Exam Foundations of Analysis 4/3/2026

① By (b), $\forall x,y\in X$,

$$d(x,y) \le \underbrace{d(x,x)}_{=0 \text{ by (a)}} + d(y,x) \Rightarrow d(x,y) \le d(y,x).$$

As x,y interchangable, $d(x,y) = d(y,x) \; \forall x,y$.

Moreover, again by (b), (a), $\forall x,z\in X$:

$$0 = d(x,x) \le d(x,z) + d(x,z) = 2d(x,z)$$

$\Rightarrow d(x,z) \ge 0 \;\; \forall x,z \in X$.

$\Rightarrow d$ metric.

② (a) We have

$$\left|\frac{\sin(nx)}{n^5}\right| \le \frac{1}{n^5} \text{ and } \sum_{n=1}^{\infty}\frac{1}{n^5} < \infty.$$

Hence

$$f(x) = \sum_{n=1}^{\infty}\frac{x\sin(nx)}{n^5} = x\sum_{n=1}^{\infty}\frac{\sin(nx)}{n^5}$$

converges uniformly on $\mathbb{R}$ by Weierstrass m-test. I.p. f is continuous.

(b) $A = f^{-1}(\underbrace{(0,\tfrac{1}{2})}_{\text{open}})$, and since f continuous,

A is open.

③ (a) Let $x, y \in (0,\infty)$. Then

$$|f(x)-f(y)| = \left|\frac{1}{1+x^2} - \frac{1}{1+y^2}\right| = \frac{|y^2-x^2|}{(1+x^2)(1+y^2)}$$

$$= \left(\frac{y}{\underbrace{(1+x^2)}_{\geq 1}(1+y^2)} + \frac{x}{(1+x^2)\underbrace{(1+y^2)}_{\geq 1}}\right)|y-x|$$

$$\leq \left(\underbrace{\frac{y}{1+y^2}}_{\leq 1} + \underbrace{\frac{x}{1+x^2}}_{\leq 1}\right)|y-x|$$

$$\leq 2|y-x|.$$

Thus, if $\varepsilon > 0$ and $\delta = \frac{\varepsilon}{2}$, then $|x-y|<\delta$ implies

$$|f(x)-f(y)| \leq 2|y-x| < 2\frac{\varepsilon}{2} = \varepsilon.$$

(b) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is an example:

Assume $\exists \delta > 0$ s.t.

$$|f(x)-f(y)| < 1 \quad \text{if} \quad |x-y| < \delta.$$

If $x > 0$ and $y = x + \frac{\delta}{2}$, then $|x-y|<\delta$ and

$$|f(x)-f(y)| = |x-y|\,|x+y| = \frac{\delta}{2}\left(2x+\frac{\delta}{2}\right)$$

$$> \delta x > 1$$

whenever $x > \frac{1}{\delta}$ — a contradiction.

4 Since f is continuous, $f \in \mathcal{R}(\alpha)$.

Furthermore, note that α is continuous on $[0,3]$ and differentiable on each of the intervals $(0,2)$ and $(2,3)$. Thus

$$\int_0^3 f \, d\alpha = \int_0^2 f \, d\alpha + \int_2^3 f \, d\alpha$$

$$= \int_0^2 f(x)\, \alpha'(x)\, dx + \int_2^3 f(x)\, \alpha'(x)\, dx$$

$$= 2\int_0^2 3x^2 \, dx + \int_2^3 3x^2 \cdot 2x \, dx$$

$$= 2\left[x^3\right]_0^2 + \frac{3}{2}\left[x^4\right]_2^3$$

$$= 16 + \frac{3}{2}(81-16) = \frac{32 + 195}{2}$$

$$= \underline{\underline{\frac{227}{2}}}$$

5 (a) Pointwise:

$$f_n(x) = \frac{x^3}{n^2} - 2\frac{x^2}{n} + e^x \to e^x \text{ as } n \to \infty.$$

Convergence is not uniform:

$$|f_n(x) - e^x| = \frac{|x^3 - 2n x^2|}{n^2} = \frac{x^2}{n^2}|x - 2n|, \quad (*)$$

and choosing $x = n^2$ gives

$$|f_n(n^2) - e^{n^2}| = n^2(n^2 - 2n) \to +\infty.$$

Hence

$$\sup_{x \in \mathbb{R}} |f_n(x) - e^x| = +\infty \quad \forall n.$$

(b) On the interval $[-1, 99]$ we have, by $(*)$,

$$|f_n(x) - e^x| \leq \frac{99^2}{n^2}(99 + 2n),$$

hence

$$\sup_{-1 \leq x \leq 99} |f_n(x) - e^x| \leq \frac{99^3}{n^2} + 2\frac{99^2}{n} \to 0$$

as $n \to \infty$. Thus on $[-1, 99]$ the convergence is uniform and, hence,

$$\lim_{n\to\infty} \int_{-1}^{99} f_n(x)\,dx = \int_{-1}^{99} e^x\,dx = e^{99} - e^{-1}.$$

⑥ F is C^1 and $F(0,2,2) = (0,0)$.

$$F'(x,y,z) = \begin{pmatrix} 1 & 3y^2 & -4 \\ e^{y/z} & xe^{y/z} & xe^y \end{pmatrix},$$

$$F'(0,2,2) = \begin{pmatrix} \underbrace{\begin{matrix} 1 & 12 \\ 2e^2 & 0 \end{matrix}}_{=: A_{(x,y)}} & \begin{matrix} -4 \\ 0 \end{matrix} \end{pmatrix}$$

Since $A_{(x,y)}$ is invertible ($\det A_{(x,y)} = -24e^2 \neq 0$), the implicit fct thm guarantees that there ex. open sets $U \subset \mathbb{R}^3$, $W \subset \mathbb{R}$, with $(0,2,2) \in U$ and $2 \in W$ s.t. $\forall z \in W$ $\exists!$ (x,y) s.t. $(x,y,z) \in U$ and

$$F(x,y,z) = 0.$$

If we write $G(z) := (x,y)$ for the solution function, then G is C^1 on W and

$$G'(2) = -\begin{pmatrix} 1 & 12 \\ 2e^2 & 0 \end{pmatrix}^{-1} \begin{pmatrix} -4 \\ 0 \end{pmatrix}$$

$$= \frac{1}{24e^2} \begin{pmatrix} 0 & -12 \\ -2e^2 & 1 \end{pmatrix} \begin{pmatrix} -4 \\ 0 \end{pmatrix}$$

$$= \frac{1}{24e^2} \begin{pmatrix} 0 \\ 8e^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/3 \end{pmatrix} \quad //$$