MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Examinator: Matthew de Courcy-Ireland

Final written exam
Foundations of Analysis
MM5021 7.5 hp
13 August 2025

Please READ CAREFULLY the general instructions:

- During the exam you CANNOT use any textbook, class notes, or any other supporting material.
- Calculators are **not allowed** during the exam.
- In all your solutions show your reasoning, explaining carefully what you are doing. Justify your answers.
- Use clear and legible writing. Write preferably in black or dark blue ink.
- Do not write two exercises on the same page.

✓ The exam comprises four tasks written on both sides of the paper.

✓ The total is 24 points. Each lettered item is worth 1 point unless otherwise indicated. (2a, 2b, and 4d are worth 2 points each.) Show your work as it may be worth partial credit.

✓ You can use earlier items to answer later ones, even without answering the former.

1. True or false? Say which and justify your answer.
(An ideal justification: if true, outline a proof; if false, give a counterexample.)

(a) For any increasing function $\alpha : [a, b] \to \mathbb{R}$, the Riemann–Stieltjes integral satisfies

$$\int_a^b 1 \, d\alpha = \alpha(b) - \alpha(a).$$

(b) The set

$$\left\{ x \in \mathbb{R} \; ; \; 1 < \int_0^x e^{t^2} dt < 2 \right\}$$

is an open subset of $\mathbb{R}$.

(c) If $\sum_{n=0}^{\infty} a_n x^n$ converges for $x = -3$, then it converges for $x = 2$ (where a sequence of real numbers a_n is given).

(d) If $f : [0, 1] \to \mathbb{R}$ is continuous, then there is a countable set $E \subset [0, 1]$ such that f is differentiable at all $x \in [0, 1] \setminus E$.

(e) If $f : [0, 1] \to \mathbb{R}$ is monotone, then there is a countable set $E \subset [0, 1]$ such that f is continuous at all $x \in [0, 1] \setminus E$.

(f) The function ψ defined on $]0, \infty[$ by $\psi(x) = x + \frac{1}{x}$ is a contraction.

Solution to task 1

(a) True. The integral $\int_a^b 1 \, d\alpha$ is a limit of Riemann sums $\sum_i \left(\alpha(x_i) - \alpha(x_{i-1})\right)$ where $a = x_0 \leq x_1 \leq \ldots \leq x_N = b$. This is a telescoping sum, where most terms cancel leaving $\alpha(x_N) - \alpha(x_0) = \alpha(b) - \alpha(a)$.

(b) True. This set is $f^{-1}(V)$ where $V =]1, 2[$ and $f(x) = \int_0^x e^{t^2} dt$. Since V is open and f is continuous, the pre-image $f^{-1}(V)$ is also open.

(c) True. Since the series converges for $x = -3$, the radius of convergence R for this power series satisfies $R \geq 3$. The series converges for all x satisfying $|x| < R$, and in particular for $x = 2$.

(d) False. We know from Rudin that there is a nowhere-differentiable continuous function. See Theorem 7.18 for a proof. □

Just knowing that there is a nowhere-differentiable continuous function was enough for this true/false question. It is of course a good thing if you sketch how the construction goes. The idea is to consider

$$f(x) = \sum_{n=0}^{\infty} \alpha(p^n x) \cdot q^n$$

where p and q are numerical parameters one can play with, and α is a periodic version of the absolute value function: $\alpha(x) = |x|$ for $-1 \leq x \leq 1$ extended periodically by $\alpha(x+2) = \alpha(x)$. We will see that choosing $p = 4$ and any q in the range $\frac{1}{2} < q < 1$ makes f continuous but nowhere differentiable.

We claim f is continuous for any parameter in the range $0 < q < 1$. Since $|\alpha(x)| \leq 1$ and $\sum q^n$ converges, the Weierstrass M-test shows that the series converges uniformly (Theorem 7.10). Therefore, since α and its rescalings are continuous, the series f is also continuous.

However, given any point x, estimates of the difference quotients show that f is not differentiable at x. There is a sequence of values $h \to 0$ for which

$$\left|\frac{f(x+h) - f(x)}{h}\right| \to \infty$$

whereas these would converge to $|f'(x)|$ if f were differentiable at x.

In more detail (much more than necessary to get the 1 point!), the proof of Theorem 7.18 uses two key properties of α. By periodicity, $\alpha(y) = 0$ for all even integers $y \in 2\mathbb{Z}$, and from properties of the absolute value, we also have

$$|\alpha(s) - \alpha(t)| \leq |s - t|$$

for any $s, t \in \mathbb{R}$. Let us now fix x and show f is not differentiable there. The difference quotients are

$$\frac{f(x+h) - f(x)}{h} = \sum_{n=0}^{\infty} q^n \frac{\big(\alpha(p^n x + p^n h) - \alpha(p^n x)\big)}{h}$$

The numerator vanishes by periodicity provided that $p^n h \in 2\mathbb{Z}$. Let us therefore choose

$$p = 4$$

and take a sequence of values

$$h = h_k = \pm\frac{1}{2}4^{-k}, \quad k = 1, 2, 3, \ldots$$

The sign is chosen depending on k so that there are no integers between $4^k x$ and $4^k(x+h)$, in other words, depending on whether $4^k x$ must be rounded up or rounded down to the nearest integer. Then the terms in the series vanish for $n > k$, while the term for $n = k$ is

$$q^k \frac{\big(\alpha(p^k x + p^k h) - \alpha(p^k x)\big)}{h} = \pm q^k \frac{p^k h}{h} = \pm q^k 4^k$$

because the choice of $\pm$ means α is evaluated within the same period for both terms. For $n < k$, we can bound the remaining terms by

$$\left| q^n \frac{\big(\alpha(p^n x + p^n h) - \alpha(p^n x)\big)}{h} \right| \leq q^n \frac{|(p^n x + p^n h) - p^n x|}{|h|} = (4q)^n.$$

By the triangle inequality,

$$\left| \sum_{n=0}^{k} q^n \frac{\big(\alpha(p^n x + p^n h) - \alpha(p^n x)\big)}{h} \right| \geq (4q)^k - \sum_{n=0}^{k-1} (4q)^n$$

which leaves a geometric series. Let use choose q so that $4q > 1$, that is, any parameter in the range $\frac{1}{4} < q < 1$. We conclude from the formula for a geometric series that

$$\left|\frac{f(x+h) - f(x)}{h}\right| \geq (4q)^k - \frac{(4q)^k - 1}{4q - 1} = \Big(1 - \frac{1}{4q-1}\Big)(4q)^k + \frac{1}{4q-1}.$$

This diverges as $k \to \infty$, as long as $4q - 1 > 1$. The simplest choice is Rudin's $q = \frac{3}{4}$, so that $4q - 1 = 2 > 1$. Any choice in the range $\frac{1}{2} < q < 1$ works.

(e) True. This is Theorem 4.30 in the text by Rudin. We let E be the set of all points where f is not continuous, and show E is countable. The proof uses two key facts: $\mathbb{Q}$ is a countable dense subset of $\mathbb{R}$, and monotone functions have limits from the left and from the right. We may assume f is increasing rather than decreasing, and then

$$\lim_{x\to b^-} f(x) < \lim_{x\to b^+} f(x)$$

for each point b where f is not continuous. The fact that $\mathbb{Q}$ is dense means there is a rational number $q(b)$ such that

$$\lim_{x\to b^-} f(x) < q(b) < \lim_{x\to b^+} f(x)$$

and although $q(b)$ is not unique, we choose one such rational number for each discontinuity b. Since f is increasing, q is also increasing and in particular $q(b) \neq q(b')$ for $b \neq b'$. Therefore E can be mapped injectively into a subset of $\mathbb{Q}$, which shows that E is countable.

(f) False. A contraction would satisfy $|\psi(x) - \psi(y)| \leq c|x - y|$ for some $c < 1$ and all $x, y \in]0, \infty[$. In this case

$$\psi(x) - \psi(y) = x - y + \frac{1}{x} - \frac{1}{y} = \Big(x - y\Big)\Big(1 - \frac{1}{xy}\Big)$$

which shows that ψ is not a contraction. For any $c < 1$, there are x and y for which $|\psi(x) - \psi(y)| > c|x - y|$. Indeed

$$1 - \frac{1}{xy} > c \Longleftrightarrow xy > \frac{1}{1-c}$$

so for example the contraction property would fail for $y = 1$ and $x = \frac{2}{1-c}$. □

There is a tempting alternative that does not quite work but is worth commenting on. You could try to argue that there are no solutions to $\psi(x) = x$ since $1/x \neq 0$, whereas a contraction mapping on a complete space must have a fixed point. However $]0, \infty[$ is not complete (a Cauchy sequence might converge to 0, which is not in the space) so the contraction mapping theorem would not apply even if ψ were a contraction. You could argue this way for a function defined by the same formula on $[1, \infty[$ instead of $]0, \infty[$.

□

2. (a) **[2 points]** Suppose X is a metric space with distance d and let

$$b(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

Show that b is a metric on X.

(b) **[2 points]** Is the following function m a metric on $\mathbb{R}$?

$$m(x, y) = \frac{|x - y|}{1 + |x - y|^2}$$

(c) Show that $[1, 2] \cup [3, 4]$ is a complete metric space, with respect to the usual distance function $|x - y|$.

(You may use, without proof, the fact that $\mathbb{R}$ is complete.)

(d) Give an example of a metric space X with a subset $E \subset X$ that is both compact and open, but $E \neq \varnothing$ and $E \neq X$.

Solution to task 2

(a) Since d is a distance, we have the symmetry property $d(x, y) = d(y, x)$ and the inequality $d(x, y) \geq 0$ with equality only for $x = y$. The same properties are inherited by b. It remains to verify the triangle inequality

$$b(x, y) \leq b(x, z) + b(z, y)$$

where x, y, z are any points in X.

Since d is a distance on X, we have

$$d(x, y) \leq d(x, z) + d(z, y).$$

Let us abbreviate

$$p = d(x, y), \quad q = d(x, z), \quad r = d(z, y)$$

which are all non-negative and satisfy $p \leq q + r$ because d is a metric. We must show

$$\frac{p}{1+p} \leq \frac{q}{1+q} + \frac{r}{1+r}$$

This claim is equivalent to

$$p(1+q)(1+r) \leq (1+p)\big(q(1+r) + r(1+q)\big)$$

that is

$$p + pq + pr + pqr \leq q + r + 2qr + pq + pr + 2pqr$$

or most simply

$$p \leq q + r + 2qr + pqr$$

This follows from $p \leq q + r$ because $2qr + pqr \geq 0$.

(b) No, m is not a metric on $\mathbb{R}$ because it does not satisfy the triangle inequality. Suppose, for a contradiction, that

$$m(x, y) \leq m(x, z) + m(z, y)$$

for all x, y, z in $\mathbb{R}$. Fix any $x \neq y$. Then as $z \to \infty$ the inequality gives

$$\frac{|x-y|}{1+|x-y|^2} \leq \frac{|x-z|}{1+|x-z|^2} + \frac{|z-y|}{1+|z-y|^2} \to 0$$

so $|x - y| = 0$, contrary to $x \neq y$. $\square$

Alternatively, instead of a proof by contradiction, one can give specific values of x, y, z for which the triangle inequality fails. One such example is

$$x = 0, \quad y = 1, \quad z = 5.$$

For those values,

$$\frac{|x-y|}{1+|x-y|^2} = \frac{1}{2}$$

compared to

$$\frac{|x-z|}{1+|x-z|^2} + \frac{|z-y|}{1+|z-y|^2} = \frac{5}{26} + \frac{4}{17} < \frac{1}{2}$$

Indeed, after multiplying by 2, the final inequality holds if and only if

$$\frac{5}{13} + \frac{8}{17} < 1 \iff 5 \times 17 + 13 \times 8 < 13 \times 17 \iff 5 \times 17 < 13 \times 9$$

which follows from $85 < 117$. We could also calculate in decimals that

$$\frac{5}{26} + \frac{4}{17} = \frac{189}{442} = 0.4276\ldots < 0.5$$

Choosing a larger value of z gives larger denominators but makes the result less than $\frac{1}{2}$ by an even wider margin. $\square$

(c) Let x_n be a Cauchy sequence in $[1,2]\cup[3,4]$. The same sequence also has the Cauchy property in $\mathbb{R}$ because we are using the same distance function. By the completeness of $\mathbb{R}$, there is some $x \in \mathbb{R}$ so that $x_n \to x$ as $n \to \infty$. Since $[1,2]\cup[3,4]$ is a finite union of closed intervals, it is a closed subset of $\mathbb{R}$. Therefore any sequence in $[1,2]\cup[3,4]$ has its limit points also in $[1,2]\cup[3,4]$. Thus the limit x shows x_n converges not only in $\mathbb{R}$ but also in the original space $[1,2]\cup[3,4]$.

We have shown all Cauchy sequences in $[1,2]\cup[3,4]$ are convergent, which is the definition of completeness.

(d) For example take $X = [1,2]\cup[3,4]$ and $E = [1,2]$. It's clear enough that E is both open and closed. Compactness follows from the Heine–Borel theorem since E is closed and bounded.

(d) Alternatively: let X be a discrete space and let E be any finite subset other than $\varnothing$ and (in case X is finite) X itself.

(d) Pitfall: several answers incorrectly proposed to let E be the Cantor set in $X = [0,1]$. This is a compact subset but not open. The Cantor set has empty interior. In that sense, it is as far as possible from being open. An open set contains a neighbourhood of every one of its points, whereas the Cantor set does not contain any.

□

3. Given $-1 \le x \le 1$, define

$$\varphi(t) = t + \frac{x^2 - t^2}{2}.$$

Define a sequence of functions on $[-1,1]$ inductively by $p_0(x) = 0$ and

$$p_{n+1}(x) = \varphi\big(p_n(x)\big) = p_n(x) + \frac{x^2 - p_n(x)^2}{2}$$

where $n \ge 0$ is an integer.

(a) What are the fixed points of φ? (Your answer should depend on x.)

(b) Show that

$$|x| - p_{n+1}(x) = \Big(|x| - p_n(x)\Big)\Big(1 - \frac{|x| + p_n(x)}{2}\Big)$$

(c) Show that for all $x \in [-1,1]$ one has $0 \le p_n(x) \le p_{n+1}(x) \le |x|$.

(d) On the interval $0 \le u \le 1$, where does $u \cdot (1-u)^n$ attain its maximum?

(e) Show that $p_n(x) \to |x|$ uniformly on $[-1,1]$ as $n \to \infty$.

(f) Calculate $\lim_{n\to\infty} \int_{-1}^{1} p_n(x)dx$.

Solution to task 3

This is close to an exercise from Rudin (23 in chapter 7).

(a) The fixed points of φ are x and $-x$. Indeed $\varphi(t) = t$ if and only if $x^2 = t^2$, that is, $t = \pm x$.

(b) On the right

$$\Big(|x| - p_n(x)\Big)\Big(1 - \frac{|x| + p_n(x)}{2}\Big) = |x| - p_n(x) - \frac{\big(|x| - p_n(x)\big)\big(|x| + p_n(x)\big)}{2}$$
$$= |x| - p_n(x) - \frac{x^2 - p_n(x)^2}{2}$$

where the remaining $|x|$ cancels with the absolute value on the other side $|x| - p_{n+1}(x)$. The claim is equivalent to

$$-p_{n+1}(x) = -p_n(x) - \frac{x^2 - p_n(x)^2}{2}$$

which is the recurrence defining p_{n+1}, multiplied by -1.

(c) Here is a proof by induction on n.

As a base case, clearly $p_0 = 0$ satisfies $0 \le p_0(x) \le |x|$. The next polynomial is $p_1(x) = \frac{x^2}{2} \ge 0 = p_0$, which also satisfies $\frac{x^2}{2} \le |x|$ for all x in the range $|x| \le 1$, so we have verified $0 \le p_n(x) \le p_{n+1}(x) \le |x|$ for $n = 0$.

Suppose $0 \le p_n(x) \le p_{n+1}(x) \le |x|$ as an induction hypothesis. We claim $0 \le p_{n+1}(x) \le p_{n+2}(x) \le |x|$. The first inequality $p_{n+1}(x) \ge 0$ is part of the induction hypothesis. The second inequality $p_{n+2} \ge p_{n+1}$ follows from the definition

$$p_{n+2}(x) = p_{n+1}(x) + \frac{x^2 - p_{n+1}(x)^2}{2}$$

because the induction hypothesis $p_{n+1}(x) \le |x|$ implies $x^2 - p_{n+1}(x)^2 \ge 0$ in the second term. Finally the inequality $p_{n+2}(x) \le |x|$ follows from the identity in (b). Indeed, one more appeal to the induction hypothesis $p_{n+1}(x) \le |x|$ gives

$$1 - \frac{|x| + p_{n+1}(x)}{2} \ge 1 - |x| \ge 0$$

for $|x| \le 1$. Then by (b) we conclude

$$|x| - p_{n+2}(x) = \Big(|x| - p_{n+1}(x)\Big)\Big(1 - \frac{|x| + p_{n+1}(x)}{2}\Big) \ge 0$$

since both factors are greater than or equal to 0.

We have shown that all three inequalities in $0 \le p_{n+1}(x) \le p_{n+2}(x) \le |x|$ follow from the previous case, completing the induction.

(d) By calculus, the maximum is attained at

$$u = \frac{1}{n+1}.$$

Indeed, setting the derivative equal to 0 gives

$$0 = (1-u)^n - u \cdot n(1-u)^{n-1} = (1-u)^{n-1}(1 - u - nu)$$

so $\frac{1}{n+1}$ is a critical point. The maximum is attained at this critical point rather than at the endpoints, because at $u = 0$ and $u = 1$ the function $u(1-u)^n$ vanishes for any $n \ge 1$. For $n = 0$, the endpoint $u = 1$ coincides with $\frac{1}{n+1}$. Thus in all cases $\frac{1}{n+1}$ is where the maximum is attained.

(e) Using the identity from (b) and the fact that $p_n(x) \ge 0$ from (c), we deduce that

$$\big||x| - p_{n+1}(x)\big| \le \big||x| - p_n(x)\big|\Big(1 - \frac{|x|}{2}\Big)$$

Substituting this into itself n times, we get

$$\begin{aligned} \big||x| - p_{n+1}(x)\big| &\le \big||x| - p_n(x)\big|\Big(1 - \frac{|x|}{2}\Big) \le \big||x| - p_{n-1}(x)\big|\Big(1 - \frac{|x|}{2}\Big)^2 \le \dots \\ &\le \big||x| - p_0(x)\big|\Big(1 - \frac{|x|}{2}\Big)^{n+1} \end{aligned}$$

Since $p_0(x) = 0$, the inequality says

$$\big||x| - p_{n+1}(x)\big| \le 2\frac{|x|}{2}\Big(1 - \frac{|x|}{2}\Big)^{n+1}$$

or, the same for n instead of $n+1$,

$$\big||x| - p_n(x)\big| \le 2u(1-u)^n$$

where $u = |x|/2$. Using (d), since the maximum is attained at $u = \frac{1}{n+1}$, we find

$$2u(1-u)^n \leq 2 \cdot \frac{1}{n+1}\Big(1 - \frac{1}{n+1}\Big)^n < \frac{2}{n+1}.$$

It follows that

$$\big||x| - p_n(x)\big| \leq 2u(1-u)^n \leq \frac{2}{n+1}$$

which gives uniform convergence. For any $\varepsilon > 0$, there is n large enough, independent of $x \in [-1, 1]$, so that $\frac{2}{n+1} < \varepsilon$. □

Alternative solution: one can also deduce uniform convergence from the fact that $p_n \leq p_{n+1}$ using Theorem 7.13 from Rudin (formally, one should replace p_n by $-p_n$ because the result is stated for decreasing rather than increasing sequences). However, that theorem has several hypotheses that need to be checked. One must prove first of all that $p_n(x) \to |x|$ pointwise, for instance by the same proof as above (which already shows the convergence is uniform, not merely pointwise). One must also mention that the domain $[-1, 1]$ is compact and that the limit function $x \mapsto |x|$ is continuous. □

(f) By uniform convergence

$$\lim_{n\to\infty} \int_{-1}^{1} p_n(x)dx = \int_{-1}^{1} \lim_{n\to\infty} p_n(x)dx$$

as we know from Rudin (Theorem 7.16). We know the limit under the integral sign from (e).

$$\lim_{n\to\infty} \int_{-1}^{1} p_n(x)dx = \int_{-1}^{1} \lim_{n\to\infty} p_n(x)dx = \int_{-1}^{1} |x|dx = 2\int_{0}^{1} xdx = 1.$$

□

4. Let $\varphi(x) = x + \sin(x)$ for $x \in \mathbb{R}$.

(a) Show that φ is a monotone function from $\mathbb{R}$ to $\mathbb{R}$.

(b) Show that φ maps the interval $[\frac{2\pi}{3}, \frac{4\pi}{3}]$ to itself.

(c) Show that φ is a contraction on the interval from (b).

(d) **[2 points]** Suppose $|\varphi(x) - \varphi(y)| \leq c|x - y|$ where $0 \leq c < 1$. Let x_n be the sequence defined inductively by

$$x_{n+1} = \varphi(x_n)$$

starting from a given x_0. Let

$$p = \lim_{n\to\infty} x_n$$

(you do not need to show the limit exists).

Show that

$$|p - x_n| \leq \frac{c^n}{1-c}|x_1 - x_0|.$$

(e) If we start from $x_0 = 3$ and want to make $|x_n - \pi| < 2^{-2025}$, how large should n be?

Solution to task 4

(a) The derivative is $\varphi'(x) = 1 + \cos(x) \geq 0$ so φ is monotone increasing (this uses elementary properties of sine and cosine, which can be found in Rudin's chapter 8, as well as a standard consequence of the mean value theorem, which is Theorem 5.11 in Rudin).

(b) Since φ is monotone increasing, $\varphi(2\pi/3) \leq \varphi(x) \leq \varphi(4\pi/3)$ for any $x \in [\frac{2\pi}{3}, \frac{4\pi}{3}]$. We have $\sin(2\pi/3) > 0$ and $\sin(4\pi/3) < 0$, or more explicitly, these values are $\pm\frac{\sqrt{3}}{2}$. Therefore $\varphi(\frac{2\pi}{3}) = \frac{2\pi}{3} + \sin(2\pi/3) \geq \frac{2\pi}{3}$ and $\varphi(4\pi/3) = \frac{4\pi}{3} + \sin(4\pi/3) \leq \frac{4\pi}{3}$. The result follows.

(c) We must show $|\varphi(x) - \varphi(y)| \le c|x - y|$ for some $c < 1$, where x and y are points from the given interval. We claim $c = 1/2$ works, and prove it using the mean value theorem. For some t between x and y

$$\varphi(x) - \varphi(y) = \varphi'(t)(x - y) = \big(1 + \cos(t)\big)(x - y)$$

Since t lies in the same interval, and $\cos(2\pi/3) = \cos(4\pi/3) = -\frac{1}{2}$, we have

$$|1 + \cos(t)| \le 1 - \frac{1}{2} = \frac{1}{2}$$

which proves the claim.

(d) This follows Rudin's proof of Theorem 9.23, page 220-221.

Fix n and compare x_n to x_{n+k} as follows. Applying the triangle inequality with a telescoping sum, we have

$$|x_{n+k} - x_n| \le \sum_{i=0}^{k-1} |x_{n+i+1} - x_{n+i}|$$

Since $x_{n+i+1} = \varphi(x_{n+i})$ and φ is a contraction by the factor c, it follows that

$$|x_{n+k} - x_n| \le \sum_{i=0}^{k-1} |x_{n+i+1} - x_{n+i}| \le |x_1 - x_0| \sum_i c^{n+i} \le \frac{c^n}{1-c}|x_1 - x_0|$$

where the last step uses a geometric series.

(e) We apply (d) with $p = \pi$ since $\sin(\pi) = 0$ gives $\varphi(\pi) = \pi$. We choose n so large that

$$\frac{c^n}{1-c}|x_1 - x_0| < 2^{-2025}$$

With $x_0 = 3$ we have $x_1 = \varphi(3) = 3 + \sin(3)$ so $|x_1 - x_0| = |\sin(3)| \le 1$. (In fact, $\sin(3) \approx 0.14112$ but without doing any calculations we can always say $|\sin| \le 1$.)

From (c), we have $c = \frac{1}{2}$. Therefore it is enough to take n so large that

$$\frac{2^{-n}}{1 - 1/2} \cdot 1 < 2^{-2025}$$

in other words $2^{2025+1} < 2^n$. One can take $n = 2027$.

□

If one has already computed π to high precision by other means, then one can see that $n = 2027$ is much larger than necessary. It appears that $n = 6$ is not enough but $n = 7$ works. The triangle inequality $|\sum(x_{n+i+1} - x_{n+i})| \le \sum |x_{n+i+1} - x_{n+i}|$ gives a worst-case estimate in (d). In practice, if the summands $x_{n+i+1} - x_{n+i}$ have different signs, then the absolute value of the sum may be much smaller than the sum of absolute values. As a result, x_n converges to π even more rapidly than promised by the proof.