Exam Math III - Foundations of Analysis 14/8/2026

① (a) Let $\{G_\alpha\}$ be an open cover of K. Then $\exists\ \alpha_0$ s.t. $x \in G_{\alpha_0}$. Moreover, as G_{α_0} is open, $\exists\ \varepsilon > 0$ s.t. $N_\varepsilon(x) \subset G_{\alpha_0}$. Since $\lim_{n\to\infty} x_n = x$, $\exists\ N$ s.t. $d(x_n, x) < \varepsilon\ \forall n \geq N$, hence

$$x_n \in N_\varepsilon(x) \subset G_{\alpha_0} \qquad \forall n \geq N.$$

Finally, as $\{G_\alpha\}$ is an open cover of K, $\exists\ \alpha_1, \dots, \alpha_{N-1}$ s.t.

$$x_j \in G_{\alpha_j} \qquad \text{for } j = 1, \dots, N-1.$$

With $\{G_{\alpha_0}, G_{\alpha_1}, \dots, G_{\alpha_{N-1}}\}$ we have found a finite subcover.

(b) Yes: if $x \in \mathbb{Q} \cap (0,1)$ then $\exists\ \varepsilon > 0$: $N_\varepsilon(x) \subset (0,1)$ as real sets. Hence $\mathbb{Q} \cap N_\varepsilon(x) \subset \mathbb{Q} \cap (0,1)$, and $\mathbb{Q} \cap N_\varepsilon(x)$ is the rational ball of radius ε centered at x.

(2) Let $M_n := \frac{1}{n^3}$. Then $\sum_{n=1}^{\infty} M_n < \infty$ and

$$\frac{e^{-nx^2}}{n^3} \leq \frac{1}{n^3} = M_n, \qquad \forall n \in \mathbb{N}.$$

$nx^2 \geq 0$

Moreover, $f_n(x) = \frac{e^{-nx^2}}{n^3}$ is continuous $\forall n \in \mathbb{N}$.

By Weierstrass test, $\sum_{n=1}^{\infty} \frac{e^{-nx^2}}{n^3}$ converges uniformly on $\mathbb{R}$

and, hence, defines a continuous fct.

Differentiability: each f_n is differentiable with

$$f_n'(x) = -2nx\, f_n(x) = -2x\, \frac{e^{-nx^2}}{n^2}.$$

Since $\frac{e^{-nx^2}}{n^2} \leq \frac{1}{n^2}$ and $\sum \frac{1}{n^2}$ converges, again by Weierstrass test,

$\sum \frac{e^{-nx^2}}{n^2}$ converges uniformly, and so does $\sum \left(-2x \frac{e^{-nx^2}}{n^2}\right) = \sum f_n'(x)$.

Hence f is differentiable with

$$f'(x) = -2x \sum_{n=1}^{\infty} \frac{e^{-nx^2}}{n^2}, \qquad x \in \mathbb{R}.$$

3) <u>On $[1,\infty)$</u>: Let $\varepsilon > 0$, $\delta := \varepsilon$. Then for $x, y \in [1,\infty)$ with $|x-y| < \delta$ we have

$$\left|\frac{1}{x} - \frac{1}{y}\right| = \frac{|y-x|}{\underbrace{xy}_{\geq 1}} \leq |y-x| < \delta = \varepsilon.$$

Thus f is uniformly continuous on $[1,\infty)$.

<u>On $(0,1)$</u>: Let $\varepsilon = 1$. Assume for a contradiction that f is uniformly continuous on $(0,1)$. Then $\exists\, \delta > 0$ s.t.

$$|x-y| < \delta \Rightarrow \left|\frac{1}{x} - \frac{1}{y}\right| < 1 \quad \forall x, y \in (0,1). \quad (*)$$

Let $x > 0$ s.t. $y = x + \frac{\delta}{2} < 1$. Then $|x-y| < \delta$ and

$$\left|\frac{1}{x} - \frac{1}{y}\right| = \frac{1}{x} - \frac{1}{y} = \frac{y-x}{xy} = \frac{\delta}{2x\left(x+\frac{\delta}{2}\right)} > \frac{\delta}{2\left(x+\frac{\delta}{2}\right)^2}.$$

If we now choose $x < \sqrt{\frac{\delta}{2}} - \frac{\delta}{2}$ (note that the RHS is positive since $\frac{\delta}{2} < 1$), then

$$x + \frac{\delta}{2} < \sqrt{\frac{\delta}{2}} \iff \left(x+\frac{\delta}{2}\right)^2 < \frac{\delta}{2},$$

hence

$$\left|\frac{1}{x} - \frac{1}{y}\right| > \frac{\delta}{2\,\frac{\delta}{2}} = 1,$$

contradicting $(*)$.

(4) In the notation of Rudin, Chapter 6, for $\varepsilon > 0$ small enough, consider first the interval $[-\varepsilon, 0]$ and its trivial partition $P: -\varepsilon < 0$. Then

$$U(P, f, \alpha) = \sup_{-\varepsilon \le x \le 0} f(x) \, (\alpha(0) - \alpha(-\varepsilon))$$

$$= \sup_{-\varepsilon \le x \le 0} f(x) \, (1 - (-\varepsilon^2)) \to f(0) \overset{=0}{} \text{ as } \varepsilon \searrow 0$$

by continuity of f. In the same way,

$$L(P, f, \alpha) \to f(0) = 0 \quad \text{as } \varepsilon \searrow 0.$$

We get

$$\int_{-1}^{1} f \, d\alpha = \lim_{\varepsilon \searrow 0} \left(\int_{-1}^{-\varepsilon} x^4 d\alpha + \int_{-\varepsilon}^{0} x^4 d\alpha + \int_0^1 x^4 d\alpha \right)$$

$$= \lim_{\varepsilon \searrow 0} \left(\int_{-1}^{-\varepsilon} x^4 (-2x) dx + \int_{-\varepsilon}^{0} x^4 d\alpha + \int_0^1 x^4 \cdot 1 \, dx \right)$$

$$= \lim_{\varepsilon \searrow 0} \left[-\frac{x^6}{3} \right]_{-1}^{-\varepsilon} + 0 + \left[\frac{x^5}{5} \right]_0^1$$

$$= \lim_{\varepsilon \searrow 0} \left(\frac{1}{3} - \frac{\varepsilon^6}{3} \right) + \frac{1}{5} = \frac{8}{15}.$$

⑤ $\underline{f_n}$: Pointwise:

$$f_n(x) = \frac{x}{1+nx} = \frac{1}{n} \frac{x}{\frac{1}{n}+x} \to 0 \text{ as } n\to\infty,$$

for all $x\in[0,1]$.

Uniformly:

$$0 \le \sup_{x\in[0,1]} |f_n(x)-0| = \sup_{x\in[0,1]} \frac{x}{1+nx} = \sup_{x\in(0,1]} \frac{1}{\underbrace{\frac{1}{x}}_{\ge 1}+n} \le \frac{1}{1+n}$$

$$\to 0 \quad \text{as } n\to\infty.$$

$\rightsquigarrow$ $f_n\to 0$ uniformly on $[0,1]$.

$\underline{g_n}$: Pointwise: For $x\neq 0$,

$$g_n(x) = \frac{nx}{1+n^2x^2} = \frac{1}{\frac{1}{nx}+nx} \to 0,$$

and for $x=0$, $g_n(x)=0$. Hence $g_n(x)\to 0$ $\forall x\in[0,1]$.

Uniformly: Note that

$$g_n'(x) = \frac{n(1+n^2x^2)-2n^3x^2}{(1+n^2x^2)^2} = n\frac{1-n^2x^2}{(1+n^2x^2)^2} = -n^3\frac{(x-\frac{1}{n})(x+\frac{1}{n})}{(1+n^2x^2)^2},$$

which has its only root in $[0,1]$ at $x=\frac{1}{n}$, with g_n' being positive to the left and negative to the right of $x=\frac{1}{n}$. In particular, g_n has a local maximum $g_n(\frac{1}{n})=\frac{1}{2}$. Since $g_n(0)=0$ and $g_n(1)=\frac{1}{\frac{1}{n}+n}\le 2$,

we have

$$\sup_{x\in[0,1]} |g_n(x)-0| = \frac{1}{2} \not\to 0 \quad \text{as } n\to\infty.$$

Thus the convergence is not uniform.

⑥ We have $f \in C^1(\mathbb{R}^2)$ and $f(\frac{\pi}{2}, \pi) = \frac{\pi}{2} \cdot 0 - 0 \cdot \pi^2 = 0$.

Moreover, w.r.t. standard basis,

$$[f'(x,y)] = \left(\sin(y) + \sin(x) y^2, \; x\cos(y) - 2\cos(x) y\right).$$

In particular, $\frac{\partial f}{\partial y}(\frac{\pi}{2}, \pi) = \frac{\pi}{2}(-1) = -\frac{\pi}{2} \neq 0$.

By implicit function thm there ex. $U \subset \mathbb{R}^2$, $W \subset \mathbb{R}$ open with $(\frac{\pi}{2}, \pi) \in U$ and $\frac{\pi}{2} \in W$ s.t. $\forall x \in W \; \exists ! y \in \mathbb{R}$ s.t.

$$(x,y) \in U \quad \text{and} \quad f(x,y) = 0.$$

Let $\varphi(x) := y$. Then $\varphi \in C^1(W)$, again by impl. fct. thm, and

$$[\varphi'(\tfrac{\pi}{2})] = -\left(\frac{\partial f}{\partial y}(\tfrac{\pi}{2}, \pi)\right)^{-1} \cdot \frac{\partial f}{\partial x}(\tfrac{\pi}{2}, \pi)$$

$$= \frac{2}{\pi} \cdot \pi^2 = \underline{\underline{2\pi}}$$