Exam MM5021 08 AGUSTI

Exercise 1

(a) Countable

$\mathbb{Q}\cap(0,1) \hookrightarrow \mathbb{Q}$ Thus $\mathbb{Q}\cap(0,1)$ is at most countable. We show that it is not finite

for all $n\geq 1$ $\frac{1}{n} \in \mathbb{Q}\cap(0,1)$

There are infinitely many elements in $\mathbb{Q}\cap(0,1)$

(b) $\mathcal{P}(\mathbb{N})$ uncountable

$\mathcal{P}(\mathbb{N}) \supseteq \{n\}$ not finite. Now we prove

$|\mathcal{P}(\mathbb{N})| = |[0,1]|$

$S = \{a_1, a_2, \dots a_3 \dots\} \longmapsto \sum_k 2^{-a_k}$

positions of 1 in the binary expansion $\longleftarrow$ x real number

Alternatively $\varphi : \mathcal{P}(\mathbb{N}) \xrightarrow{\sim} \{\text{sequences in } \{0,1\}\}$

$A \longmapsto \varphi_A : \mathbb{N} \to \{0,1\}$

$\varphi_A(n) = 1 \iff n \in A$

(c) From easy combinatoric we know that the FINITE number of subset of $\{1,2,3\}$ with k elements is $\binom{3}{k}$

$|\mathcal{P}(\{1,2,3\})| = \binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 1+3+3+1 = 8$

which is finite

(d) COUNTABLE

We can see that this is not uncountable. Call A this set and let

$f : A \longrightarrow \mathbb{N}$

$\{a_n\} \longmapsto \sum_n a_i \cdot 2^i$

is injective because a natural # has

only one binary expansion.

~~Byo~~ We can see that this is bijective.
~~Or Alternatively~~ Or one can note that for all $k \in \mathbb{N}$

$$S_k = \{a_n\} \qquad a_i = \begin{cases} 0 & i \neq k \\ 1 & i = k \end{cases}$$

gives ∞-many distinct elements

Exercise 2

(a) Let $x \in \mathbb{R}$ $\quad d(x,0) = \frac{|x|}{|x|+1} < 1$

$\mathbb{R} \subseteq N_1(0)$ is bounded

$\Rightarrow E \subseteq \mathbb{R} \Rightarrow E \subseteq N_1(0)$ is bounded

(b) Suppose that a_n is Cauchy with respect to d_E. Fix $\varepsilon > 0$

Observe that the function

$g(x) = \frac{|x|}{|x|+1}$ is continuous in $x = 0$

and has $g(0) = 0$. In particular

we can assume $\frac{|x|}{|x|+1} < \varepsilon$ when $|x| < \delta$

Since a_n is Cauchy there is an $N > 0$ such that $|a_n - a_m| < \delta$ for all $n, m > N$

we conclude that

$$\frac{|a_n - a_m|}{|a_n - a_m| + 1} < \varepsilon \quad \text{for all } n, m > N.$$

Conversely let a_n to be cauchy wrt d And fix $\varepsilon > 0$. Then there is an $N > 0$ Such that

$$d(a_n, a_m) = \frac{|a_n - a_m|}{|a_n - a_m| + 1} \overset{(1)}{<} \frac{\varepsilon}{\varepsilon + 1}$$

for all $n, m \geq N$. But

(1) is equivalent to

$$|a_n - a_m| < \varepsilon$$

for all $n, m > N$.

(c) & (d) in another page.

<u>Exercise 3</u> we have that $f(x) = x$ is continuous and so in $R(\alpha)$

we note that

$$\alpha(x) = 20\, I(x - \tfrac{1}{3}) + 50\, I(x - \tfrac{2}{3}) + 30 + 9x^2$$

we use linearity

$$\int_0^1 x\, d\alpha = \int_0^1 x\, d(30 + 9x^2) + 20\int_0^1 x\, dI(x - \tfrac{1}{3}) + 50\int_0^1 x\, I(x - \tfrac{2}{3})$$

$$= \int_0^1 18x^2 dx + 20 \cdot \frac{1}{3} + 50\, \frac{2}{3}$$

$$= 6 + \frac{20}{3} + \frac{100}{3} = 46$$

Exercise 4

(a) $\forall x$ $\forall \varepsilon > 0$ $\exists$ δ ~~not depending on x~~ possibly depending of x

such that for all $k > \delta$ ~~[illegible]~~

~~[illegible]~~ $\left| \sum_{n=1}^{k} f_n(x) - f(x) \right| < \varepsilon$ ~~[illegible]~~

(b) $\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \xrightarrow[n \to +\infty]{} 0 < 1$

for all $x \in \mathbb{R}$ ~~[illegible]~~

· Thus $\limsup_{n} \frac{x^{n+1}}{(n+1)!} \frac{n!}{x^n} < 1$ for all $x \in \mathbb{R}$

and the series converges

(c) $[0,1]$ is compact.

- for all k we have $g_k := \sum_{n=0}^{k} \frac{x^{2n}}{n!}$ is a continuous function
- for all $x \in [0,1]$ $\{g_k(x)\}$ is a monotone sequence

$\Longrightarrow$ g_k converges uniformly on $[0,1]$

(d) We use uniform convergence and the fact that we can exchange the sign of limits and $\int$

$$\int_0^1 e^{x^2} = \int_0^1 \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} dx \overset{\text{unif conv}}{=} \sum_{n=0}^{\infty} \int_0^1 \frac{x^{2n}}{n!} dx$$

$$= \sum_{n=0}^{\infty} \left[\frac{1}{2n+1} x^{2n+1} \frac{1}{n!} \right]_0^1 = \sum_{n=0}^{\infty} \frac{1}{(2n+1)} \frac{1}{n!}$$

Exercise 2

(c) Let $\{a_n\}$ be cauchy in $(\mathbb{R}, d)$

$\Longrightarrow$ by (b) it is cauchy in $(\mathbb{R}, d_E)$ which is complete so $a_n \longrightarrow a$ in $\mathbb{R}\, d_E$

we show that $a_n \to a$ also in $(\mathbb{R}, d)$

Let $\varepsilon > 0$ (assume $\varepsilon < 1$) since all points have distance and chose N such that

$$|a_n - a| \overset{(2)}{<} \frac{\varepsilon}{1-\varepsilon} \quad \text{for all } n > N$$

$$(2) \iff \frac{|a_n - a|}{|a_n - a| + 1} < \varepsilon \quad \text{for all } n > N.$$

So that $a_n \to a$ in $(\mathbb{R}, d)$

(d) Let $N_r(p)$ a neighbourhood in $(\mathbb{R}, d)$

Let $q \in N_r(p)$ we want to show that there is a $\delta > 0$

$$|x - q| < \delta \implies d(x, p) < r$$

Let $\rho := r - d(p, q) > 0$

Since $\frac{|x|}{|x|+1}$ is continuous in $x = 0$ and in 0 has value 0 we can assume that there is $\delta > 0$ such that

$$d(x, q) = \frac{|x-q|}{|x-q|+1} < \rho \quad \text{if } |x - q| < \delta$$

this implies $d(x, p) < \rho + d(p, q) < r$

for all x with $|x - q| < \delta$

$\Rightarrow N_\delta^E(x) \subseteq N_r(p)$

Exercise 6 we compute the Jacobian matrix of F:

$$\begin{pmatrix} 1 & \frac{1}{y} & 2 \\ 1 & 2y & e^{z} \end{pmatrix}$$

in (0,1,1) we have that this is

$$\begin{pmatrix} \underbrace{\begin{matrix} 1 & 1 \\ 1 & 2 \end{matrix}} & \begin{matrix} 2 \\ e \end{matrix} \end{pmatrix}$$ & get $A_x = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

$A_z = \begin{pmatrix} 2 \\ e \end{pmatrix}$

Since the matrix $A_x = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ is invertible with inverse $\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$ we have that locally around (0,1,1) (x y) can be defined as a function of z. That is

$(x, y) = f(z)$

(This follow from the theorem of the implicit function)

Now we have that

$$f'(1) = -A_x^{-1} \cdot A_z$$

$$= -\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ e \end{pmatrix} = \begin{pmatrix} -4+e \\ 2-e \end{pmatrix}$$