STOCKHOLM UNIVERSITY
Department of Mathematics
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Final written exam
Foundations of Analysis
MM5021 7.5 ECTS
9 August 2023

Please READ CAREFULLY the general instructions:

- During the exam you CAN NOT use any textbook, class notes, or any other supporting material.
- Calculators are **not allowed** during the exam.
- In all your solutions show your reasoning, explaining carefully what you are doing. Justify your answers.
- Use natural language, not just mathematical symbols.
- Use clear and legible writing. Write preferably with a ball-pen or a pen (black or dark blue ink).
- Don't write two exercises in the same page.

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1. Let (X,d) be a metric space and assume that $(x_n)_n$ is a sequence in that metric space satisfying that the subsequences $(x_{2n})_n$, $(x_{2n+1})_n$ and $(x_{3n})_n$ converge, with respective limits ℓ_1, ℓ_2 and ℓ_3.

   (a) [**2 pt**] Prove that $\ell_1 = \ell_2 = \ell_3$;

   (b) [**2 pt**] Prove that the sequence $(x_n)_n$ converges.

2. Let (X, d) be a metric space.

   (a) [**2 pt**] Fix $x_0 \in X$ and $\delta > 0$. Define
   $$E_<^\delta = \{x \in X : \mathrm{d}(x,x_0) < \delta\} \quad \text{and} \quad E_>^\delta = \{x \in X : \mathrm{d}(x,x_0) > \delta\}.$$
   Prove in detail that these two sets are open.

   (b) [**2 pt**] Prove that if E_1, E_2 are disjoint non-empty open subsets of X, then they are separated.

   (c) [**1 pt**] Prove that every connected metric space with at least two different points is uncountable. *Hint: Use parts (a) and (b).*

3. [**3 pt**] Let $(\phi_n)_n$ be a sequence of smooth real-valued functions with the property that for all $n \geq 1$:

   (i) $\phi_n(t) \geq 0$, (ii) $\phi_n(t) = 0$ for $|t| \geq \frac{1}{n}$, (iii) $\int_{-1}^{1} \phi_n(t) dt = 1$.

   Let $f : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function on $\mathbb{R}$. Define, for all $n \geq 1$ and for $x \in \mathbb{R}$
   $$f_n(x) := \int_{-1}^{1} f(x-t)\phi_n(t)\mathrm{d}t.$$
   Show that f_n converges uniformly to f. Write explicitly where you use the uniform continuity of f.

4. [**3pt**] Consider the series of functions given by
   $$\sum_{n\geq 1} \frac{(-1)^n}{n^3} \cos nx.$$
   Show that, as a function of x, it is continuous and differentiable on $\mathbb{R}$, and its derivative is also continuous on $\mathbb{R}$. Argument fully your answer.

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5. (a) **[2pt]** Let $a < b$ be two real numbers, and α be a monotonically increasing function on $[a,b]$. Let P be a partition of $[a,b]$, and let P^* be a refinement of P. Prove that for every bounded function $f : [a,b] \to \mathbb{R}$

$$U(P, f, \alpha) \geq U(P^*, f, \alpha).$$

(b) **[2pt]** Give a definition of upper limit of a sequence $(x_n)_n$ of real numbers. Calculate, using the definition you gave, the upper limit of the sequence $(x_n)_n$ where

$$x_n := \frac{1}{n^2} + (-1)^n, \quad n \geq 1.$$

6. Determine which of the following statements are true, and which are false. Explain your reasoning, by giving a proof or a counterexample to each statement. Each answer is graded over one point.

i. Let $E \subset \mathbb{R}$ be bounded, nonempty, and suppose $\sup E \notin E$. Then the set E is infinite.

ii. If a sequence $(x_n)_n$ in a metric space (X, d) satisfies that $\mathrm{d}(x_n, x_{n+1}) < \frac{1}{n^2}$ for all $n \geq 1$, then it is Cauchy.

iii. Let Y be a metric space. If $f : (0,1) \to Y$ is a continuous function between $(0,1)$ and Y, and $E \subset Y$ is closed, then $f^{-1}(E)$ is a closed subset of $\mathbb{R}$.

iv. The set S of all infinite sequences $(s_n)_{n \geq 1}$ with $s_n \in \{☺, ☹\}$ is uncountable.

v. No closed set in a metric space can be written as an intersection of open sets.

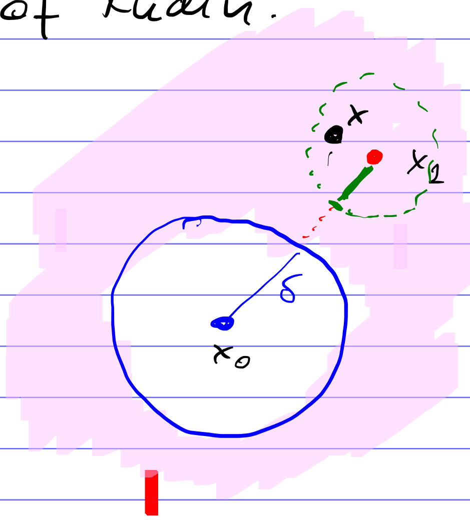
Hints to solutions

1) a) Consider the subsequences $(x_{6n})_n$ and $(x_{6n+3})_n$ and appeal to the fact that if a sequence converges so it does any subsequence and has the same limit.

b) Use the definition of convergence and what is proven in a)

2) See problem 19 in Chapter 2 of Rudin.

3) a) If $E_> = \emptyset$, it's open. So WLOG we can assume that $E_> \neq \emptyset$.
Given $x_1 \in E_>$ we know that
$$d(x_1, x_0) > \delta$$
Let. $\varepsilon := (d(x_1, x_0) - \delta)/2 > 0$



$E_< = N_r(x_0)$ is open (seen in the course)

b) It suffices to prove $\overline{E_1} \cap E_2 = \emptyset$

Note
$$\overline{E_1} \cap E_2 = E_1' \cap E_2$$

Show that if $x \in E_1' \cap E_2$, then $E_1 \cap E_2 \neq \emptyset$ leading to a contradiction

c) Let $\delta \in \left(0, \frac{d(x_0, x_1)}{2}\right)$ with $x_0, x_1 \in X$
$x_0 \neq x_1$.

So
$$X = E_<^\delta \cup E_>^\delta \cup \{x \in X : d(x_0, x) = \delta\}$$

Since $E_<^\delta$ and $E_>^\delta$ are separated, non empty open sets then
$$\exists x_\delta : d(x_0, x_\delta) = \delta$$

Since $\left(0, \frac{d(x_0, x_1)}{2}\right)$ is uncountable, we deduce that X contains an uncountable number of elements

3) Check the proof of the Stone-Weierstrass theorem and follow the strategy of the proof.

4) Use the Weierstrass M-theorem and Theorem 7.12 to show the continuity
Use theorem 7.17 to prove the differentiability and to calculate the derivative

5) a) See Theorem 6.4. in the coursebook
b) Determine the set of all subsequential limits and use the definition to calculate the $\limsup_n x_n$

6) i) TRUE
ii) TRUE
iii) FALSE
iv) TRUE
v) FALSE