MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
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Make up assignment
MM5021 Foundations of Analysis
7.5 hp
August 8th, 2024

Please read carefully the general instructions:

- During the exam any textbook, class notes, or any other supporting material is forbidden.
- In particular, calculators are not allowed during the exam.
- In all your solutions show your reasoning, explaining carefully what you are doing. Justify your answers. A correct answer without proper justification will not award full points.
- Use natural language, not just mathematical symbols.
- Use clear and legible writing. Write preferably with a ball-pen or a pen (black or dark blue ink).
- A maximum score of 24 points can be achieved.

GOOD LUCK!

1. **Cardinality** Classify each of the following sets as either FINITE, COUNTABLE, or UNCOUNTABLE, providing a brief justification for your answer

   (a) (1 pt) $\mathbb{Q} \cap (0, 1)$

   (b) (1 pt) The power set of the Natural numbers $\mathcal{P}(\mathbb{N})$.

   (c) (1 pt) The power set of $\{1, 2, 3\}$.

   (d) (1 pt) The set of sequences a_n on the set $\{0, 1\}$ such that there is $N \in \mathbb{Z}$ (every sequence has its own N), $N \geq 0$ such that $a_n = 0$ for all $n \geq N$.

2. **Topology of metric spaces** In this exercise you can assume that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{|x|}{|x|+1}$ is continuous when we consider the Euclidean distance in both domain and codomain. Observe that $\lim_{x \to 0} g(x) = 0$.
   Consider the real line with the following distance function

$$d(x, y) = \frac{|x - y|}{|x - y| + 1}$$

   (a) (1 pt) Show that every $E \subseteq \mathbb{R}$ is bounded with respect of the distance d.

   (b) (2 pts) Show that a sequence a_n is Cauchy with respect to d if and only if it is so with respect the Euclidean distance $d_E(x, y) = |x - y|$.

   (c) (2 pts) Deduce from (b) that $(\mathbb{R}, d)$ is a complete metric space.

   (d) (2 pts) Show that neighbourhood with respect to d are open with respect the Euclidean distance $d_E(x, y) = |x - y|$.

3. **Stieltjes integral.** (3 pts) Let $\alpha : [0, 1] \to \mathbb{R}$ the function defined by

$$\alpha(x) = \begin{cases} 30 + 9x^2 & \text{if } x \leq \frac{1}{3}, \\ 50 + 9x^2 & \text{if } \frac{2}{3} \geq x > \frac{1}{3}, \\ 100 + 9x^2 & \text{if } x > \frac{2}{3}. \end{cases}$$

   Show that $f(x) = x \in \mathcal{R}(\alpha)$ on $[0, 1]$ and compute

$$\int_0^1 x d\alpha.$$

4. **Series of functions** Let (X, d_X) and consider $\mathbb{R}$ with the Euclidean distance function $d_E(x, y) = |x-y|$ functions $f_n : E \to \mathbb{R}$ with $E \subseteq X$. Consider also $f : E \to \mathbb{R}$ another function.

   (a) (1 pt) Define formally, using ε and N, what it means that the series of function s$\sum_{n=1}^{+\infty} f_n$ converges pointwise to f.

   (b) (2 pts) Use the ratio test to show that the power series

$$\sum_{n=0}^{+\infty} \frac{x^n}{n!}$$

   converges (to e^x) for all $x \in \mathbb{R}$.

   (c) (2 pts) Show that the series

$$\sum_{n=0}^{+\infty} \frac{x^{2n}}{n!}$$

   converges uniformly (to e^{x^2}) for all $x \in [0, 1]$.

(d) (2 pts) Show that

$$\int_0^1 e^{x^2}\,dx = \sum_{n=0}^{+\infty} \frac{1}{(2n+1)n!}$$

In order to get points you have to justify every step.

5. **Implicit functions** (3 pts) Let us consider the function $F : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$F(x, y, z) = (x + \ln(y) + 2z - 2, x + y^2 + e^z - 1 - e).$$

Show that around the point (0,1,1) $F(x, y, z) = 0$ defines x and y as a functions of z, that is $(x, y) = f(z)$, with $f : \mathbb{R} \to \mathbb{R}^2$. Compute $f'(1)$.