MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Examinator: Matthew de Courcy-Ireland

Final written exam
Foundations of Analysis
MM5021 7.5 hp
13 August 2025

Please READ CAREFULLY the general instructions:

- During the exam you CANNOT use any textbook, class notes, or any other supporting material.
- Calculators are **not allowed** during the exam.
- In all your solutions show your reasoning, explaining carefully what you are doing. Justify your answers.
- Use clear and legible writing. Write preferably in black or dark blue ink.
- Do not write two exercises on the same page.

✓ The exam comprises four tasks written on both sides of the paper.

✓ The total is 24 points. Each lettered item is worth 1 point unless otherwise indicated. (2a, 2b, and 4d are worth 2 points each.) Show your work as it may be worth partial credit.

✓ You can use earlier items to answer later ones, even without answering the former.

1. True or false? Say which and justify your answer.
(An ideal justification: if true, outline a proof; if false, give a counterexample.)

   (a) For any increasing function $\alpha : [a, b] \to \mathbb{R}$, the Riemann–Stieltjes integral satisfies
   $$\int_a^b 1 \, d\alpha = \alpha(b) - \alpha(a).$$

   (b) The set
   $$\left\{ x \in \mathbb{R} \; ; \; 1 < \int_0^x e^{t^2} dt < 2 \right\}$$
   is an open subset of $\mathbb{R}$.

   (c) If $\sum_{n=0}^{\infty} a_n x^n$ converges for $x = -3$, then it converges for $x = 2$ (where a sequence of real numbers a_n is given).

   (d) If $f : [0, 1] \to \mathbb{R}$ is continuous, then there is a countable set $E \subset [0, 1]$ such that f is differentiable at all $x \in [0, 1] \setminus E$.

   (e) If $f : [0, 1] \to \mathbb{R}$ is monotone, then there is a countable set $E \subset [0, 1]$ such that f is continuous at all $x \in [0, 1] \setminus E$.

   (f) The function ψ defined on $]0, \infty[$ by $\psi(x) = x + \frac{1}{x}$ is a contraction.

2. (a) [**2 points**] Suppose X is a metric space with distance d and let
   $$b(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$
   Show that b is a metric on X.

   (b) [**2 points**] Is the following function m a metric on $\mathbb{R}$?
   $$m(x, y) = \frac{|x - y|}{1 + |x - y|^2}$$

   (c) Show that $[1, 2] \cup [3, 4]$ is a complete metric space, with respect to the usual distance function $|x - y|$.
   (You may use, without proof, the fact that $\mathbb{R}$ is complete.)

   (d) Give an example of a metric space X with a subset $E \subset X$ that is both compact and open, but $E \neq \varnothing$ and $E \neq X$.

[exam continues on next page]

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3. Given $-1 \le x \le 1$, define
$$\varphi(t) = t + \frac{x^2 - t^2}{2}.$$
Define a sequence of functions on $[-1, 1]$ inductively by $p_0(x) = 0$ and
$$p_{n+1}(x) = \varphi\big(p_n(x)\big) = p_n(x) + \frac{x^2 - p_n(x)^2}{2}$$
where $n \ge 0$ is an integer.

   (a) What are the fixed points of φ? (Your answer should depend on x.)

   (b) Show that
$$|x| - p_{n+1}(x) = \Big(|x| - p_n(x)\Big)\Big(1 - \frac{|x| + p_n(x)}{2}\Big)$$

   (c) Show that for all $x \in [-1, 1]$ one has $0 \le p_n(x) \le p_{n+1}(x) \le |x|$.

   (d) On the interval $0 \le u \le 1$, where does $u \cdot (1 - u)^n$ attain its maximum?

   (e) Show that $p_n(x) \to |x|$ uniformly on $[-1, 1]$ as $n \to \infty$.

   (f) Calculate $\lim_{n\to\infty} \int_{-1}^{1} p_n(x)dx$.

4. Let $\varphi(x) = x + \sin(x)$ for $x \in \mathbb{R}$.

   (a) Show that φ is a monotone function from $\mathbb{R}$ to $\mathbb{R}$.

   (b) Show that φ maps the interval $[\frac{2\pi}{3}, \frac{4\pi}{3}]$ to itself.

   (c) Show that φ is a contraction on the interval from (b).

   (d) **[2 points]** Suppose $|\varphi(x) - \varphi(y)| \le c|x - y|$ where $0 \le c < 1$. Let x_n be the sequence defined inductively by
$$x_{n+1} = \varphi(x_n)$$
starting from a given x_0. Let
$$p = \lim_{n\to\infty} x_n$$
(you do not need to show the limit exists).
Show that
$$|p - x_n| \le \frac{c^n}{1 - c}|x_1 - x_0|.$$

   (e) If we start from $x_0 = 3$ and want to make $|x_n - \pi| < 2^{-2025}$, how large should n be?