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Sketch of solutions

1. Problem: Determine all entire functions f for which (i) $f(1) = f'(1) = 0$ and (ii) $|f(z)| \leq |z-1|^2$ for all $z \in \mathbb{C}$.

   Solution: An entire function has an everywhere convergent power series expansion around $z = 1$: $f = \sum_0^\infty a_n(z-1)^n$. Condition (i) is equivalent to $a_0 = a_1 = 0$, and hence $f(z) = (z-1)^2 g(z)$, where $g(z)$ is another entire function. The condition (ii) now implies that $|g(z)| \leq 1$; so g is a bounded entire function, and hence by Liouville's theorem, constant. Answer: any such function is $f(z) = c(z-1)^2$ where $|c| \leq 1$.

2. Problem: Describe the region in the w-plane that is the image of

$$D := \{z \in \mathbb{C} \ : |z| < 1, \ \mathrm{Re} z > 0, \ \mathrm{Im} z > 0\},$$

   under the mapping

$$w = \frac{z^2+1}{z^2-1}.$$

   Solution: The mapping is the composition of $z \mapsto u = z^2$ and the Möbius transformation $w = m(u) = \frac{u+1}{u-1}$. The first map takes D to the part of $|z| < 1$ that is in the upper halfplane(by considering what happens to the polar representations of complex numbers under squaring). This semi-disk is bounded by the circle $|z| = 1$ and the real axis. Next step is to see what happens to these two curves under the the Möbius transformation, so compute some values- $M(\infty) = 1, M(-1) = 0, M(1) = \infty$ and use that these kind of transformations take lines/circles to lines/circles. The three values determine a line(containing ∞) through 0,1, that is the real axis. So M takes the real axis to the real axis. Next $M(i) = -i$ so M takes the three point $-1, 1, i$ on $|z| = 1$ to the three points $0, \infty, -i$ and the image has to be the imaginary axis(allternatively:one could have used that M is conformal). These two lines bound the four quadrants, but since $M(0) = -1$ and $M(i) = -i$ have to be boundary points, the image of the semi-disc has to be the third quadrant. The third quadrant is then the answer.

3. Problem: Determine the residue for each pole of the function

$$f(z) = \frac{2 + z^7}{z^4(z+1)^3}.$$

Use the result to calculate

$$\int_C f(z)dz,$$

where C is the curve $|\text{Re } z| + |\text{Im } z| = 0.5$, oriented counterclockwise.

Solution: The residues at the two poles $z = 0$(order 4) and $z = -1$(order 3) can be computed either by the formulas

$$Res(f, 0) = \frac{d^3}{dz^3}(z^4 f(z))|_{z=0} = \text{ long calculation } = -20$$

$$Res(f, 1) = \frac{d^3}{dz^2}((z+1)^3 f(z))|_{z=0} = \text{ long calculation } = 17,$$

or else by expanding the function in a Laurent series. We exemplify by one of the residues. At $z = 0$ we first get the following calculation by the standard Maclaurin expansion:

$$(z+1)^{-3} = 1 + (-3)z + \frac{(-3)(-4)}{2}z^2 + \frac{(-3)(-4)(-5)}{3!}z^3 + ... =$$

$$1 - 3z + 6z^2 - 10z^3 + 20z^4 + ...$$

Clearly $\frac{z^7}{z^4(z+1)^3} = \frac{1}{(z+1)^3}$ is an analytic function at $z = 0$, and will therefore not contribute to the residue, so we can compute

$$f(z) = \frac{2 + z^7}{z^4(z+1)^3} = 2(z^{-4} - 3z^{-3} + 6z^{-2} - 10z^{-1} + 20 + ...).$$

This gives the result $Res(f, 0) = -20$ again.

The only pole that lies inside the curve C is $z = 0$, so the integral will be $-40\pi i$ by the residue theorem.

4. Problem: Calculate the integral

$$\int_{-\infty}^{\infty} \frac{\sin \pi x}{x(1-x^2)} dx.$$

Note that the function to be integrated actually has limits for the zeros of the denominator $x = 0, \pm 1$(check by l'Hopital e.g.). Let the path Γ_R be the following: it starts in $-R < -1$ goes to a small halfcircle clockwise around -1, continues along the x-axis to a small halfcircle clockwise around 0, continues along the x-axis to a small halfcircle clockwise around 1 and then along the x-axis to R. The integral is the limit of the integral along Γ_R as $R \to \infty$ and the halvcircles shrink to respective center.

We are going to rewrite

$$\sin(\pi z) = \frac{e^{i\pi z} - e^{-i\pi z}}{2i} \tag{1}$$

The main technical idea is that $|e^{i\pi z}| = e^{-Im\pi z} \leq 1$ in the upper halfplane, while $|e^{-i\pi z}| \leq 1$ in the lower halfplane.

Then consider the integral of $\frac{e^{i\pi z}}{z(1-z^2)}$ on the curve Ω that consists of first going along Γ_R and then continuing along a half circle C_R^+ of radius R from R to $-R$. There are no poles inside Ω so the integral is 0. and since $e^{i\pi z}$ is bounded on C_R^+, this means by the residue theorem and a standard argument, that the integral on Γ_R is 0.

Now we do a similar argument for $\frac{e^{-i\pi z}}{z(1-z^2)}$. Let Ω_- be the path that begins as Γ_R and continues clock-wise along a half circle C_R^- of radius R from R to $-R$. This time the simple poles $-1, 1, 0$ are inside Ω_-, and they have residues 1/2,1/2,1. The integral along C_R^- has the limit 0, since $e^{-i\pi z}$ is bounded, and the residue theorem thus gives that the integral along Γ_R has the limit $-4\pi i$(the minus sign since we are going clockwise). Now we can use (1) to conclude that the limit is 2π.

5. Problem: Determine the Laurent series of the function

$$\frac{z^2}{z^2 - 4z + 3},$$

in the annulus $1 < |z| < 3$.

Solution: first do a partial fractions decomposition:

$$\frac{z^2}{z^2 - 4z + 3} = 1 + (\frac{9}{2})\frac{1}{z-3} - (\frac{1}{2})\frac{1}{z-1}. \tag{2}$$

Then use the geometric series :

$$\frac{1}{z-3} = (\frac{-1}{3})\frac{1}{1-(z/3)} = \frac{-1}{3}\sum_{n=0}^{\infty}(z/3)^n.$$

Again:

$$\frac{1}{z-1} = (\frac{1}{z})\frac{1}{1-1/z} = \sum_{n=1}^{\infty}(1/z)^n.$$

Substituting these two expressions into (2) solves the problem.

6. Problem:

(a) Show that a function $f(z)$, which is analytic in a neighbourhood N of $z = 0$, may be developed in a series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n (1+z)^{-n}, .$$

where the series is convergent and the equality is true in some neighbourhood N_1 of $z = 0$.

(b) Determine the first three non-zero coefficients in such a series expression for $\sin z$.

Solution:a) The map $z \mapsto w = M(z) = z/(1+z)$ is 1-1 onto and analytic everywhere except at $z = -1$, and takes $z = 0$ to $w = 0$. Its inverse function is $z = M^{-1}(w) = w/(1-w)$ has similar properties. If $g(w) = \sum_{n=0}^{\infty} a_n w^n$, then the power series identity in the problem says formally that

$$f(z) = g(M(z)) \iff f(M^-1(w)) = g(w) \tag{3}$$

. Now we can argue as follows: by continuity there is a small neighbourhood N_1 of $w = 0$ that is mapped by M^{-1} into N, and consequently the composite map $g(w) := f(M^{-1}(w))$ is analytic in N_1. If we develop g in a power series

$$g(w) = \sum_{n=0}^{\infty} a_n w^n,$$

and use (3) we are done.

b) The coefiicients are determined by $\sin(w/(1-w)) = \sum_{n=0}^{\infty} a_n w^n$.

Now

$$\sin(w/(1-w)) = \sin(w + w^2 + w^3 + ...) =$$

$$(w+w^2+w^3+...)-(w+w^2+w^3+...)^3/3!+(w+w^2+w^3+...)^5/5!+... =$$

$$w + w^2 + (5/6)w^3 + ...,$$

so $a_0 = 0, a_1, a_2 = 1, a_3 = 5/6$.