Department of Mathematics
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MA 5021 - Real analysis

Fall 2024 - final exam

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- **No** use of textbook, notes, or calculators is allowed.
- Some problems have multiple parts. You may use the results of an earlier part even if you did not do it.
- Unless told otherwise, you may quote results that you learned during the class. When you do, state precisely the result that you are using.
- Be sure to justify your answers, and show clearly all steps of your solutions.

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1. For each of the following statements determine if it is true or false. Give a brief justification or a counterexample.

   (a) (1 point) Suppose $f\colon [a,b] \to \mathbb{R}$ is a continuous function, and $U \subset [a,b]$ is open in $[a,b]$. Then $f(U)$ is an open subset of $\mathbb{R}$.

   Answer: False. For example, let f be the inclusion function; $f(x) = x$ for all $a \leq x \leq b$. Then $[a,b]$ is an open subset of itself, but $f([a,b]) = [a,b]$ is not an open subset of $\mathbb{R}$.

   (b) (1 point) Suppose $f\colon [a,b] \to \mathbb{R}$ is a continuous function, and $C \subset [a,b]$ is closed in $[a,b]$. Then $f(C)$ is a closed subset of $\mathbb{R}$.

   Answer: True. If C is a closed subset of $[a,b]$ then C is compact, and therefore $f(C)$ is compact, and therefore closed.

   (c) (1 point) If $U \subset \mathbb{R}$ is an open subset then $\operatorname{int}(\overline{U}) = U$.

   Answer: False. For example, let $U = (1,2) \cup (2,3)$. Then $\overline{U} = [1,3]$, and $\operatorname{int}(\overline{U}) = (1,3) \neq U$.

   (d) (1 point) Let $f_n\colon [a,b] \to \mathbb{R}$ be a sequence of differentiable functions. Suppose that $\{f_n\}$ converges uniformly to a differentiable function f. Then $\{f_n'\}$ converges uniformly to f'.

   Answer: False. Let $[a,b]$ be an arbitrary interval, and define $f_n(x) = \frac{\sin(nx)}{n}$. Then it is clear that f_n converges uniformly to the zero function, but the sequence $f_n'(x) = \cos(nx)$ clearly does not converge to the zero function.

2. (a) (2 points) Let $U \subset \mathbb{R}$ be an interval and let $f\colon U \to \mathbb{R}$ be a continuous 1-1 function. Prove that f is monotonic (either increasing or decreasing).

   Answer: Assume that f is continuous and $1-1$. First, we prove that f is monotonic on any closed interval contained in U. Suppose $a, b \in U$, with $a < b$. Since f is $1-1$, $f(a) \neq f(b)$, so either $f(a) < f(b)$ or $f(a) > f(b)$.

   Claim: if $f(a) < f(b)$ then for all $t \in]a,b[$, $f(a) < f(t) < f(b)$, and if $f(a) > f(b)$ then for all $t \in]a,b[$ it holds that $f(a) > f(t) > f(b)$.

   Proof. Suppose, for example, that $f(a) < f(b)$ and $a < t < b$. Suppose, by contradiction, that it is not true that $f(a) < f(t) < f(b)$. Then either $f(t) > f(b)$ or $f(t) < f(a)$. Suppose $f(t) > f(b)$. Then also $f(a) < f(t)$. Choose a z such that

$$\max(f(a), f(b)) < z < f(t).$$

By the intermediate value theorem there exist points p, q such that $a < p < t < q < b$ such that $f(p) = f(q) = z$, contradicting the assumption that f is 1-1. The case when $f(t) < f(a)$ is handled similarly. Finally, the case when $f(a) > f(b)$ is handled similarly as well. This proves the claim. $\square$

It follows from the claim that if $a \leq c < d \leq b$ and $f(a) < f(b)$ then $f(c) < f(d)$ and conversely if $f(a) > f(b)$ then $f(c) > f(d)$. So we have shown that f is monotonic on $[a, b]$, for any closed interval $[a, b] \subset U$. Now we prove that f is monotonic on U. Suppose not. Then there exist points $p < q$ in U such that $f(p) > f(q)$ and points $s < t$ in U such that $f(s) < f(t)$. Let $a = \min(p, q, s, t)$ and $b = \max(p, q, s, t)$. Then f is not monotonic on $[a, b]$, contradicting the claim.

(b) (2 points) Let $U \subset \mathbb{R}$ be an open interval, and suppose $f \colon U \to \mathbb{R}$ is a differentiable function. Suppose that exists points $a, b \in U$ such that $f'(a) < 0$ and $f'(b) > 0$. Prove that there exists a point $\xi \in U$ for which $f'(\xi) = 0$. Notice that we are not assuming that f' is continuous. Hint: try to mimic the proof of Rolle's theorem.

Answer: Let us assume that $a < b$. The case $a > b$ is proved similarly. Since $f'(a) < 0$ it follows that there exists a $\delta > 0$ such that for all $x \in (a, a + \delta)$, $f(x) < f(a)$. Similarly, there exists a $\delta > 0$ such that for all $x \in (b - \delta, b)$, $f(x) < f(b)$.

Since f is continuous on $[a, b]$, there exists a $\xi \in [a, b]$ such that $f(\xi) \leq f(x)$ for all $x \in [a, b]$. It follows from the previous paragraph that in fact $a < \xi < b$. So ξ is a point in (a, b) and f attains a local minimum at ξ. It follows that $f'(\xi) = 0$.

3. (4 points) Let $f \colon \mathbb{R}^n \to \mathbb{R}$ be a function satisfying the following conditions:

   1. f is continuous.
   2. $f(\bar{0}) > 0$
   3. For all $\bar{x} \in \mathbb{R}^n$, $f(\bar{x}) \leq \frac{1}{\|\bar{x}\|^2 + 1}$.

Prove that f attains a maximum. That is, prove that there exists an $\bar{x} \in \mathbb{R}^n$ such that $f(\bar{x}) \geq f(\bar{y})$ for all $\bar{y} \in \mathbb{R}^n$.

Answer: Consider the set

$$D = \left\{ \bar{x} \in \mathbb{R}^n \mid \|\bar{x}\| \leq \frac{1}{\sqrt{f(\bar{0})}} \right\}.$$

Then D is a compact set, and therefore f attains a maximum on D. There exists a point $\bar{x} \in D$ such that $f(\bar{x}) \geq f(\bar{y})$ for all $\bar{y} \in D$. In particular, $f(\bar{x}) \geq f(\bar{0})$.

We claim that $f(\bar{x}) \geq f(\bar{y})$ for all $\bar{y} \in \mathbb{R}^n$. We already know this for $\bar{y} \in D$, so it remains to verify the claim for $\bar{y} \notin D$. In this case $\|\bar{y}\| > \frac{1}{\sqrt{f(\bar{0})}}$, so $\frac{1}{\|\bar{y}\|^2} < f(\bar{0})$, and we have the inequalities

$$f(\bar{y}) \leq \frac{1}{\|\bar{y}\|^2 + 1} < \frac{1}{\|\bar{y}\|^2} < f(\bar{0}) \leq f(\bar{x}).$$

This is what we wanted to prove.

4. (4 points) Let $f \colon [a, b] \to \mathbb{R}$ be a bounded function and let $\alpha \colon [a, b] \to \mathbb{R}$ be a monotonic increasing function. Let P and Q be two partitions of the interval $[a, b]$. Prove, using just the

definitions of upper and lower sums, that

$$L(P,f,\alpha) \le U(Q,f,\alpha).$$

Answer: First we consider the case when P and Q are the same partition. Suppose P consists of points $a = t_0 \le t_1 \le \cdots \le t_n = b$. For $i = 1, \dots n$ let m_i and M_i denote the infimum and the supremum of f on the interval $[t_{i-1}, t_i]$. Also let $\Delta\alpha_i = \alpha(t_i) - \alpha(t_{i-1})$. By definition

$$L(P,f,\alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i, \quad \text{and} \quad U(P,f,\alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i.$$

Since for all i we have $m_i \le M_i$ and $\Delta\alpha_i \ge 0$, it follows that $L(P,f,\alpha) \le U(P,f,\alpha)$.

Next we claim that if P' is a refinement of P then $L(P,f,\alpha) \le L(P',f,\alpha)$ and $U(P,f,\alpha) \ge U(P',f,\alpha)$.

We will prove the claim for lower sums. The proof for upper sums is exactly the same. Recall that P' is a refinement of P if P' is obtained by adding finitely many points to P. It is enough to prove the claim when P' is obtained by adding one point to P.

So, let P be as above, and suppose P' consists of points $a = t_0 \le t_1 \le \cdots \le t_{j-1} \le \xi \le t_j \le \cdots t_n = b$. For all $i = 1, \dots, n$, let $m_i, M_i, \Delta\alpha_i$ be defined as above. Also let m_j', m_j'' be the infimum of f on the intervals $[t_{j-1}, \xi]$ and $[\xi, t_j]$ respectively. Finally, let $\Delta\alpha_j' = \alpha(\xi) - \alpha(x_{j-1})$ and $\Delta\alpha_j'' = \alpha(x_j) - \alpha(\xi)$. It is clear that we have inequalities $m_j \le m_j', m_j''$, and equality $\Delta\alpha_j = \Delta\alpha_j' + \Delta\alpha_j''$.

Now we have the following

$$\begin{aligned} L(P,f,\alpha) &= m_1\Delta\alpha_1 + \cdots + m_j\Delta\alpha_j + \cdots + m_n\Delta\alpha_n = \\ &= m_1\Delta\alpha_1 + \cdots + m_j\Delta\alpha_j' + m_j\Delta\alpha_j'' + \cdots + m_n\Delta\alpha_n \le \\ &\le m_1\Delta\alpha_1 + \cdots + m_j'\Delta\alpha_j' + m_j''\Delta\alpha_j'' + \cdots + m_n\Delta\alpha_n = L(P',f,\alpha). \end{aligned}$$

Finally, given two partitions P and Q, we can form their common refinement $P \cup Q$. We have the following inequalities

$$L(P,f,\alpha) \le L(P\cup Q,f,\alpha) \le U(P\cup Q,f,\alpha) \le U(Q,f,\alpha).$$

5. (4 points) Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} xy\sin\left(\frac{1}{x^2+y^4}\right) & (x,y) \ne (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Is f differentiable at $(0,0)$? Be sure to justify your answer.

Answer: Yes, f is differentiable at $(0,0)$, and moreover $f'(0,0)$ is the zero transformation. To prove this, we have to prove that

$$\lim_{(x,y)\to(0,0)} \frac{|f(x,y) - f(0,0)|}{\sqrt{x^2+y^2}} = 0.$$

This means proving that

$$\lim_{(x,y)\to(0,0)} \frac{\left|xy\sin\left(\frac{1}{x^2+y^4}\right)\right|}{\sqrt{x^2+y^2}} = 0.$$

It follows from the inequalities $(x\pm y)^2 \geq 0$ that $|xy| \leq \frac{x^2+y^2}{2}$. Therefore, we have inequalities

$$0 \leq \frac{\left|xy\sin\left(\frac{1}{x^2+y^4}\right)\right|}{\sqrt{x^2+y^2}} \leq \frac{\left|\sqrt{x^2+y^2}\sin\left(\frac{1}{x^2+y^4}\right)\right|}{2}.$$

Since $\lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2} = 0$, and $\sin\left(\frac{1}{x^2+y^4}\right)$ is bounded, it follows that

$$\lim_{(x,y)\to(0,0)} \frac{\left|\sqrt{x^2+y^2}\sin\left(\frac{1}{x^2+y^4}\right)\right|}{2} = 0$$

and therefore also

$$\lim_{(x,y)\to(0,0)} \frac{\left|xy\sin\left(\frac{1}{x^2+y^4}\right)\right|}{\sqrt{x^2+y^2}} = 0.$$

Remark: Note that we are not saying that f is *continuously* differentiable at $(0,0)$.

6. (a) (2 points) Show that there exists an open subset $W \subset \mathbb{R}^2$ containing $(1,1)$, and a differentiable function $F = (f_1, f_2)$ from W to $\mathbb{R}^2$, that satisfies

   1. $f_1(1,1) = f_2(1,1) = 1$
   2. For all $(x,y) \in W$, $xy + xf_1(x,y) + yf_2(x,y) = 3$ and $f_1(x,y)^2 f_2(x,y) + x^2y = 2$.

   Answer: Let us define the function $G = (g_1, g_2)\colon \mathbb{R}^4 \to \mathbb{R}^2$ as follows

   $$g_1(x,y,u,v) = xy + xu + yv - 3; \quad g_2(x,y,u,v) = u^2v + x^2y - 2.$$

   It is clear that $G(1,1,1,1) = (0,0)$. Let us calculate the derivatives of G with respect to u and v at $(1,1,1,1)$.

   $$\begin{array}{ll} \frac{\partial g_1}{\partial u} = x = 1 & \frac{\partial g_1}{\partial v} = y = 1 \\ \frac{\partial g_2}{\partial u} = 2uv = 2 & \frac{\partial g_2}{\partial v} = u^2 = 1. \end{array}$$

   It is clear that the matrix $\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$ is invertible, and therefore the answer follows by the implicit function theorem.

   (b) (2 points) Show that F is invertible in some neighbourhood of $(1,1)$.

   Answer: Now let us calculate the derivatives of G with respect to x and y at $(1,1,1,1)$.

   $$\begin{array}{ll} \frac{\partial g_1}{\partial x} = y + u = 2 & \frac{\partial g_1}{\partial y} = x + v = 2 \\ \frac{\partial g_2}{\partial x} = 2xy = 2 & \frac{\partial g_2}{\partial y} = x^2 = 1. \end{array}$$

   Once again, the matrix of derivatives $\begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix}$ is invertible. By the implicit function theorem, the matrix of derivatives of the implicitly defined function F at $(1,1)$ is

$$-\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix}.$$

This is a product of invertible matrices, so it is invertible. Moreover, it is clear that F is continuously differentiable near $(1,1)$. Therefore, by the inverse function theorem, F is invertible (i;.e., 1-1) on some open neighbourhood of $(1,1)$.