Department of Mathematics
Stockholm University

MA 5021 - Real analysis

Fall 2024 - final exam

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- **No** use of textbook, notes, or calculators is allowed.
- Some problems have multiple parts. You may use the results of an earlier part even if you did not do it.
- Unless told otherwise, you may quote results that you learned during the class. When you do, state precisely the result that you are using.
- Be sure to justify your answers, and show clearly all steps of your solutions.

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1. For each of the following statements determine if it is true or false. Give a brief justification or a counterexample.
   (a) (1 point) Suppose $f\colon [a,b] \to \mathbb{R}$ is a continuous function, and $U \subset [a,b]$ is open in $[a,b]$. Then $f(U)$ is an open subset of $\mathbb{R}$.
   (b) (1 point) Suppose $f\colon [a,b] \to \mathbb{R}$ is a continuous function, and $C \subset [a,b]$ is closed in $[a,b]$. Then $f(C)$ is a closed subset of $\mathbb{R}$.
   (c) (1 point) If $U \subset \mathbb{R}$ is an open subset then $\operatorname{int}(\overline{U}) = U$.
   (d) (1 point) Let $f_n\colon [a,b] \to \mathbb{R}$ be a sequence of differentiable functions. Suppose that $\{f_n\}$ converges uniformly to a differentiable function f. Then $\{f_n'\}$ converges uniformly to f'.

2. (a) (2 points) Let $U \subset \mathbb{R}$ be an interval and let $f\colon U \to \mathbb{R}$ be a continuous 1-1 function. Prove that f is monotonic (either increasing or decreasing).
   (b) (2 points) Let $U \subset \mathbb{R}$ be an open interval, and suppose $f\colon U \to \mathbb{R}$ is a differentiable function. Suppose that exists points $a, b \in U$ such that $f'(a) < 0$ and $f'(b) > 0$. Prove that there exists a point $\xi \in U$ for which $f'(\xi) = 0$. Notice that we are not assuming that f' is continuous. Hint: try to mimic the proof of Rolle's theorem.

3. (4 points) Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a function satisfying the following conditions:
   1. f is continuous.
   2. $f(\bar{0}) > 0$
   3. For all $\bar{x} \in \mathbb{R}^n$, $f(\bar{x}) \le \frac{1}{\|\bar{x}\|^2+1}$.

   Prove that f attains a maximum. That is, prove that there exists an $\bar{x} \in \mathbb{R}^n$ such that $f(\bar{x}) \ge f(\bar{y})$ for all $\bar{y} \in \mathbb{R}^n$.

4. (4 points) Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function and let $\alpha\colon [a,b] \to \mathbb{R}$ be a monotonic increasing function. Let P and Q be two partitions of the interval $[a,b]$. Prove, using just the definitions of upper and lower sums, that
$$L(P, f, \alpha) \le U(Q, f, \alpha).$$

5. (4 points) Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$
$$f(x,y) = \begin{cases} xy\sin(\frac{1}{x^2+y^4}) & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

   Is f differentiable at $(0,0)$? Be sure to justify your answer.

6. (a) (2 points) Show that there exists an open subset $W \subset \mathbb{R}^2$ containing $(1,1)$, and a differentiable function $F = (f_1, f_2)$ from W to $\mathbb{R}^2$, that satisfies

   1. $f_1(1,1) = f_2(1,1) = 1$
   2. For all $(x,y) \in W$, $xy + xf_1(x,y) + yf_2(x,y) = 3$ and $f_1(x,y)^2 f_2(x,y) + x^2 y = 2$.

   (b) (2 points) Show that F is invertible in some neighbourhood of $(1,1)$.