi}{4} < x \leq \frac{\pi}{2} \end{cases}.$$

Sketch solutions:

We first decompose the discontinuous function $\alpha(x)$ as $\alpha = \alpha_1 + \alpha_2$ where

$$\alpha_1(x) = \sin x$$

and

$$\alpha_2(x) = \begin{cases} 0 & 0 \leq x \leq \frac{\pi}{4} \\ 1 & \frac{\pi}{4} < x \leq \frac{\pi}{2} \end{cases}$$

The function α_1 is monotone increasing on $[0, \pi/4]$ and continuously differentiable, with derivative $\alpha_1'(x) = \cos x$. Hence

$$\int_0^{\pi/2} f(x) d\alpha_1(x) = \int_0^{\pi/2} x^2 \cos x dx$$

and using integration by parts, we obtain

$$\int_0^{\pi/2} f(x) d\alpha_1(x) = \frac{\pi^2}{4} - 2.$$

Since α_2 is a step function, with jump at $x = \pi/4$, we have

$$\int_0^{\frac{\pi}{2}} x^2 d\alpha_2(x) = \frac{\pi^2}{16}.$$

Then

$$\int f d\alpha = \int f d\alpha_1 + \int f d\alpha_2 = \frac{5\pi^2}{16} - 2.$$

4. Let $f \colon [-\pi, \pi]$ be a continuously differentiable function. Define a sequence $\{\hat{f}(n)\}_{n \in \mathbb{Z}}$ via

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

What, if anything, can you say about

- (a) the boundedness of the sequence $\{\hat{f}(n)\}$?
- (b) the relation between the sequences $\{\hat{f}(n)\}$ and $\{\widehat{f'}(n)\}$?
- (c) the limiting behavior of $\{\hat{f}(n)\}$ as $n \to \infty$?

Sketch solutions:

(a) Note that

$$|\hat{f}(n)| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-nx} dx \right| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)||e^{-inx}| dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)| dx$$

and the integral on the right is finite since $|f|$ is a continuous function. Hence $\{\hat{f}(n)\}_{n \in \mathbb{Z}}$ is a bounded sequence.

(b) The formulation of the problem was meant to include the requirement that f be periodic. Then, by integration by parts, we get

$$\widehat{f'}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f'(x) e^{-inx} dx = \frac{1}{2\pi} [f(x) e^{-inx}]_{-\pi}^{\pi} + \frac{in}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

By periodicity, $\widehat{f'}(n) = in\hat{f}(n)$.

(c) Note that the argument from (a) shows that $\{\widehat{f'}(n)\}$ is a bounded sequence. Then,

$$|\hat{f}(n)| = \left| \frac{1}{n} \widehat{f'}(n) \right| \leq \frac{C}{n}$$

Hence $\lim_{n \to \infty} \hat{f}(n) = 0$.