MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Jonathan Rohleder

Written examination in
Mathematics III – Foundations of Analysis
10 January 2026

Max score is 30p; grade of E guaranteed at 15p. Admission to the oral exam guaranteed at 21p. Appropriate amount of details required for full marks. No resources are allowed. GOOD LUCK!

1. **(5p)** Let $X = \mathbb{Q}$ be the set of rational numbers, equipped with the metric

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Investigate whether the metric space (X,d) is complete. Moreover, find all compact sets in (X,d).

2. **(4p)** Determine all $x \in \mathbb{R}$ for which the series

$$\sum_{k=1}^{\infty} \frac{x^{4k}}{k^2 9^k}$$

converges.

3. **(5p)** Let X and Y be metric spaces and assume that $f : X \to Y$ is uniformly continuous. Furthermore, let $\{x_n\}$ be a Cauchy sequence in X. Prove that $\{f(x_n)\}$ is a Cauchy sequence in Y. Does this conclusion remain true if f is only required to be continuous?

4. **(6p)** (a) For $0 \leq x \leq 2$ let

$$f(x) = \frac{1}{1+x^2} \quad \text{and} \quad \alpha(x) = \begin{cases} x & \text{for } 0 \leq x < 1, \\ x^2 + 2 & \text{for } 1 \leq x \leq 2. \end{cases}$$

Prove that $f \in \mathscr{R}(\alpha)$ and compute the Riemann–Stieltjes integral $\int_0^2 f \mathrm{d}\alpha$.

(b) Prove or disprove the following statement: If $\alpha, \beta : [a,b] \to \mathbb{R}$ are monotonically increasing functions such that $\alpha(x) \leq \beta(x)$ for each $x \in [a,b]$, then

$$\int_a^b f \mathrm{d}\alpha \leq \int_a^b f \mathrm{d}\beta$$

holds for each continuous, non-negative function $f : [a,b] \to \mathbb{R}$.

5. **(5p)** Define $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = x^n(1-x)^n$, $n \in \mathbb{N}$.
(a) Prove that $\{f_n\}$ converges uniformly on $[0,1]$.
(b) Compute $\lim_{n\to\infty} \int_0^1 f_n(x)\mathrm{d}x$.

6. **(5p)** Consider the function

$$f : \mathbb{R}^3 \to \mathbb{R}, \quad f(x,y,z) = z^3 + 2xy - 4xz + 2y - 1.$$

Show that in a neighborhood of $(x,y) = (1,1)$ the equation $f(x,y,z) = 0$ defines a differentiable function $z = \varphi(x,y)$ such that $\varphi(1,1) = 1$. Moreover, compute the partial derivatives of first order of φ at the point $(1,1)$.