MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Jonathan Rohleder

Written examination in
Mathematics III – Foundations of Analysis
15 October 2026

Max score is 30p; grade of E guaranteed at 15p; admission to the oral exam guaranteed at 20p. Appropriate amount of details required for full marks. No resources allowed. GOOD LUCK!

1. **(5p)** In a metric space X, let K be compact and $A \subseteq K$. Suppose every $x \in K$ has an open neighbourhood containing only finitely many points of A. Show that A is finite.

2. **(5p)** Show that
$$f(x) = \sum_{n=1}^{\infty} \frac{\sin(\frac{x}{n})\cos(nx)}{n^3}$$
is a continuous function on $\mathbb{R}$. Is it even differentiable? If so, provide $f'(x)$.

3. **(5p)** Show that $f(x) = \cos(\frac{1}{x})$ is uniformly continuous on $[1, \infty)$ but not on $(0, 1)$.

4. Compute the following Riemann–Stieltjes integrals:
(3p) $\int_{-1}^{1} f \mathrm{d}\alpha$ with $f(x) = x^3 + 4$ and $\alpha(x) = \begin{cases} x, & x \leq 0, \\ 2, & x > 0; \end{cases}$
(2p) $\int_{0}^{1} f \mathrm{d}\alpha$ with $f(x) = x^2$ and $\alpha(x) = e^x$.

5. **(5p)** Investigate whether the sequences of functions $\{f_n\}$ and $\{g_n\}$ defined by
$$f_n : [0,1] \to \mathbb{R}, \quad f_n(x) := \frac{nx^2}{1 + n^2 x}, \quad n \in \mathbb{N},$$
and
$$g_n : [0,1] \to \mathbb{R}, \quad g_n(x) := \frac{3 + n^4 x^2}{7 + 5n^2 x + 6n^6 x^3}, \quad n \in \mathbb{N},$$
converge pointwise and uniformly, respectively.

6. **(5p)** Consider the function
$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x, y) = x^2 + y^3 + xy - 3.$$
Show that in a neighborhood of $x = 1$ the equation $f(x, y) = 0$ defines a differentiable function $y = \varphi(x)$ satisfying $\varphi(1) = 1$. Moreover, compute the derivative $\varphi'(1)$.