MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Jonathan Rohleder

Written examination in
Mathematics III – Foundations of Analysis
14 August 2026

Max score is 30p; grade of E guaranteed at 15p; admission to the oral exam guaranteed at 20p. Appropriate amount of details required for full marks. No resources allowed. GOOD LUCK!

1. **(5p)** (a) Let X be a metric space and $\{x_n\}$ a sequence in X such that $\lim_{n\to\infty} x_n = x \in X$. Prove that
$$K := \{x_n : n \in \mathbb{N}\} \cup \{x\}$$
is compact in X.
(b) Is $\mathbb{Q} \cap (0,1)$ open in $\mathbb{Q}$ (equipped with the Euclidean metric)? Motivate your answer!

2. **(5p)** Show that
$$f(x) = \sum_{n=1}^{\infty} \frac{e^{-nx^2}}{n^3}$$
is a continuous function on $\mathbb{R}$. Is it even differentiable? If so, provide $f'(x)$.

3. **(5p)** Show that $f(x) = \frac{1}{x}$ is uniformly continuous on $[1,\infty)$ but not on $(0,1)$.

4. **(5p)** Compute the Riemann–Stieltjes integral $\int_{-1}^{1} f \, \mathrm{d}\alpha$, where $f(x) = x^4$ and
$$\alpha(x) = \begin{cases} -x^2 & \text{for } -1 \le x < 0, \\ x+1 & \text{for } 0 \le x \le 1. \end{cases}$$

5. **(5p)** Investigate whether the sequences of functions $\{f_n\}$ and $\{g_n\}$ defined by
$$f_n : [0,1] \to \mathbb{R}, \quad f_n(x) := \frac{x}{1+nx}, \quad n \in \mathbb{N},$$
and
$$g_n : [0,1] \to \mathbb{R}, \quad g_n(x) := \frac{nx}{1+n^2x^2}, \quad n \in \mathbb{N},$$
converge pointwise and uniformly, respectively.

6. **(5p)** Consider the function
$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x,y) = x\sin(y) - \cos(x)y^2.$$
Show that in a neighborhood of $x = \frac{\pi}{2}$ the equation $f(x,y) = 0$ defines a differentiable function $y = \varphi(x)$ satisfying $\varphi(\frac{\pi}{2}) = \pi$. Moreover, compute the derivative $\varphi'(\frac{\pi}{2})$.