Exam Foundations of Analysis 15/10/2026 – Solution Sketches

1. For each $x \in K$ choose an open neighbourhood U_x of x such that $U_x \cap A$ is finite. Then $\{U_x\}_{x\in K}$ is an open cover of K. By compactness there exist $x_1, \dots, x_m \in K$ with $K = U_{x_1} \cup \cdots \cup U_{x_m}$. Since $A \subseteq K$,

$$A = \bigcup_{j=1}^{m} \left(U_{x_j} \cap A\right),$$

which is a finite union of finite sets. Hence A is finite.

2. Let $u_n(x) = \frac{\sin(x/n)\cos(nx)}{n^3}$. Each u_n is continuous on $\mathbb{R}$ and

$$|u_n(x)| \le \frac{1}{n^3} \quad \text{for all } x \in \mathbb{R}.$$

Since $\sum_n n^{-3} < \infty$, the Weierstrass M-test shows that $\sum_n u_n$ converges uniformly on $\mathbb{R}$. As a uniform limit of continuous functions, f is continuous.

Each u_n is C^1 with

$$u_n'(x) = \frac{\cos(x/n)\cos(nx)}{n^4} - \frac{\sin(x/n)\sin(nx)}{n^2}, \qquad |u_n'(x)| \le \frac{1}{n^4} + \frac{1}{n^2}.$$

The right-hand side is summable, so $\sum_n u_n'$ converges uniformly on $\mathbb{R}$ by the M-test. Since moreover $\sum_n u_n(0)$ converges, the theorem on termwise differentiation yields that f is differentiable with

$$f'(x) = \sum_{n=1}^{\infty} \left(\frac{\cos(\frac{x}{n})\cos(nx)}{n^4} - \frac{\sin(\frac{x}{n})\sin(nx)}{n^2} \right), \qquad x \in \mathbb{R}.$$

3. *On $[1,\infty)$:* For $x \ge 1$ we have

$$|f'(x)| = \frac{|\sin(1/x)|}{x^2} \le 1.$$

By the mean value theorem, $|f(x) - f(y)| \le |x - y|$ for all $x, y \ge 1$. Thus if $\varepsilon > 0$ is arbitrary and $\delta := \varepsilon$, then $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$. Hence f is uniformly continuous on $[1,\infty)$.

On $(0,1)$: Let $x_n = \frac{1}{2\pi n}$ and $y_n = \frac{1}{(2n+1)\pi}$, $n \in \mathbb{N}$. Then $x_n, y_n \in (0,1)$ and $|x_n - y_n| \le \frac{1}{2\pi n} \to 0$, but

$$|f(x_n) - f(y_n)| = |\cos(2\pi n) - \cos((2n+1)\pi)| = 2.$$

Hence for $\varepsilon = 1$ no $\delta > 0$ works, and f is not uniformly continuous on $(0,1)$.

4. *First integral:* Write $\alpha = \alpha_1 + \alpha_2$ with

$$\alpha_1(x) = \begin{cases} x, & x \le 0, \\ 0, & x > 0, \end{cases} \qquad \alpha_2(x) = \begin{cases} 0, & x \le 0, \\ 2, & x > 0. \end{cases}$$

Here α_1 is continuous and piecewise C^1 with $\alpha_1' = 1$ on $(-1,0)$ and $\alpha_1' = 0$ on $(0,1)$, and α_2 is a jump of height 2 at 0. Since f is continuous,

$$\int_{-1}^{1} f \, \mathrm{d}\alpha = \int_{-1}^{0} (x^3 + 4)\, \mathrm{d}x + 2f(0) = \left[\frac{x^4}{4} + 4x\right]_{-1}^{0} + 8 = \frac{15}{4} + 8 = \frac{47}{4}.$$

Second integral: Since α is C^1 with $\alpha'(x) = e^x$, integration by parts (twice) gives

$$\int_0^1 x^2 \, \mathrm{d}\alpha = \int_0^1 x^2 e^x \, \mathrm{d}x = \Big[e^x(x^2 - 2x + 2)\Big]_0^1 = e - 2.$$

5. *The sequence* $\{f_n\}$*:* For $x \in (0, 1]$,

$$0 \le f_n(x) = \frac{nx^2}{1 + n^2 x} \le \frac{nx^2}{n^2 x} = \frac{x}{n} \le \frac{1}{n},$$

and $f_n(0) = 0$. Hence $\sup_{x\in[0,1]} |f_n(x)| \le \frac{1}{n} \to 0$, so $f_n \to 0$ pointwise and uniformly on $[0, 1]$.

The sequence $\{g_n\}$*:* We have $g_n(0) = \frac{3}{7}$ for all n. For $x \in (0, 1]$,

$$g_n(x) = \frac{3n^{-6} + x^2 n^{-2}}{7n^{-6} + 5xn^{-4} + 6x^3} \longrightarrow \frac{0}{6x^3} = 0.$$

Thus $g_n \to g$ pointwise, where $g(0) = \frac{3}{7}$ and $g(x) = 0$ for $x \in (0, 1]$. Each g_n is continuous on $[0, 1]$ (the denominator is at least 7), but g is not continuous at 0; hence the convergence is not uniform.

6. The function f is a polynomial and hence C^1 on $\mathbb{R}^2$. Moreover,

$$f(1, 1) = 1 + 1 + 1 - 3 = 0, \qquad \frac{\partial f}{\partial y}(x, y) = 3y^2 + x, \qquad \frac{\partial f}{\partial y}(1, 1) = 4 \neq 0.$$

By the implicit function theorem there exist open sets $U \subset \mathbb{R}^2$, $W \subset \mathbb{R}$ with $(1, 1) \in U$, $1 \in W$, and a C^1 function φ with $\varphi(1) = 1$ such that for each $x \in W$ there exists a unique y such that $(x, y) \in U$ and $f(x, y) = 0$. Set $y = \varphi(x)$. Then, since $\frac{\partial f}{\partial x}(x, y) = 2x + y$,

$$\varphi'(1) = -\frac{\frac{\partial f}{\partial x}(1, 1)}{\frac{\partial f}{\partial y}(1, 1)} = -\frac{3}{4}.$$