Exam Foundations of Analysis 10/01/2026

1. $(\mathbb{Q}, d)$ is complete: Let $\{x_n\}$ Cauchy seq.
Then, in particular, $\exists N \in \mathbb{N}$ s.t.

$$d(x_n, x_m) < \frac{1}{2} \quad \forall n, m \geq N.$$

$\Rightarrow \quad x_n = x_m \quad \forall n, m \geq N \Rightarrow \exists x \in \mathbb{Q}: x_n = x \; \forall n \geq N.$

In particular, $d(x_n, x) = 0 \; \forall n \geq N$, thus
$\lim_{n \to \infty} x_n = x.$

For compactness, note that each set $E \subset \mathbb{Q}$ is open.
Namely, if $x \in E$, then $B_{\frac{1}{2}}(x) = \{x\} \subset E$.

Now we show that $K \subset \mathbb{Q}$ is compact w.r.t. d iff
K is finite.

Namely, if K is finite, then K is compact in any metric.
Conversely, let $K \subset \mathbb{Q}$ compact. Then

$$U_x := \{x\}, \quad x \in K,$$

form an open cover of K, and if $x_1, \dots, x_n$ are
finitely many points in K, then by compactness,

$$K \subset \bigcup_{i=1}^{n} U_{x_i} = \bigcup_{i=1}^{n} \{x_i\}.$$

Thus K is finite.

2. $$\sum_{k=1}^{\infty} \frac{x^{4k}}{k^2 9^k} = \sum_{k=1}^{\infty} \frac{1}{k^2}\left(\frac{x^4}{9}\right)^k = \sum_{k=1}^{\infty} \underbrace{\frac{1}{k^2}}_{=:c_k} t^k$$

with $t = \frac{x^4}{9}$. This is a power series with a radius of convergence R. We have

$$\sqrt[k]{|c_k|} = \left(\frac{1}{\sqrt[k]{k}}\right)^2 \longrightarrow 1 \text{ as } k \to \infty.$$

Hence series converges if $|t| < 1$ and diverges if $|t| > 1$.

That is, for $-\sqrt{3} < x < \sqrt{3}$, the series converges, and for $|x| > \sqrt{3}$ the series diverges.

$\underline{x = \pm\sqrt{3}}$: $$\sum_{k=1}^{\infty} \frac{x^{4k}}{k^2 9^k} = \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty.$$

$\Rightarrow$ Series converges if and only if $|x| \leq \sqrt{3}$.

3. Let $\{x_n\}$ Cauchy in X and let $\varepsilon > 0$. Then $\exists\, \delta > 0$:

$$d(x,y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon. \qquad (*)$$

By Cauchy property, $\exists\, N \in \mathbb{N}$:

$$d(x_n, x_m) < \delta \quad \forall n, m \geq N.$$

Thus, by $(*)$,

$$d(f(x_n), f(x_m)) < \varepsilon \qquad \forall n, m \geq N.$$

$\Rightarrow$ $\{f(x_n)\}$ is Cauchy in Y.

This is false if f is merely continuous: Take $X = (0,1]$ with Euclidean metric, $Y = \mathbb{R}$, $f(x) = \frac{1}{x}$. Then $x_n = \frac{1}{n}$ form a Cauchy sequence in X, and f is continuous on X. But $f(x_n) = n$ $\forall n$, and this is not Cauchy in $\mathbb{R}$.

4. (a) For $\varepsilon > 0$, we have, in the notation of Rudin, chapter 6, on the interval $[1-\varepsilon, 1]$ and for a partition $P = \{1-\varepsilon, s, 1\}$,

$$U(P, f, \alpha) = \sup_{1-\varepsilon \le x \le s} f(x) \; (s-1+\varepsilon) + \sup_{s \le x \le 1} f(x) \; (3-s)$$

$$\xrightarrow[(\Rightarrow s \to 1)]{\varepsilon \searrow 0} f(1) \cdot 0 + f(1) \cdot 2,$$

and in the same way, $L(P, f, \alpha) \to 2 f(1)$.

Moreover, as α is continuous on $[0, 1-\varepsilon] \cup [1,2]$, $f \in \mathcal{R}(\alpha)$ and

$$\int_0^2 f \, d\alpha = \lim_{\varepsilon \searrow 0} \left(\int_0^{1-\varepsilon} \frac{1}{1+x^2} dx + \int_{1-\varepsilon}^1 \frac{1}{1+x^2} d\alpha \right)$$

$$+ \int_1^2 \frac{1}{1+x^2} \alpha'(x) \, dx$$

$$= \lim_{\varepsilon \searrow 0} \arctan(1-\varepsilon) + 2 f(1)$$

$$+ \int_1^2 \frac{2x}{1+x^2} dx$$

$$= \arctan(1) + 1 + \left[\ln(1+x^2) \right]_1^2$$

$$= \frac{\pi}{4} + 1 + \ln \frac{5}{2}.$$

(b) False – let $[a,b] = [0,1]$, $\alpha(x) = x$, $\beta(x) = 1$, $f(x) = 1$. Then $\alpha(x) \le \beta(x) \; \forall x \in [0,1]$, and $\int_0^1 f \, d\alpha = 1 > 0 = \int_0^1 f \, d\beta$.

(5.) (a) For $0 \le x \le \frac{1}{2}$,

$$|f_n(x)| = \underbrace{|x|^n}_{\le \left(\frac{1}{2}\right)^n} \underbrace{|1-x|^n}_{\le 1} \le \left(\frac{1}{2}\right)^n,$$

and for $\frac{1}{2} \le x \le 1$,

$$|f_n(x)| = \underbrace{|x|^n}_{\le 1} \underbrace{|1-x|^n}_{\le \left(\frac{1}{2}\right)^n} \le \left(\frac{1}{2}\right)^n.$$

Hence

$$\sup_{0 \le x \le 1} |f_n(x)| \le \frac{1}{2^n} \to 0 \text{ as } n \to \infty.$$

$\Rightarrow$ $f_n \to 0$ uniformly on $[0,1]$.

(b) Since $f_n \to 0$ uniformly,

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 \left(\lim_{n\to\infty} f_n(x)\right) dx = 0.$$

⑥ f is a polynomial, hence $f \in C^1(\mathbb{R}^3)$.

In the standard basis,

$$[f'(x,y,z)] = (2y-4z,\ 2x+2,\ 3z^2-4x).$$

In particular, $\frac{\partial f}{\partial z}(1,1,1) = -1 \neq 0$.

By the implicit function thm, there ex. open sets $U \subset \mathbb{R}^3$, $W \subset \mathbb{R}^2$, with $(1,1,1) \in U$ and $(1,1) \in W$ such that for each $(x,y) \in W$ $\exists!\, z \in \mathbb{R}$ s.t. $(x,y,z) \in U$ and $f(x,y,z) = 0$.

Setting $\varphi(x,y) := z$, we have $\varphi \in C^1(W)$ and

$$[\varphi'(x,y)] = -\frac{1}{-1}(2y-4z,\ 2x+2)\Big|_{(x,y,z)=(1,1,1)}$$

$$= (-2, 4).$$

That is,

$$\frac{\partial \varphi}{\partial x}(1,1) = -2 \quad \text{and} \quad \frac{\partial \varphi}{\partial y}(1,1) = 4.$$