MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Jonathan Rohleder

Tentamensskrivning i
Matematik III Komplex Analys
7.5 hp
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No calculators, books, or other resources allowed. Max score on each problem is 5p; grade of E guaranteed at 15p. Appropriate amount of details required for full marks.

As the exam questions were individualized, we provide solutions to one set of example questions.

1. Find all solutions to the equation $3\tan z + 4i = ie^{2iz}$.

   Solution: The equation is equivalent to

$$-3i\frac{e^{iz}-e^{-iz}}{e^{iz}+e^{-iz}} = ie^{2iz} - 4i.$$

   After multiplying by $e^{iz}+e^{-iz}$ and simplifying we arrive at the equation $e^{3iz} = 7e^{-iz}$, which in turn is equivalent to $e^{4iz} = 7$. This is satisfied if and only if

$$4iz = \log 7 = \operatorname{Log} 7 + 2k\pi i, \qquad k \in \mathbb{Z},$$

   which leads to $z = -\frac{i}{4}\operatorname{Log} 7 + \frac{k\pi}{2}$, $k \in \mathbb{Z}$.

2. Calculate all Laurent series expansions of the function

$$f(z) = \frac{1}{z(z-i)^2}$$

   centered at $z_0 = i$.

   Solution: The function has two singularities, a pole of order two at 0 and a pole of order one at i. Hence we get Laurent series expansions in two regions, namely for $|z-i| < 1$ and for $|z-i| > 1$. Note that $f(z)$, written in partial fractions, equals

$$f(z) = \frac{1}{z-i} - \frac{i}{(z-i)^2} - \frac{1}{z}.$$

   Now in $0 < |z-i| < 1$ we get

$$-\frac{1}{z} = -\frac{1}{z-i+i} = \frac{i}{1-\frac{i-z}{i}} = \sum_{j=0}^{\infty} i^{1-j}(-1)^j(z-i)^j$$

   and thus

$$f(z) = \frac{1}{z-i} - \frac{i}{(z-i)^2} + \sum_{j=0}^{\infty} i^{1-j}(-1)^j(z-i)^j.$$

   On the other hand, in $|z-i| > 1$ we have

$$-\frac{1}{z} = \frac{1}{i-z}\frac{1}{1-\frac{i}{i-z}} = (i-z)^{-1}\sum_{j=0}^{\infty} i^j(i-z)^{-j} = \sum_{j=0}^{\infty} i^j(-1)^{j+1}(z-i)^{-j-1}$$

   and thus

$$f(z) = \frac{1}{z-i} - \frac{i}{(z-i)^2} + \sum_{j=0}^{\infty} i^j(-1)^{j+1}(z-i)^{-j-1} = \sum_{j=2}^{\infty} i^j(-1)^{j+1}(z-i)^{-j-1}.$$

3. Use residue calculus to determine the value of the integral

$$\int_0^{2\pi} \frac{1}{2+\sin x} \mathrm{d}x.$$

Solution: We have

$$\int_0^{2\pi} \frac{1}{2+\sin x} \mathrm{d}x = \int_0^{2\pi} \frac{2i}{4i+e^{ix}-e^{-ix}} \mathrm{d}x = \int_\Gamma \frac{2i}{4i+z-\frac{1}{z}} \frac{1}{iz} \mathrm{d}z = 2\int_\Gamma \frac{1}{z^2+4iz-1} \mathrm{d}z$$

for the contour Γ describing the unit circle and its parametrization $x \mapsto e^{ix}$ for $x \in [0, 2\pi]$. The integrand has poles of order one at $(-2 \pm \sqrt{3})i$, of which $z_0 := (-2+\sqrt{3})i$ lies inside the unit circle. Thus for $f(z) = \frac{1}{z^2+4iz-1}$ we have

$$\int_0^{2\pi} \frac{1}{2+\sin x} \mathrm{d}x = 2(2\pi i)\mathrm{Res}\,(f, z_0) = 4\pi i \lim_{z\to z_0} \frac{1}{z-(-2-\sqrt{3})i} = \frac{4\pi i}{2\sqrt{3}i} = \frac{2\pi}{\sqrt{3}}.$$

4. Determine the number of zeroes of $z^5 - 3z^4 - 2$ in the disk $|z| < 2$.

Solution: In order to apply Rouché's theorem we define

$$f(z) := 3z^4 \qquad \text{and} \qquad g(z) := z^5 - 3z^4 - 2.$$

Then f has four zeroes in $|z| < 2$, namely a zero of order four at the origin. Moreover, on the boundary $|z| = 2$ we have

$$|f(z) - g(z)| = |z^5 - 2| \le |z|^5 + 2 = 2^5 + 2 = 34 < 48 = 3 \cdot 2^4 = |f(z)|.$$

It follows that g has the same number of zeroes in $|z| < 2$ as f, that is, four.

5. (a) Show that, for $A, B \in \mathbb{R}$ constant, the function $A \operatorname{Arg} z + B$ is harmonic in the right half-plane $\operatorname{Re} z > 0$.

   (b) Construct a Möbius transformation of the unit disk $|z| < 1$ onto the right half-plane $\operatorname{Re} z > 0$ such that the upper half-disk is mapped onto the first quadrant $\operatorname{Re} z > 0, \operatorname{Im} z > 0$.

   (c) Find a harmonic function u in the unit disk $|z| < 1$ that satisfies $u = \phi$ on the boundary $|z| = 1$, where

$$\phi = \begin{cases} 2 & \text{on the upper half-circle,} \\ -2 & \text{on the lower half-circle.} \end{cases}$$

Solution: (a) The function $-iA \operatorname{Log} z + B = -iA \operatorname{Log}|z| + A \operatorname{Arg} z + B$ is analytic in the right half-plane and has $A \operatorname{Arg} z + B$ as its real part, which is then harmonic.

(b) Such a Möbius transformation has to map the unit circle onto the imaginary axis. This can be done by, e.g., mapping -1 to 0, i to i and 1 to ∞. The corresponding Möbius transformation with these properties is

$$f(z) = \frac{1+z}{1-z}.$$

It maps the point 0 which is inside the disc to the point 1 which is inside the right-half plane and, thus, has the desired properties. For further use in (c) we note that f maps the upper half-circle to the "positive imaginary half-axis".

(c) According to (a), a harmonic function in the right half-plane having the value 2 on the "positive imaginary half-axis" and -2 on the negative one is $4 \operatorname{Arg} z/\pi$. As conformal mappings transform harmonic functions into harmonic functions,

$$u(z) = 4 \operatorname{Arg} \frac{1+z}{1-z},$$

and this function has the desired boundary values.

6. Compute the integral

$$\iint_{\partial_0 P} \frac{1}{2zw-3}\,\mathrm{d}z\,\mathrm{d}w,$$

where $\partial_0 P = \{(z,w) : |z| = |w| = 1\}$ is the distinguished boundary of the unit polydisk centered at the origin, taken with the usual orientation.

Solution: We rewrite the integral as

$$\iint_{\partial_0 P} \frac{1}{2zw-3}\,\mathrm{d}z\,\mathrm{d}w = \int_{|w|=1} \frac{1}{2w} \int_{|z|=1} \frac{1}{z-\frac{3}{2w}}\,\mathrm{d}z\,\mathrm{d}w,$$

and the integrand of the inner integral has, for fixed w with $|w|=1$, its only singularity at $3/(2w)$ which has modulus $3/2$, that is, the singularity is outside the contour $|z|=1$. Consequently, the integrand of the inner integral is analytic inside the contour for each w with $|w|=1$ and, thus, the inner integral is always zero. As a result,

$$\iint_{\partial_0 P} \frac{1}{2zw-3}\,\mathrm{d}z\,\mathrm{d}w = 0.$$