STOCKHOLM UNIVERSITY
Department of Mathematics
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Final written exam
Foundations of Analysis
MM5021 7.5 ECTS
13 January 2024

Please READ CAREFULLY the general instructions:

- During the exam you CAN NOT use any textbook, class notes, or any other supporting material.
- Calculators are **not allowed** during the exam.
- In all your solutions show your reasoning, explaining carefully what you are doing. Justify your answers.
- Use natural language, not just mathematical symbols.
- Use clear and legible writing. Write preferably with a ball-pen or a pen (black or dark blue ink).
- **Do not write two exercises on the same page.**
- The written exam gives up to 24 points. A minimum of 15 points (including bonus points) guarantees a pass grade. A minimum of 21 points in the written exam guarantees access to the oral exam.
- The list of the anonymous codes of those who qualify for the oral exam will be published on the course page by Friday 19/1 the latest. The oral exam will take place in Albano premises on **Thursday 25/1** (and eventually also Friday 26/1).

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1. Determine if the following sets are countable or not:
   (a) **[2pt]** The family of all functions from $\mathbb{N}$ to $\{0,1,2\}$ i.e. $\{f : f : \mathbb{N} \to \{0,1,2\}\}$.
   (b) **[1pt]** The set of all affine lines in the plane, which are perpendicular to the vector $(-2,1)$ and have a nonempty intersection with the set $\{(x,y) : x = 0,\ y \in \mathbb{Q}\}$.

2. Let (X,d) be a metric space, and let E, K be two subsets of X.
   (a) **[1pt]** Define what it means for a set $K \subset X$ to be compact.
   (b) **[1pt]** Define what it means to be a limit point of E.
   (c) **[2pt]** Let $E \subset K$ and assume that K is compact. Prove in detail that E has no limit point in K if, and only if E is finite.

3. **[3pt]** Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in a complete metric space (X,d), for which there exists a positive constant $C < 1$ such that for all $n \geq 2$,
$$\mathrm{d}(x_{n+1},x_n) \leq C\mathrm{d}(x_n,x_{n-1}).$$
   Prove that the sequence $\{x_n\}_{n=1}^{\infty}$ is convergent.

4. (a) **[3pt]** Given $n \in \mathbb{N}$, let $f_n(x) = x(1-x^2)^n$ defined for $x \in [0,1]$. Study the pointwise and uniform convergence of the sequence $\{f_n\}_{n\geq 1}$ on $[0,1]$.
   (b) **[2pt]** Let $\{f_n\}_{n\geq 1}$ be a sequence of real-valued bounded functions defined on $[0,1]$ for which
$$\limsup_n \left(\|f_n\|_\infty \left(n(\log n)^2\right)\right) = C \in [0,\infty),$$
   where $\|f_n\|_\infty = \sup_{x\in[0,1]} |f(x)|$. Prove that the series $\sum f_n$ converges uniformly.

5. **[3pt]** Consider the following system of equations
$$\begin{cases} xu + y^2 v = 0 \\ xv^4 - y^3 u^5 = 0. \end{cases}$$
   Show that near the point $(x_0,y_0,u_0,v_0) = (1,-1,0,1)$ we can write u,v as a $\mathcal{C}^1$ function of x,y. Calculate $\frac{\partial v}{\partial y}(1,-1)$.

Please turn page ⟶

6. Determine which of the following statements are true and which are false. Explain your reasoning, by giving a proof or a counterexample to each statement. Each answer is graded over one point. Please summarise your answers in the form that you can find in the cover sheet.

   i. There exists two increasing functions $g, h : [0, 2024] \to \mathbb{R}$ such that the function $f(x) = g(x) - h(x)$ has an uncountable number of discontinuities.

   ii. If $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = 1/(2+x^2)$ then, for every closed set $K \subset \mathbb{R}$, $f^{-1}(K)$ is a closed set of $\mathbb{R}$.

   iii. Let $a < b$ be two real numbers. There exist two bounded functions $f, g : [a, b] \to \mathbb{R}$ for which $\underline{\int_a^b}(f+g)\mathrm{d}x > \underline{\int_a^b} f dx + \underline{\int_a^b} g\mathrm{d}x$.

   iv. If the power series with complex coefficients $\sum_{k\geq 0} a_k z^k$ converges for $z = 5i$, it also converges for $z = 1 - 2i$.

   v. For any real-valued continuous function $f : [0, 1] \to \mathbb{R}$, there exists a real-valued polynomial P such that for all $t \in [0, 1]$, $-(10^{-2024} + 2P(t)) \leq f(t) \leq 10^{-2024} - 2P(t)$.

   vi. Let $f \in \mathcal{R}(\alpha)$ in $[a, b]$ for a given increasing and bounded function α. Then, there exists a partition P of $[a, b]$, $a = x_0 < x_1 < \ldots < x_n = b$, such that for all $i \in \{1, \ldots, n\}$ and for all $s_i, t_i \in [x_{i-1}, x_i]$ we have that $\sum_{i=1}^{n} |f(t_i) - f(s_i)|\Delta\alpha_i \leq 10^{-2024}$.

MM5021 - Hints for solutions. 240113.

1) a) No.

b) Yes

2) a) See the coursebook

b) See the coursebook

c) Use the definition of limit point to show that if E is finite then E has no limit point.

For the other implication, use theorem 2.37. (the proof of it)

(3) Check the proof of theorem 9.23

(4) (a) Show that it converges pointwise and uniformly to zero.

(b) Use the alternative definition of limsup, the M-test of Weierstrass (Thm. 7.10) and thm. 3.29 to deduce the uniform convergence of the series.

(5) Use the implicit function theorem.

(6) i) True

ii) False

iii) True.

iv) True

v) True.

vi) True.