MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Jonathan Rohleder

Written examination in
Mathematics III – Foundations of Analysis
4 March 2026

Max score is 30p; grade of E guaranteed at 15p. Admission to the oral exam guaranteed at 21p. Appropriate amount of details required for full marks. No resources are allowed. GOOD LUCK!

1. **(5p)** Let X be a non-empty set and let $d : X \times X \to \mathbb{R}$ be a mapping such that for all $x, y, z \in X$ the following hold:

   (a) $d(x, y) = 0$ if and only if $x = y$;

   (b) $d(x, y) \leq d(x, z) + d(y, z)$.

   Show that d is a metric on X.

2. **(5p)** (a) Consider the series of functions given by

   $$f(x) = \sum_{n=1}^{\infty} \frac{x \sin(nx)}{n^5}, \quad x \in \mathbb{R}.$$

   Show that f is a continuous function on $\mathbb{R}$.

   (b) For the function f defined in (a), investigate whether the set

   $$\mathscr{A} := \{x \in \mathbb{R} : 0 < f(x) < 1/2\}$$

   is open or closed or both or neither of them.

3. **(6p)** (a) Show that the function

   $$f : (0, \infty) \to \mathbb{R}, \quad f(x) = \frac{1}{1 + x^2},$$

   is uniformly continuous.

   (b) Provide an example of a function which is continuous but not uniformly continuous. Justify your answer.

4. **(4p)** For $0 \leq x \leq 3$ let

   $$f(x) = 3x^2 \quad \text{and} \quad \alpha(x) = \begin{cases} 2x & \text{for } 0 \leq x \leq 2, \\ x^2 & \text{for } 2 < x \leq 3. \end{cases}$$

   Prove that $f \in \mathscr{R}(\alpha)$ and compute the Riemann–Stieltjes integral $\int_0^3 f \mathrm{d}\alpha$.

5. **(5p)** (a) Investigate pointwise and uniform convergence of the sequence of functions

   $$f_n(x) = (x^3 - 2nx^2 + n^2 e^x) n^{-2}, \quad x \in \mathbb{R}, n \in \mathbb{N}.$$

   (b) Compute $\lim_{n\to\infty} \int_{-1}^{99} f_n(x) \mathrm{d}x$ for the functions f_n defined in (b).

6. **(5p)** Consider the function $F : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

   $$F(x, y, z) = (x + y^3 - 4z, x e^y z).$$

   Show that around the point $(0, 2, 2)$, the equation $F(x, y, z) = (0, 0)$ defines x and y as functions of z, that is $(x, y) = G(z)$ for a differentiable vector-valued function G. Compute $G'(2)$.